Research Article

Ran Zhang, Xiao-Chuan Xu, Chuan-Fu Yang* and Natalia Pavlovna Bondarenko

Determination of the impulsive Sturm–Liouville operator from a set of eigenvalues

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Abstract: In this work, we consider the inverse spectral problem for the impulsive Sturm–Liouville problem on $(0, \pi)$ with the Robin boundary conditions and the jump conditions at the point $\frac{\pi}{2}$. We prove that the potential $M(x)$ on the whole interval and the parameters in the boundary conditions and jump conditions can be determined from a set of eigenvalues for two cases: (i) the potential $M(x)$ is given on $(0, \frac{(1+\alpha)\pi}{4})$; (ii) the potential $M(x)$ is given on $(\frac{(1+\alpha)\pi}{4}, \pi)$, where $0 < \alpha < 1$, respectively. It is also shown that the potential and all the parameters can be uniquely recovered by one spectrum and some information on the eigenfunctions at some interior point.

Keywords: Inverse spectral problem, interior inverse problem, Sturm–Liouville operator, spectrum, uniqueness

MSC 2010: 34A55, 34B24, 47E05

1 Introduction

Define

$$\rho(x) = \begin{cases} 1, & x < \frac{\pi}{2}, \\ \alpha^2, & x > \frac{\pi}{2}, \end{cases} \qquad (0 < \alpha < 1).$$

Consider the following impulsive Sturm–Liouville problem:

$$ly := -y''(x) + M(x)y(x) = \lambda\rho(x)y(x), \quad x \in (0, \tfrac{\pi}{2}) \cup (\tfrac{\pi}{2}, \pi), \tag{1.1}$$

with the boundary conditions

$$U(y) := y'(0) - hy(0) = 0, \tag{1.2}$$
$$V(y) := y'(\pi) + Hy(\pi) = 0, \tag{1.3}$$

***Corresponding author: Chuan-Fu Yang,** Department of Applied Mathematics, School of Science, Nanjing University of Science and Technology, Nanjing, 210094, Jiangsu, P. R. China, e-mail: chuanfuyang@njust.edu.cn
Ran Zhang, Department of Applied Mathematics, School of Science, Nanjing University of Science and Technology, Nanjing, 210094, Jiangsu, P. R. China, e-mail: ranzhang9203@163.com
Xiao-Chuan Xu, School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing, 210044, Jiangsu, P. R. China, e-mail: xiaochuanxu@126.com
Natalia Pavlovna Bondarenko, Department of Applied Mathematics and Physics, Samara National Research University, Moskovskoye Shosse 34, Samara 443086; and Department of Mechanics and Mathematics, Saratov State University, Astrakhanskaya 83, Saratov 410012, Russia, e-mail: bondarenkonp@info.sgu.ru

and the jump conditions

$$\begin{cases} y\left(\frac{\pi}{2}+0\right) = \beta y\left(\frac{\pi}{2}-0\right), \\ y'\left(\frac{\pi}{2}+0\right) = \beta^{-1} y'\left(\frac{\pi}{2}-0\right) + a y\left(\frac{\pi}{2}-0\right). \end{cases} \tag{1.4}$$

Here λ is the spectral parameter, $M(x)$ is a real-valued function in $L^2(0,\pi)$, h, H, β, a are real, and $\beta > 0$. Problem (1.1)–(1.4), denoted by $L = L(M(x), \rho(x), h, H, \beta, a)$, is called a boundary value problem for the Sturm–Liouville equation with the discontinuity conditions at $\frac{\pi}{2}$.

The boundary value problems with a discontinuous point inside the interval frequently appear in mathematics, physics, geophysics, and other aspects of natural sciences (see [1, 2, 6, 10, 14]). Generally, such problems are related to discontinuous material characters of a intermediary. This kind of problem has been studied by many authors (see, e.g., [3, 5, 7, 23, 27]).

In general, for reconstructing the potential on the whole interval and all parameters about the Sturm–Liouville operator, it is necessary to specify two spectra of the problem with different boundary conditions (see, e.g., [16, 17, 26]). Hochstadt and Lieberman (see [8]) showed that if the potential $M(x)$ is known a priori on the half-interval $(\frac{\pi}{2}, \pi)$, then a single spectrum is sufficient to determine $M(x)$ on the half-interval $(0, \frac{\pi}{2})$. This is the so-called half-inverse problem which has been generalized into many cases (see [9, 11, 18, 20–22, 24, 25] and the references therein).

