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# Transfer operators for coupled analytic maps

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*Abstract*. We consider analytically coupled circle maps (uniformly expanding and analytic) on the $\mathbb{Z}^d$-lattice with exponentially decaying interaction. We introduce Banach spaces for the infinite-dimensional system that include measures whose finite-dimensional marginals have analytic, exponentially bounded densities. Using residue calculus and 'cluster expansion'-like techniques we define transfer operators on these Banach spaces. We get a unique (in the considered Banach spaces) probability measure that exhibits exponential decay of correlations.

## 0. *Introduction*

Coupled map lattices were introduced by Kaneko (cf. [**12**] for a review) as systems that are mixing w.r.t. spatio–temporal shifts. Bunimovich and Sinai proved in [**6**] (cf. also the remarks on this in [**3**]) the existence of an invariant measure and its exponential decay of correlations for a one-dimensional lattice of weakly coupled maps by constructing a Markov partition and relating the system to a two-dimensional spin system.

Bricmont and Kupiainen extend this result in [**2**–**4**] to coupled circle maps over the $\mathbb{Z}^d$-lattice with analytic and Hölder-continuous weak interaction, respectively. They use a 'polymer' or 'cluster'-expansion for the Perron–Frobenius operator for the finite-dimensional subsystems over $\Lambda \subset \mathbb{Z}^d$ and write the $n$th iterate of this operator applied to the constant function one in terms of potentials for a $(d+1)$-dimensional spin system. Taking the limit as $n \to \infty$ and $\Lambda \to \mathbb{Z}^d$ they get existence and uniqueness (among measures with certain properties) of the invariant probability measure and exponential decay of correlations.

Baladi *et al* define in [**1**], for infinite-dimensional systems over the $\mathbb{Z}^d$-lattice, transfer operators on a Frechet space and, for $d = 1$, on a Banach space; they study the spectral properties of these operators, viewing the coupled operator as a perturbation of the uncoupled operator in the Banach case.

In [**13**] Keller and Künzle consider periodic or infinite one-dimensional lattices of weakly coupled maps of the unit interval. In particular they define transfer operators on the space $BV$ of measures whose finite-dimensional marginals have densities of bounded variation and prove the existence of an invariant probability measure. For the infinite-dimensional system they further show that for a small perturbation of the uncoupled map any invariant measure in $BV$ is close (in a specified sense) to what they found. Coupled map lattices with multi-dimensional local systems of the hyperbolic type have been studied by Pesin and Sinai [**16**], Jiang [**8, 9**], Jiang and Mazel [**10**], Jiang and Pesin [**11**] and Volevich [**18, 19**]. Detailed surveys on coupled map lattices can be found in [**5, 11, 3**].

In the above papers (except [**1, 13**]) the analysis has been performed only for Banach spaces defined for finite subsets $\Lambda$ of the lattice, and the (weak) limit of the invariant measure for $\Lambda \to \mathbb{Z}^d$ was taken afterwards. Here we present a new point of view in which a natural Banach space and transfer operators are defined for the infinite lattice of weakly coupled analytic maps (§1). The space contains consistent families of analytic densities over finite subsets of $\mathbb{Z}^d$. We take a weighted sup-norm so that the sup-norms of the densities for the subsystems over finitely many (say $N$) lattice points is bounded exponentially in $N$ (§2). We identify an ample subset of this space with a set of *rca* (regular, countably additive) measures (§4) that contains the unique invariant probability density (§2). We derive exponential decay of correlations for this measure and a certain class of observables from (the proof of) the spectral properties of our transfer operators (§§2 and 7). The operator for the coupled system and also the invariant measure are (for a small interaction) in fact perturbations of their counterparts in the uncoupled case. So the mixing properties are inherited from the single site systems. §8 contains the proofs.

Our approach provides a natural setting for an analysis of the full $\mathbb{Z}^d$ Perron–Frobenius operator in terms of cluster expansions over finite subsets of the lattice. Using residue calculus we introduce an integral representation for the Perron–Frobenius operator for finite-dimensional subsystems (§3) which yields a uniform control over the perturbation and also gives rise to an easy approach to stochastic perturbation (cf. [**15**]) which, however, we do not consider here.