Nabiev and Amirov (see [14]) studied the boundary value problem $L = L(M(x), \rho(x), h, H, 1, 0)$, where $\beta = 1$ and $a = 0$, and gave some integral representations for the solutions of equation (1.1). In 2008, Shieh and Yurko (see [19]) gave the uniqueness theorem of the half-inverse problem for the problem $L = L(M(x), 1, h, H, \beta, a)$, where $\rho(x) \equiv 1$, β, a and H are assumed to be known a priori. In [28], Yurko studied the problem $L = L(M(x), \rho(x), h, H, \beta, a)$, and proved that the potential $M(x)$ and the coefficients in the boundary conditions and the jump conditions can be uniquely determined from the Weyl-type function or from two spectra.

In 2001, Mochizuki and Trooshin (see [12]) studied the problem $L(M(x), 1, h, H, 0, 1)$, where $\rho(x) \equiv 1$, $a = 1$ and $\beta = 0$, and proved that a set of values of the logarithmic derivative of eigenfunctions at some an internal point and spectrum can uniquely determine the potential $M(x)$ on $(0, \pi)$. They used the same method for reconstructing the potential for Dirac operator (see [13]). Yang (see [25]) considered the problem $L(M(x), 1, h, H, \beta, a)$, where $\rho(x) \equiv 1$, and showed that the potential $M(x)$ can uniquely be determined by a set of values of eigenfunctions at some an internal point $\frac{\pi}{2}$ and one spectrum. For the Dirac operator, the similar problems were studied in [6, 24].

In [15], Ozkan, Keskin and Cakmak considered the problem $L(M(x), \rho(x), 0, 0, \beta, 0)$, where h, H and a are assumed to be zero. They showed that if the potential $M(x)$ is prescribed on $(0, \frac{\pi}{2})$ (see Figure 1), then only one spectrum is sufficient to determine $M(x)$ on the interval $(0, \pi)$ and $\rho(x)$, β. The assumptions proposed in [15] to reconstruct the potential are overdetermined. In fact, it is enough to assume that the potential $M(x)$ is given on a smaller interval $(0, \frac{(1+\alpha)\pi}{4})$ (see Figure 2).

In this paper, we consider the problem $L = L(M(x), \rho(x), h, H, \beta, a)$ and prove that if the potential $M(x)$ on $(0, \frac{(1+\alpha)\pi}{4})$ (see Figure 2) and h are given, then only a single spectrum is sufficient to determine $M(x)$ on $(0, \pi)$, $\rho(x)$, H, β and a. We also consider the case that the potential $M(x)$ is given on the right "half-interval" (see Figure 3), and prove a uniqueness theorem. Also it is shown that potential $M(x)$ on $(0, \pi)$, $\rho(x)$, β, a, h

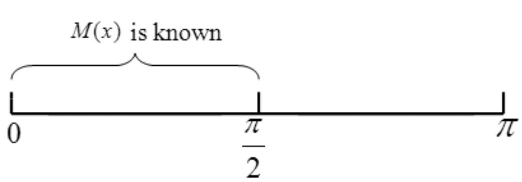

Figure 1: The case in [15].

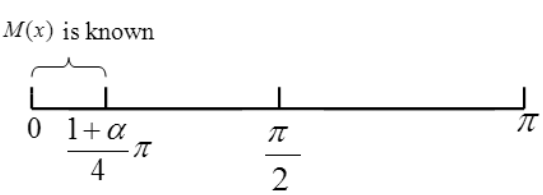

Figure 2: Case (i) in this paper.

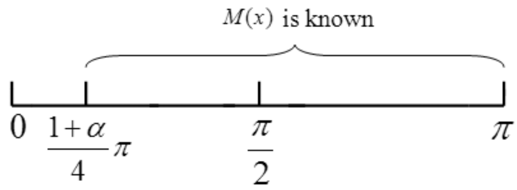

Figure 3: Case (ii) in this paper.

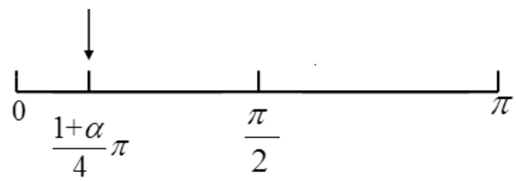

Figure 4: The interior point.

and H can be uniquely determined by one spectrum and some information on eigenfunctions at the internal point $\frac{(1+\alpha)\pi}{4}$ (see Figure 4).

2 Preliminaries

Let $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be the solutions of equation (1.1), satisfying the initial conditions $\varphi(0,\lambda)=1$, $\varphi'(0,\lambda)=h$, $\psi(\pi,\lambda)=1$, $\psi'(\pi,\lambda)=-H$ and the jump condition (1.4). Denote

$$\sigma(x)=\int_0^x\sqrt{\rho(t)}\,dt,\qquad k=\sqrt{\lambda},\qquad \tau=\operatorname{Im}k.$$

Lemma 2.1. *The following asymptotic relations hold as $|k|\to\infty$. For $\frac{\pi}{2}<x<\pi$,*

$$\varphi(x,\lambda)=\beta^{+}\cos k\sigma(x)+\beta^{-}\cos k(\pi-\sigma(x))+O(k^{-1}\exp(|\tau|\sigma(x))),\tag{2.1}$$

$$\varphi'(x,\lambda)=-k\alpha\beta^{+}\sin k\sigma(x)+k\alpha\beta^{-}\sin k(\pi-\sigma(x))+O(\exp(|\tau|\sigma(x)).\tag{2.2}$$

For $0<x<\frac{\pi}{2}$,

$$\psi(x,\lambda)=A^{+}\cos k(\sigma(\pi)-\sigma(x))+A^{-}\cos k(\sigma(\pi)+\sigma(x)-\pi)+O(k^{-1}\exp(|\tau|(\sigma(\pi)-\sigma(x)))),\tag{2.3}$$

$$\psi'(x,\lambda)=kA^{+}\sin k(\sigma(\pi)-\sigma(x))-kA^{-}\sin k(\sigma(\pi)+\sigma(x)-\pi)+O(\exp(|\tau|(\sigma(\pi)-\sigma(x)))),\tag{2.4}$$

where $\beta^{\pm}=\frac{1}{2}(\beta\pm\frac{1}{\alpha\beta})$, $A^{\pm}=\frac{1}{2}(\frac{1}{\beta}\pm\alpha\beta)$.

Proof. Let us prove (2.1) and (2.2). Relations (2.3) and (2.4) can be obtained similarly. For $0<x<\frac{\pi}{2}$ the solution $\varphi(x,\lambda)$ satisfies the following standard asymptotic formulas (see, e.g., [5]) as $|k|\to\infty$:

$$\varphi(x,\lambda)=\cos kx+O(k^{-1}\exp(|\tau|x)),\tag{2.5}$$

$$\varphi'(x,\lambda)=-k\sin kx+O(\exp(|\tau|x)).\tag{2.6}$$

Consider the solutions $C(x,\lambda)$ and $S(x,\lambda)$ of equation (1.1) on $(\frac{\pi}{2},\pi)$, satisfying the initial conditions $C(\frac{\pi}{2},\lambda)=1$, $C'(\frac{\pi}{2},\lambda)=0$, $S(\frac{\pi}{2},\lambda)=0$, $S'(\frac{\pi}{2},\lambda)=1$. They satisfy the following standard asymptotic relations:

$$C(x,\lambda)=\cos k\alpha(x-\tfrac{\pi}{2})+O(k^{-1}\exp(|\tau|(x-\tfrac{\pi}{2}))),\tag{2.7}$$

$$C'(x,\lambda)=-k\alpha\sin k\alpha(x-\tfrac{\pi}{2})+O(\exp(|\tau|(x-\tfrac{\pi}{2}))),\tag{2.8}$$

$$S(x,\lambda)=\frac{\sin k\alpha(x-\frac{\pi}{2})}{k\alpha}+O(k^{-2}\exp(|\tau|(x-\tfrac{\pi}{2}))),\tag{2.9}$$

$$S'(x,\lambda)=\cos k\alpha(x-\tfrac{\pi}{2})+O(k^{-1}\exp(|\tau|(x-\tfrac{\pi}{2}))),\tag{2.10}$$

as $|k|\to\infty$.

For $\frac{\pi}{2}<x<\pi$, the solution $\varphi(x,\lambda)$ can be represented in the following form:

$$\varphi(x,\lambda)=D_1(\lambda)C(x,\lambda)+D_2(\lambda)S(x,\lambda).\tag{2.11}$$