Our ‘cluster expansion’ combinatorics (§5) uses ideas from the work of Maes and Van Moffaert [**15**] who have introduced a simplified (compared to that in [**2**]) polymer expansion. Apart from the analysis of the one-dimensional operator, which is fairly standard and for which we refer to for example [**2**], the paper should be self-contained.

## 1. *General setting*

We consider coupled map lattices in the following setting: the state space is $M = (S^1)^{\mathbb{Z}^d}$ where $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is the unit circle in the complex plane and $d$ a positive integer.

The map $S : M \to M$ is the composition $S = F \circ T^\epsilon$ of a coupling map $T^\epsilon$ depending on a (small) non-negative parameter $\epsilon$ and another parameter for the decay of interaction (cf. (1)) with an (uncoupled) map $F$ that acts on each component of $M$ separately. We make the following assumptions.

*Assumption 1.* $F(\mathbf{z}) = (f_p(z_p))_{p\in\mathbb{Z}^d}$ where $f_p : S^1 \to S^1$ are real analytic and expanding (i.e. $f_p' \geq \lambda_0 > 1$) maps that extend for some $\delta_1$ holomorphically to the interior of an annulus $A_{\delta_1} \stackrel{\text{def}}{=} \{z \in \mathbb{C} \mid -\delta_1 \leq \ln|z| \leq \delta_1\}$ and the family of Perron–Frobenius operators $\mathcal{L}_{f_p}$ for the single site systems uniformly satisfies a condition specified in §5.1 below (31).

(We need some more definitions to specify these conditions, but note that they are, in particular, satisfied if all $f_p$ are the same.)

We write $T^\epsilon : M \to M$ as $T^\epsilon(\mathbf{z}) = (T_p^\epsilon(\mathbf{z}))_{p\in\mathbb{Z}^d}$ and $T_p^\epsilon(\mathbf{z}) = z_p \exp[2\pi i \epsilon g_p(\mathbf{z})]$ with $g_p(\mathbf{z}) = \sum_{k=1}^{\infty} g_{p,k}(\mathbf{z})$. The function $g_{p,k}$ is real valued on $(S^1)^{\mathbb{Z}^d}$ and depends only on those $z_q$ with $\|p-q\| \le k$ (neighbours of distance at most k) where $\|p\| \stackrel{\text{def}}{=} \sum_{l=1}^{d} |p_l|$. We write $B_k(p) = \{q \in \mathbb{Z}^d \mid \|p-q\| \le k\}$ and also denote by $g_{p,k}$ the function from the finite-dimensional torus $(S^1)^{B_k(p)}$ to $\mathbb{R}$. We assume the following for the functions $g_{p,k}$.

*Assumption 2.* For all $p \in \mathbb{Z}^d$ and $k \ge 1$ each map $g_{p,k}$ extends to a holomorphic map $g_{p,k} : A_{\delta_1}^{B_k(p)} \to \mathbb{C}$ and its sup-norm (of modulus) is exponentially bounded by

$$\|g_{p,k}\|_{A_{\delta_1}^{B_k(p)}} \le c_1 \exp(-c_2 k^d) \tag{1}$$

with $c_1 > 0$ and $c_2$ larger than a certain constant specified in (100).

The parameter $c_1$ is actually redundant as it is multiplied by $\epsilon$ in the definition of $T_p^\epsilon$. We also have $\exp(-c_2 k^d) \le \exp(-\xi)\exp(-c_2^* k^d)$ for $c_2^* = c_2 - \xi$, $\xi > 0$, i.e. for any $\epsilon$ we can make the interaction small just by taking $c_2$ large. However, once we have chosen $c_2$ large enough to guarantee the convergence of the infinite sums in our analysis we can consider perturbations of the uncoupled map depending on the parameter $\epsilon$ only.

With the metric

$$d_\gamma(\mathbf{x}, \mathbf{y}) \stackrel{\text{def}}{=} \sup_{p\in\mathbb{Z}^d} \gamma^{\|p\|} \|x_p - y_p\| \tag{2}$$

for $0 < \gamma < 1$ $(M, d_\gamma)$ is a compact metric space. Its topology is the product topology on $(S^1)^{\mathbb{Z}^d}$. The Borel $\sigma$-algebra $\mathcal{B}$ on $M$ is the same as the product $\sigma$-algebra. $F$ and $T^\epsilon$ are continuous and measurable. Let $\mathcal{C}(M)$ denote the space of real-valued continuous functions on $(M, d_\gamma)$ with the sup-norm and $\mu$ the Lebesgue (product) measure on $M$.

For $\Lambda_1 \subseteq \Lambda_2 \subseteq \mathbb{Z}^d$, with $\Lambda_1$ finite and an integrable function $\psi$ on $M$ depending only on the $\Lambda_2$-coordinates, we define the projection

$$(\pi_{\Lambda_1}\psi)(\mathbf{z}_{\Lambda_1}) \stackrel{\text{def}}{=} \int_{(S^1)^{\Lambda_2\setminus\Lambda_1}} d\mu^{\Lambda_2\setminus\Lambda_1}(\mathbf{z}_{\Lambda_2\setminus\Lambda_1})\psi(\mathbf{z}_{\Lambda_1} \vee \mathbf{z}_{\Lambda_2\setminus\Lambda_1}). \tag{3}$$

## 2. *Main results*

For finite $\Lambda \subset \mathbb{Z}^d$ let $H(A_\delta^\Lambda)$ be the space of continuous functions on the closed polyannulus $A_\delta^\Lambda$ that are holomorphic on its interior and write $\|\cdot\|_\Lambda$ for the sup-norm (of modulus) on $H(A_\delta^\Lambda)$. Let $\mathcal{F}$ be the set of all finite subsets (including $\emptyset$) of $\mathbb{Z}^d$. We denote by $\mathcal{H}$ the vectorspace of all consistent families $\phi = (\phi_\Lambda)_{\Lambda\in\mathcal{F}}$ of functions $\phi_\Lambda \in H(A_\delta^\Lambda)$. Consistency means $\pi_{\Lambda_1}\phi_{\Lambda_2} = \phi_{\Lambda_1}$ for $\Lambda_1 \subseteq \Lambda_2 \in \mathcal{F}$. We write $\mu(\phi) \stackrel{\text{def}}{=} \phi_\emptyset$.

We want to define a norm on a (sufficiently large) subspace of $\mathcal{H}$ that should at least contain 'product densities' such as $h = (h_\Lambda)_{\Lambda\in\mathcal{F}}$ with $h_\Lambda(\mathbf{z}) = \prod_{p\in\Lambda} h_p(z_p)$, where $h_p \in H(A_\delta^{\{p\}})$ is the invariant probability density for the single system over $\{p\}$ (cf. §5.1).

Because of (32) the sup-norm $\|h_{\Lambda_1}\|_{\Lambda_1}$ does not grow faster than exponentially in $|\Lambda_1|$. Therefore we take a weighted sup-norm. For $0 < \vartheta < 1$ we define

$$\|\phi\|_\vartheta \stackrel{\text{def}}{=} \sup_{\Lambda\in\mathcal{F}} \vartheta^{|\Lambda|} \|\phi_\Lambda\|_\Lambda \tag{4}$$

and set $\mathcal{H}_\vartheta \stackrel{\text{def}}{=} \{\phi \in \mathcal{H} \mid \|\phi\|_\vartheta < \infty\}$. Then $(\mathcal{H}_\vartheta, \|\cdot\|_\vartheta)$ is a Banach space. In fact, if $(\phi^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{H}_\vartheta, \|\cdot\|_\vartheta)$ then for each $\Lambda \in \mathcal{F}$ the sequence $(\phi^n_\Lambda)_{n\in\mathbb{N}}$ is Cauchy in the Banach space $(H(A^\Lambda_\delta), \|\cdot\|_{A^\Lambda_\delta})$ and so converges to $\phi_\Lambda$. Consistency of $(\phi_\Lambda)_{\Lambda\in\mathcal{F}}$ follows from taking the limit (as $n \to \infty$) of $\pi_{\Lambda_1}\phi^n_{\Lambda_2} = \phi^n_{\Lambda_1}$ using the continuity of $\pi_{\Lambda_1}$ for any $\Lambda_1 \subseteq \Lambda_2 \in \mathcal{F}$. Analogously we define for $\Lambda \in \mathcal{F}$ the weighted norm on spaces $\mathcal{H}_{\Lambda,\vartheta}$ of consistent sub-families $(\phi_{\Lambda_1})_{\Lambda_1\subseteq\Lambda}$:

$$\|\phi\|_{\Lambda,\vartheta} \stackrel{\text{def}}{=} \sup_{\Lambda_1\subseteq\Lambda} \vartheta^{|\Lambda_1|}\|\phi_{\Lambda_1}\|_{\Lambda_1}. \tag{5}$$

We get the same (topological) vector space as $(H(A^\Lambda_\delta), \|\cdot\|_\Lambda)$, but the constants for the estimates of the norms are unbounded as $|\Lambda|$ increases.

For given $\Lambda_1 \subseteq \Lambda_2 \in \mathcal{F}$ and $N \in \mathbb{N}$ we have a map,

$$\pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_2}\circ T^{\Lambda_2,\epsilon}} \circ \pi_{\Lambda_2} : (\mathcal{H}_\vartheta, \|\cdot\|_\vartheta) \to (\mathcal{H}_{\Lambda_1,\vartheta}, \|\cdot\|_{\Lambda_1,\vartheta}), \tag{6}$$

where $\mathcal{L}^N_{F^{\Lambda_2}\circ T^{\Lambda_2,\epsilon}}$ is the Perron–Frobenius operator for the finite-dimensional system over $\Lambda_2$ (cf. §3) with fixed boundary conditions (not included in the notation). The following definition of transfer operators for the infinite system does not depend on the choice of the boundary conditions.

THEOREM 2.1. *For $\vartheta$, $\epsilon$ sufficiently small, $c_2$, $N_0$ sufficiently large and any $\Lambda_1 \in \mathcal{F}$:*

(1) *The limit*

$$\pi_{\Lambda_1} \circ \mathcal{L}^N_{F\circ T^\epsilon} \stackrel{\text{def}}{=} \lim_{\Lambda_2\to\mathbb{Z}^d} \pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_2}\circ T^{\Lambda_2,\epsilon}} \circ \pi_{\Lambda_2} \tag{7}$$

$\in L((\mathcal{H}_\vartheta, \|\cdot\|_\vartheta), (\mathcal{H}_{\Lambda_1,\vartheta_N}, \|\cdot\|_{\Lambda_1,\vartheta_N}))$ *exists for suitably chosen* $0 < \vartheta_1 \le \cdots \le \vartheta_{N_0} = \vartheta_{N_0+1} = \cdots = \vartheta$ *and the family of these operators is uniformly (in $\Lambda_1$) bounded. This defines operators*

$$\mathcal{L}^N_{F\circ T^\epsilon} \in L((\mathcal{H}_\vartheta, \|\cdot\|_\vartheta), (\mathcal{H}_{\vartheta_N}, \|\cdot\|_{\vartheta_N})) \quad \textit{by} \quad (\mathcal{L}^N_{F\circ T^\epsilon}\phi)_{\Lambda_1} \stackrel{\text{def}}{=} \pi_{\Lambda_1} \circ \mathcal{L}^N_{F\circ T^\epsilon}\phi.$$

*In particular for $N \ge N_0$ we have $\mathcal{L}^N_{F\circ T^\epsilon} \in L(\mathcal{H}_\vartheta, \|\cdot\|_\vartheta)$.*