Substituting (2.5), (2.6) and (2.11) into the jump conditions (1.4), we get

$$D_1(\lambda)=\beta\cos\tfrac{k\pi}{2}+O(k^{-1}\exp(|\tau|\tfrac{\pi}{2})),\tag{2.12}$$

$$D_2(\lambda)=-\beta^{-1}k\sin\tfrac{k\pi}{2}+O(\exp(|\tau|\tfrac{\pi}{2})),\tag{2.13}$$

as $|k| \to \infty$. Substituting asymptotics (2.12), (2.13) together with (2.7), (2.8), (2.9), (2.10) into (2.11) and into the relation

$$\varphi'(x,\lambda) = D_1(\lambda)C'(x,\lambda) + D_2(\lambda)S'(x,\lambda), \quad \tfrac{\pi}{2} < x < \pi,$$

we arrive at (2.1) and (2.2). □

Define

$$\langle \varphi(x,\lambda), \psi(x,\lambda)\rangle := \varphi(x,\lambda)\psi'(x,\lambda) - \varphi'(x,\lambda)\psi(x,\lambda).$$

It is easy to verify that if $y(x)$ and $z(x)$ satisfy equation (1.1) and the jump condition (1.4), then $\langle y,z\rangle$ is independent of x, and

$$\langle y,z\rangle\big|_{x=\frac{\pi}{2}-0} = \langle y,z\rangle\big|_{x=\frac{\pi}{2}+0}.$$

Denote

$$\Delta(\lambda) = \langle \varphi,\psi\rangle = V(\varphi) = -U(\psi). \tag{2.14}$$

The function $\Delta(\lambda)$ is called the characteristic function of L, which is entire in λ, and it has an at most countable set of zeros $\{\lambda_n\}_{n\ge 0}$.

Lemma 2.2. *The following statements hold:*

(1) *The zeros $\{\lambda_n\}_{n\ge0}$ of the characteristic function $\Delta(\lambda)$ coincide with the eigenvalues of the boundary value problem L.*

(2) *The functions $\varphi(x,\lambda_n)$ and $\psi(x,\lambda_n)$ are corresponding eigenfunctions and exists a sequence $\{\delta_n\}$, $\delta_n \ne 0$, such that*

$$\psi(x,\lambda_n) = \delta_n\varphi(x,\lambda_n). \tag{2.15}$$

(3) *For each $n \ge 0$, the eigenvalues λ_n and the corresponding eigenfunctions $\varphi(x,\lambda_n)$, $\psi(x,\lambda_n)$ are real.*

Proof. Similar to the proof of [14], so we omit the proof. □

Next, we denote by $L^2((0,\pi);\rho(x))$ a space which has the inner product

$$(\varphi,\psi) = \int_0^\pi \varphi(x)\psi(x)\rho(x)\,dx.$$

Let $\alpha_n (n \ge 0)$ be the normalized constants, which are defined as

$$\alpha_n := \int_0^\pi \rho(x)\varphi^2(x,\lambda_n)\,dx \quad \text{for all } n \ge 0.$$

Lemma 2.3. *The following relation holds:*

$$\alpha_n\delta_n = -\dot\Delta(\lambda_n), \tag{2.16}$$

where $\dot\Delta(\lambda_n) = \frac{d}{d\lambda}\Delta(\lambda)$.

Proof. Note that

$$-\varphi''(x,\lambda) + M(x)\varphi(x,\lambda) = \lambda\rho(x)\varphi(x,\lambda)$$

and

$$-\psi''(x,\lambda_n) + M(x)\psi(x,\lambda_n) = \lambda_n\rho(x)\psi(x,\lambda_n).$$

Multiplying the two equations by $\psi(x,\lambda_n)$ and $\varphi(x,\lambda)$, respectively, and subtracting the second equation from the first equation, it follows that

$$\psi''(x,\lambda_n)\varphi(x,\lambda) - \varphi''(x,\lambda)\psi(x,\lambda_n) = \frac{d}{dx}\langle\varphi(x,\lambda),\psi(x,\lambda_n)\rangle = (\lambda-\lambda_n)\rho(x)\varphi(x,\lambda)\psi(x,\lambda_n).$$

Integrating the above equality from 0 to π and considering the jump point, we can obtain that

$$\begin{aligned}\langle\varphi(x,\lambda),\psi(x,\lambda_n)\rangle\big|_0^{\frac{\pi}{2}-0} + \langle\varphi(x,\lambda),\psi(x,\lambda_n)\rangle\big|_{\frac{\pi}{2}+0}^{\pi} &= (\lambda-\lambda_n)\int_0^\pi \rho(x)\varphi(x,\lambda)\psi(x,\lambda_n)\,dx\\ &= -H\varphi(\pi,\lambda) - \varphi'(\pi,\lambda) - \psi'(0,\lambda_n) + h\psi(0,\lambda_n)\\ &= -\Delta(\lambda).\end{aligned}$$

Dividing the two sides by $\lambda - \lambda_n$ and letting $\lambda \to \lambda_n$ yields

$$\int_0^\pi \rho(x)\varphi(x,\lambda_n)\psi(x,\lambda_n)\,dx = -\dot{\Delta}(\lambda_n).$$

Combining (2.15) with the definition of α_n, we arrive at (2.16). □

Remark 2.1. It follows from Lemma 2.3 that all eigenvalues λ_n are simple.