*In the case of finite-range interaction we can define a linear map $\mathcal{L}_{F\circ T^\epsilon}$ on $\mathcal{H}$ in the same way, i.e. if $r$ is the range of interaction we set for any $\Lambda_1 \in \mathcal{F}$*

$$\pi_{\Lambda_1} \circ \mathcal{L}_{F\circ T^\epsilon} \stackrel{\text{def}}{=} \pi_{\Lambda_1} \circ \mathcal{L}_{F^{\Lambda_2}\circ T^{\Lambda_2,\epsilon}} \circ \pi_{\Lambda_2} \tag{8}$$

*where $\Lambda_2 = B_r(\Lambda_1)$.*

(2) *There is an $F \circ T^\epsilon$-invariant, non-negative probability measure $\nu^*$. It is unique in the set of non-negative probability measures whose marginal densities can be identified with a $\nu = (\nu_{\Lambda_1})_{\Lambda_1\in\mathcal{F}} \in \mathcal{H}_\vartheta$.*

*In $L(\mathcal{H}_\vartheta, \|\cdot\|_\vartheta)$ the sequence $(\mathcal{L}^N_{F\circ T^\epsilon})_{N\ge N_0}$ converges exponentially fast:*

$$\|\mathcal{L}^N_{F\circ T^\epsilon} - \mu(\cdot)\nu^*\|_{L((\mathcal{H}_\vartheta,\|\cdot\|_\vartheta))} \le c_3\tilde{\eta}^N \tag{9}$$

*for some $c_3 > 0$ and $0 < \tilde{\eta} < 1$.*

*Remarks. (1)* The relation between measures and elements of $\mathcal{H}$ is explained in §4, in particular in (23).

*(2)* A formula for $\nu$ is given in (59).

For the invariant measure $\nu$ we have exponential decay of correlations for spatio–temporal shifts on the system.

Let $(e_1, \ldots, e_d)$ be a linearly-independent system of unit vectors in $\mathbb{Z}^d$. We define translations $\tau_{e_i}(p) \stackrel{\text{def}}{=} p + e_i$ for $p \in \mathbb{Z}^d$ and $(\tau_{e_i}(z))_p \stackrel{\text{def}}{=} z_{\tau_{e_i}(p)}$ for $z \in M$.

In the following theorem we denote by $\tau$ (acting on $M$ from the right) compositions $\tau = \tau_1 \circ \cdots \circ \tau_{m(\tau)}$ and by $\sigma$ a composition of spatio–temporal shifts (on $M$): $\sigma = \sigma_1 \circ \cdots \circ \sigma_{m(\sigma)+n(\sigma)}$ with $\sigma_i \in \{S, \tau_{e_1}, \ldots, \tau_{e_d}\}$. We denote by $n(\sigma)$ the number of factors $S$ and by $m(\sigma)$ the number of spatial translations in this product. For a translation-invariant system, i.e. $f_p = f$ and $g_p(\mathbf{z}) = g_{\tau_{e_i}^{-1}(p)}(\tau_{e_i}(z))$ for all $p \in \mathbb{Z}^d$ and $i = 1, \ldots, d$, the time-shift $S$ commutes with the translations.

THEOREM 2.2. *For $\vartheta$,$\epsilon$ as in Theorem 2.1 and $c_2$ sufficiently large there is a $\kappa \in (0,1)$ such that for all non-empty $\Lambda_1, \Lambda_2 \in \mathcal{F}$ the following holds with the constant $c(\Lambda_1, \Lambda_2, \kappa) \stackrel{\text{def}}{=} \kappa^{-\max\{\|p-q\|:p\in\Lambda_1, q\in\Lambda_2\}}$.*

(1) *If $g \in \mathcal{C}((S^1)^{\Lambda_1})$ and $f \in \mathcal{C}((S^1)^{\Lambda_2})$ then*

$$\left| \int_M d\nu^* g f - \left( \int_M d\nu^* g \right) \left( \int_M d\nu^* f \right) \right| \le c_4 \vartheta^{-|\Lambda_1|-|\Lambda_2|} \|g\|_\infty \|f\|_\infty \kappa^{\operatorname{dist}(\Lambda_1,\Lambda_2)},$$

*where* $\operatorname{dist}(\Lambda_1, \Lambda_2) \stackrel{\text{def}}{=} \min\{\|p-q\| : p \in \Lambda_1, q \in \Lambda_2\}$.

(2) *If $g \in \mathcal{C}((S^1)^{\Lambda_1})$ and $f \in \mathcal{H} \cap \mathcal{C}((S^1)^{\Lambda_2})$ then*

$$\left| \int_M d\nu^* g \circ \tau \circ S^n f - \left( \int_M d\nu^* g \circ \tau \right) \left( \int_M d\nu^* f \right) \right| \le c(\Lambda_1, \Lambda_2, \kappa) c_5^{|\Lambda_1|+|\Lambda_2|} \|g\|_\infty \|f\|_{\Lambda_2} \kappa^{m(\tau)} \tilde{\eta}^n \tag{10}$$

*with suitable $c_5$ and $\tilde{\eta}$ as in Theorem 2.1.*

(3) *If the system is translation-invariant and $g$ and $f$ are as in (2), then*

$$\left| \int_M d\nu^* g \circ \sigma f - \left( \int_M d\nu^* g \right) \left( \int_M d\nu^* f \right) \right| \le c(\Lambda_1, \Lambda_2, \kappa) c_5^{|\Lambda_1|+|\Lambda_2|} \|g\|_\infty \|f\|_{\Lambda_2} \kappa^{m(\sigma)} \tilde{\eta}^{n(\sigma)}. \tag{11}$$

(4) *If $g, f \in \mathcal{C}(M)$ then*

$$\lim_{\max\{m(\tau),n\}\to\infty} \left| \int_M d\nu^* g \circ \tau \circ S^n f - \left( \int_M d\nu^* g \circ \tau \right) \left( \int_M d\nu^* f \right) \right| = 0. \tag{12}$$

(5) *If the system is translation-invariant and $g, f \in \mathcal{C}(M)$ then*

$$\lim_{\max\{m(\sigma),n(\sigma)\}\to\infty} \int_M d\nu^* g \circ \sigma f = \left( \int_M d\nu^* g \right) \left( \int_M d\nu^* f \right). \tag{13}$$

*Remarks. (1)* Theorem 2.2(5) means that for a translation-invariant system $\nu$ is mixing w.r.t. spatio–temporal shifts. According to (3), the decay of correlations for observables $g$ and $h$ as specified in (2) is exponentially fast.

*(2)* The proof of Theorem 2.2 shows that the statements hold for any $\kappa \in (0,1)$ if $\epsilon$ is sufficiently small and $c_2$ sufficiently large (both depending on $\kappa$). So a small interaction leads to small spatial correlations.

3. *Finite-dimensional systems*

We first consider 'finite-dimensional versions' of the maps $F$, $T^{\epsilon}$, etc. Let $\xi = (\xi_p)_{p\in\mathbb{Z}^d} \in M$ be a fixed configuration. For a finite subset $\Lambda \subset \mathbb{Z}^d$ we define $T^{\Lambda,\epsilon} : A_\delta^\Lambda \to \mathbb{C}^\Lambda$ by

$$(T^{\Lambda,\epsilon}(\mathbf{z}_\Lambda))_p \stackrel{\text{def}}{=} z_p \exp(2\pi i\epsilon g_p(\mathbf{z}_\Lambda \vee \xi_{\Lambda^C})), \tag{14}$$

where $\mathbf{z}_\Lambda \vee \xi_{\Lambda^C} \in M$ agrees with $\mathbf{z}_\Lambda$ on its $\Lambda$-sites and with $\xi_{\Lambda^C}$ on its $\Lambda^C$-sites. We do not specify $\xi_{\Lambda^C}$ in the notation of $T^{\Lambda,\epsilon}$. The restriction of F to $A_\delta^\Lambda$ is denoted by $F^\Lambda$.

With the following two propositions we ensure that for sufficiently small $\delta$ and $\epsilon$ (independent of $\Lambda$ and $\mathbf{z}_{\Lambda^C}$), the image of $A_\delta^\Lambda$ w.r.t. $F^\Lambda \circ T^{\Lambda,\epsilon}$ contains a larger polyannulus (cf. [**2**]) and the image of the boundary, $F^\Lambda \circ T^{\Lambda,\epsilon}(\partial A_\delta^\Lambda)$, has positive distance from $A_\delta^\Lambda$.