From (2.1), (2.2) and (2.14), we have that as $|k| \to \infty$,

$$\Delta(\lambda) = \alpha k[\beta^+ \sin k\sigma(\pi) - \beta^- \sin k(\pi - \sigma(\pi))] + O(\exp|\tau|\sigma(\pi)). \tag{2.17}$$

Define the sector $S_{\varepsilon,k^*} := \{k \in \mathbb{C} : |k| \geq k^*,\ \varepsilon < \arg k < \pi - \varepsilon\}$ for $\varepsilon > 0$, $k^* > 0$. The asymptotic formula (2.17) implies

$$|\Delta(k^2)| \geq C_{\varepsilon,k^*}|k|\exp(|\tau|\sigma(\pi)), \quad k \in S_{\varepsilon,k^*}, \tag{2.18}$$

where C_{ε,k^*} is a constant.

3 Extension of the half-inverse problem and proofs

Together with the problem L we consider a boundary value problem $\tilde{L} = L(\tilde{M}(x), \tilde{\rho}(x), \tilde{h}, \tilde{H}, \tilde{\beta}, \tilde{a})$ of the same form but with the different coefficients $\tilde{M}(x), \tilde{\rho}(x), \tilde{h}, \tilde{H}, \tilde{\beta}$, and $\tilde{a}$. We agree that if a certain symbol υ denotes an object related to L, then $\tilde{\upsilon}$ denote the analogous object related to $\tilde{L}$.

Theorem 3.1. *If $\lambda_n = \tilde{\lambda}_n$ for all $n \geq 0$, $M(x) = \tilde{M}(x)$ on $(0, \frac{(1+\alpha)}{4}\pi)$ and $h = \tilde{h}$, then $M(x) = \tilde{M}(x)$ almost everywhere on $(0,\pi)$, $H = \tilde{H}$, $\rho(x) = \tilde{\rho}(x)$, $\beta = \tilde{\beta}$ and $a = \tilde{a}$.*

Theorem 3.2. *If $\lambda_n = \tilde{\lambda}_n$ for all $n \geq 0$, $M(x) = \tilde{M}(x)$ on $(\frac{(1+\alpha)}{4}\pi, \pi)$, $a = \tilde{a}$ and $H = \tilde{H}$, then $M(x) = \tilde{M}(x)$ almost everywhere on $(0,\pi)$, $\rho(x) = \tilde{\rho}(x)$, $\beta = \tilde{\beta}$ and $h = \tilde{h}$.*

In order to prove the both Theorems 3.1 and 3.2 we need the following lemma.

Lemma 3.3. *If $\lambda_n = \tilde{\lambda}_n$ for all $n \geq 0$, then $\rho(x) = \tilde{\rho}(x)$ and $\beta = \tilde{\beta}$.*

Proof. The characteristic functions $\Delta(\lambda)$ and $\tilde{\Delta}(\lambda)$ are entire functions of λ of order $\frac{1}{2}$. By the Hadamard factorization theorem, they can be uniquely determined by their zeros up to multiplicative constants. Since their zeros coincide, i.e., $\lambda_n = \tilde{\lambda}_n$ for all $n \geq 0$, we have $\Delta(\lambda) = C\tilde{\Delta}(\lambda)$, where $C \neq 0$ is a constant. In view of (2.17) and the similar asymptotic formula for $\tilde{\Delta}(\lambda)$, we conclude that $\alpha = \tilde{\alpha}$, $\beta^+ = C\tilde{\beta}^+$ and $\beta^- = C\tilde{\beta}^-$, so

$$\frac{1}{2}\Big(\beta \pm \frac{1}{\alpha\beta}\Big) = \frac{C}{2}\Big(\tilde{\beta} \pm \frac{1}{\alpha\tilde{\beta}}\Big).$$

Consequently, $\beta = C\tilde{\beta}$, $\beta^{-1} = C\tilde{\beta}^{-1}$. Since $\beta > 0$ and $\tilde{\beta} > 0$, we get $\beta = \tilde{\beta}$. □

Proof of Theorem 3.1. Let the boundary value problems L and $\tilde{L}$ satisfy the conditions of Theorem 3.1. By virtue of Lemma 3.3, $\alpha = \tilde{\alpha}$ and $\beta = \tilde{\beta}$. For brevity, denote $c = \frac{1+\alpha}{4}\pi$. Relation (2.14) implies

$$\Delta(\lambda) = \varphi(c,\lambda)\psi'(c,\lambda) - \varphi'(c,\lambda)\psi(c,\lambda).$$

Substituting $\lambda = \lambda_n$, we get

$$\varphi(c,\lambda_n)\psi'(c,\lambda_n) - \varphi'(c,\lambda_n)\psi(c,\lambda_n) = 0, \quad n \geq 0.$$

Consequently, if $\varphi(c,\lambda_n) \neq 0$, we have

$$\frac{\psi'(c,\lambda_n)}{\psi(c,\lambda_n)} = \frac{\varphi'(c,\lambda_n)}{\varphi(c,\lambda_n)}, \quad n \geq 0. \tag{3.1}$$

A similar relation holds for $\tilde{L}$:

$$\frac{\tilde{\psi}'(c,\lambda_n)}{\tilde{\psi}(c,\lambda_n)} = \frac{\tilde{\varphi}'(c,\lambda_n)}{\tilde{\varphi}(c,\lambda_n)}, \quad n \geq 0. \tag{3.2}$$

Since $M(x) = \tilde{M}(x)$ on $(0, c)$ and $h = \tilde{h}$, we have $\varphi(x, \lambda) \equiv \tilde{\varphi}(x, \lambda)$ for $x \in [0, c]$. Hence relations (3.1) and (3.2) yield

$$\psi'(c, \lambda_n)\tilde{\psi}(c, \lambda_n) - \psi(c, \lambda_n)\tilde{\psi}'(c, \lambda_n) = 0, \quad n \geq 0. \tag{3.3}$$

Note that $\varphi(c, \lambda_n) = 0$ implies $\psi(c, \lambda_n) = \tilde{\psi}(c, \lambda_n) = 0$, so this case also leads to (3.3).

Thus, we have proved that the entire function

$$H(\lambda) := \psi'(c, \lambda)\tilde{\psi}(c, \lambda) - \psi(c, \lambda)\tilde{\psi}'(c, \lambda)$$

has zeros $\{\lambda_n\}_{n\geq 0}$. Consequently, the function $\frac{H(\lambda)}{\Delta(\lambda)}$ is entire. In view of (2.3), (2.4) and similar relations for $\tilde{\psi}(x, \lambda)$, we have

$$H(\lambda) = O(\exp(2|\tau|(\sigma(\pi) - \sigma(c)))) = O(\exp(|\tau|\sigma(\pi))), \quad |\lambda| \to \infty.$$

Together with (2.18), the latter estimate yields

$$\left|\frac{H(k^2)}{\Delta(k^2)}\right| \leq C_{\varepsilon,k^*}|k|^{-1}, \quad k \in S_{\varepsilon,k^*},$$

for some positive constants ε and k^*. Applying the Phragmen–Lindelöf Theorem [4], we show that the function $\frac{H(\lambda)}{\Delta(\lambda)}$ is bounded in the whole λ-plane. Then by Liouville's Theorem, we conclude that $H(\lambda) \equiv 0$. Hence

$$\frac{\psi(c, \lambda)}{\psi'(c, \lambda)} = \frac{\tilde{\psi}(c, \lambda)}{\tilde{\psi}'(c, \lambda)}. \tag{3.4}$$

Note that $\frac{\psi(c,\lambda)}{\psi'(c,\lambda)}$ is the Weyl function, defined in [28], of the boundary value problem for equation (1.1) on the interval (c, π) with the boundary conditions $y'(c) = 0$, $V(y) = 0$ and the jump conditions (1.4). It has been shown in [28] that the Weyl function uniquely specifies the function $M(x)$ on (c, π) and the coefficients a, H. Consequently, relation (3.4) implies $M(x) = \tilde{M}(x)$ a.e. on (c, π), $a = \tilde{a}$, $H = \tilde{H}$, so the assertion of the theorem is proved. □

Proof of Theorem 3.2. By Lemma 3.3 and the conditions of Theorem 3.2, we have $\alpha = \tilde{\alpha}$, $\beta = \tilde{\beta}$, $H = \tilde{H}$, $a = \tilde{a}$, $M(x) = \tilde{M}(x)$ on (c, π). Consequently, $\psi(x, \lambda) \equiv \tilde{\psi}(x, \lambda)$ on (c, π). Using (3.1) and (3.2), we show that

$$\varphi'(c, \lambda_n)\tilde{\varphi}(c, \lambda_n) - \varphi(c, \lambda_n)\tilde{\varphi}'(c, \lambda_n) = 0, \quad n \geq 0,$$

so the entire function

$$G(\lambda) := \varphi'(c, \lambda)\tilde{\varphi}(c, \lambda) - \varphi(c, \lambda)\tilde{\varphi}'(c, \lambda)$$

has zeros $\{\lambda_n\}_{n\geq 0}$. In view of the asymptotic formulas (2.5), (2.6) and similar relations for $\tilde{\varphi}(x, \lambda)$, we have

$$G(\lambda) = O(\exp(2|\tau|c)) = O(\exp(|\tau|\sigma(\pi))), \quad |\lambda| \to \infty.$$

Following the proof of Theorem 3.1 and applying the Phragmen–Lindelöf Theorem to the entire function $\frac{G(\lambda)}{\Delta(\lambda)}$, we show that $G(\lambda) \equiv 0$, so

$$\frac{\varphi(c, \lambda)}{\varphi'(c, \lambda)} = \frac{\tilde{\varphi}(c, \lambda)}{\tilde{\varphi}'(c, \lambda)}.$$

The fraction $\frac{\varphi(c,\lambda)}{\varphi'(c,\lambda)}$ is the Weyl function of the boundary value problem for equation (1.1) on $(0, c)$ with boundary conditions $U(y) = 0$, $y'(c) = 0$ and without discontinuity (see [5]). By [5, Theorem 1.4.7], the Weyl function uniquely specifies $M(x)$ on $(0, c)$ and the coefficient h, so Theorem 3.2 is proved. □

4 An interior inverse problem

We consider the interior inverse problem for the same boundary value problem $L = L(M(x), \rho(x), h, H, \beta, a)$ and obtain the corresponding result. To this end, we introduce a sequence $\{\kappa_n\}_{n\geq 0}$ defined by

$$\kappa_n = \frac{d}{dx} \log|\varphi(x, \lambda_n)|_{x=c},$$

where $c = \frac{(1+\alpha)\pi}{4}$.

Theorem 4.1. *If, for all $n \geq 0$, $\lambda_n = \tilde{\lambda}_n$ and $\kappa_n = \tilde{\kappa}_n$, then $M(x) = \tilde{M}(x)$ almost everywhere on $(0, \pi)$, $H = \tilde{H}$, $h = \tilde{h}$, $\rho(x) = \tilde{\rho(x)}$, $\beta = \tilde{\beta}$ and $a = \tilde{a}$.*

Proof. Firstly, the assumption that $\lambda_n = \tilde{\lambda}_n$ can determine $\alpha = \tilde{\alpha}$ and $\beta = \tilde{\beta}$ by Lemma 3.3. From $\kappa_n = \tilde{\kappa}_n$, we see that

$$\frac{\varphi(c, \lambda_n)}{\varphi'(c, \lambda_n)} = \frac{\tilde{\varphi}(c, \lambda_n)}{\tilde{\varphi}'(c, \lambda_n)}.$$

Then the entire function

$$G(\lambda) = \varphi'(c, \lambda)\tilde{\varphi}(c, \lambda) - \varphi(c, \lambda)\tilde{\varphi}'(c, \lambda)$$

has zeros $\{\lambda_n\}_{n\geq 0}$. Similarly to the proof of Theorem 3.2, we have that $M(x) = \tilde{M}(x)$ on $(0, c)$ and $h = \tilde{h}$.

Once we get that $M(x) = \tilde{M}(x)$ on $(0, c)$ and $h = \tilde{h}$, by Theorem 3.1 we have that $M(x) = \tilde{M}(x)$ a.e. on $(0, \pi)$, $H = \tilde{H}$, and $a = \tilde{a}$. This completes the proof. □

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References

[1] R. K. Amirov, On Sturm–Liouville operators with discontinuity conditions inside an interval, *J. Math. Anal. Appl.* **317** (2006), no. 1, 163–176.

[2] R. S. Anderssen, The effect of discontinuities in density and shear velocity on the asymptotic overtone structure of tortional eigenfrequencies of the Earth, *Geophys J. R. Astr. Soc.* **50** (1997), 303–309.

[3] L.-E. Andersson, Inverse eigenvalue problems with discontinuous coefficients, *Inverse Problems* **4** (1988), no. 2, 353–397.

[4] S. A. Buterin, G. Freiling and V. A. Yurko, Lectures in the theory of entire functions, Schriftenreihe der Fakultät für Matematik SM-UDE-779, Duisbug–Essen University, 2014.

[5] G. Freiling and V. Yurko, *Inverse Sturm–Liouville Problems and Their Applications*, Nova Science, Huntington, 2001.

[6] Y. Guo, G. Wei and R. Yao, Inverse problem for interior spectral data of discontinuous Dirac operator, *Appl. Math. Comput.* **268** (2015), 775–782.

[7] O. H. Hald, Discontinuous inverse eigenvalue problems, *Comm. Pure Appl. Math.* **37** (1984), no. 5, 539–577.

[8] H. Hochstadt and B. Lieberman, An inverse Sturm–Liouville problem with mixed given data, *SIAM J. Appl. Math.* **34** (1978), no. 4, 676–680.

[9] R. O. Hryniv and Y. V. Mykytyuk, Half-inverse spectral problems for Sturm–Liouville operators with singular potentials, *Inverse Problems* **20** (2004), no. 5, 1423–1444.

[10] R. J. Krueger, Inverse problems for nonabsorbing media with discontinuous material properties, *J. Math. Phys.* **23** (1982), no. 3, 396–404.

[11] O. Martinyuk and V. Pivovarchik, On the Hochstadt–Lieberman theorem, *Inverse Problems* **26** (2010), no. 3, Article ID 035011.

[12] K. Mochizuki and I. Trooshin, Inverse problem for interior spectral data of the Sturm–Liouville operator, *J. Inverse Ill-Posed Probl.* **9** (2001), no. 4, 425–433.

[13] K. Mochizuki and I. Trooshin, Inverse problem for interior spectral data of the Dirac operator on a finite interval, *Publ. Res. Inst. Math. Sci.* **38** (2002), no. 2, 387–395.

[14] A. A. Nabiev and R. K. Amirov, On the boundary value problem for the Sturm–Liouville equation with the discontinuous coefficient, *Math. Methods Appl. Sci.* **36** (2013), no. 13, 1685–1700.

[15] A. S. Ozkan, B. Keskin and Y. Cakmak, Uniqueness of the solution of half inverse problem for the impulsive Sturm Liouville operator, *Bull. Korean Math. Soc.* **50** (2013), no. 2, 499–506.

[16] W. Rundell and P. E. Sacks, Reconstruction techniques for classical inverse Sturm–Liouville problems, *Math. Comp.* **58** (1992), no. 197, 161–183.

[17] W. Rundell and P. E. Sacks, Reconstruction of a radially symmetric potential from two spectral sequences, *J. Math. Anal. Appl.* **264** (2001), no. 2, 354–381.

[18] L. Sakhnovich, Half-inverse problems on the finite interval, *Inverse Problems* **17** (2001), no. 3, 527–532.

[19] C.-T. Shieh and V. A. Yurko, Inverse nodal and inverse spectral problems for discontinuous boundary value problems, *J. Math. Anal. Appl.* **347** (2008), no. 1, 266–272.

[20] C. Willis, Inverse Sturm–Liouville problems with two discontinuities, *Inverse Problems* **1** (1985), no. 3, 263–289.
[21] X.-C. Xu and C.-F. Yang, Inverse spectral problems for the Sturm–Liouville operator with discontinuity, *J. Differential Equations* **262** (2017), no. 3, 3093–3106.
[22] X.-C. Xu and C.-F. Yang, Reconstruction of the Sturm–Liouville operator with discontinuities from a particular set of eigenvalues, *Appl. Math. J. Chinese Univ. Ser. B* **33** (2018), no. 2, 225–233.
[23] C.-F. Yang, An interior inverse problem for discontinuous boundary-value problems, *Integral Equations Operator Theory* **65** (2009), no. 4, 593–604.
[24] C.-F. Yang, Hochstadt–Lieberman theorem for Dirac operator with eigenparameter dependent boundary conditions, *Nonlinear Anal.* **74** (2011), no. 7, 2475–2484.
[25] C.-F. Yang and X.-P. Yang, An interior inverse problem for the Sturm–Liouville operator with discontinuous conditions, *Appl. Math. Lett.* **22** (2009), no. 9, 1315–1319.
[26] V. A. Yurko, Integral transforms connected with discontinuous boundary value problems, *Integral Transform. Spec. Funct.* **10** (2000), no. 2, 141–164.
[27] V. A. Yurko, On boundary value problems with discontinuity conditions inside an interval, *Differ. Uravn.* **36** (2000), no. 8, 1139–1140, 1152.
[28] V. A. Yurko, Inverse spectral problems for Sturm–Liouville operators with complex weights, *Inverse Probl. Sci. Eng.* **26** (2018), no. 10, 1396–1403.