For $\Lambda \subset \mathbb{Z}^d$ we have the metric $d_\Lambda$ on $(S^1)^\Lambda$ defined by

$$d_\Lambda(\mathbf{z}, \mathbf{w}) \stackrel{\text{def}}{=} \sup\{|z_p - w_p| \mid p \in \Lambda\}. \tag{15}$$

PROPOSITION 3.1. *For all $c_7 \in (0, 1)$, sufficiently small $\delta$ and $\epsilon$ (depending on $c_7$), and arbitrary $\Lambda \in \mathcal{F} \setminus \{\emptyset\}$, $T^{\Lambda,\epsilon}$ maps $A_\delta^\Lambda$ biholomorphically onto its image and $T^{\Lambda,\epsilon}(A_\delta^\Lambda) \supset A_{c_7\delta}^\Lambda$, i.e. the image contains a sufficiently thick polyannulus. Also $T^{\Lambda,\epsilon}(\partial A_\delta^\Lambda) \cap A_{c_7\delta}^\Lambda = \emptyset$, i.e. the image of the boundary (the same as the boundary of the image) does not intersect the smaller polyannulus.*

PROPOSITION 3.2. *Let the expanding maps $f_p : S^1 \to S^1$ satisfy Assumption 1 for some $\delta_1$ and an expansion constant $\lambda_0$ and let $1 < \lambda < \lambda_0$. Then for all sufficiently small $\delta$ ($0 < \delta < \delta_0$) and all finite $\Lambda \subset \mathbb{Z}^d$ the map $F^\Lambda : A_\delta^\Lambda \to \mathbb{C}^\Lambda$ is locally biholomorphic, $A_{\lambda\delta}^\Lambda \subset F^\Lambda(A_\delta^\Lambda)$, i.e. the image contains a thicker polyannulus, and furthermore all $\mathbf{z} \in A_{\lambda\delta}^\Lambda$ have the same number of pre-images. We also have $A_{\lambda\delta}^\Lambda \cap F^\Lambda(\partial A_\delta^\Lambda) = \emptyset$.*

Combining Propositions 3.1 and 3.2 we have for fixed $c_7$ (from Proposition 3.1) and (small) $\delta$

$$F^\Lambda \circ T^{\Lambda,\epsilon}(A_\delta^\Lambda) \supset A_{c_7\lambda\delta}^\Lambda \tag{16}$$

and

$$F^\Lambda \circ T^{\Lambda,\epsilon}(\partial A_\delta^\Lambda) \cap A_{c_7\lambda\delta}^\Lambda = \emptyset. \tag{17}$$

In particular, if we choose $c_7 > 1/\lambda$ there is a disc of radius $(c_7\lambda - 1)\delta > 0$ around each point in $A_\delta^\Lambda$ that is entirely contained in $F^\Lambda \circ T^{\Lambda,\epsilon}(A_\delta^\Lambda)$. We will need this for Cauchy estimates. From now on we keep $\delta$ fixed.

In the next proposition we establish a special representation of the Perron–Frobenius operator for our finite system with $(S^1)^N = (S^1)^\Lambda$, $S^\epsilon = F^\Lambda \circ T^{\Lambda,\epsilon}$, $\psi$ continuous (the proposition holds also for $\psi \in L^\infty(M)$) and $\phi$ continuous on the closed polyannulus $A_{\delta_1}^\Lambda$ and analytic in its interior.

First we give the definition of the Perron–Frobenius operator (cf for example [**14**]).

*Definition 3.1.* Let $\lambda$ be a measure on a metric space $M$ (with the Borel $\sigma$-algebra) and let $S : M \to M$ be a measurable map which is non-singular w.r.t. $\lambda$ (i.e. for all measurable

$A \in M$, $\lambda(A) = 0$ implies $\lambda(S^{-1}(A)) = 0$). The Perron–Frobenius operator $\mathcal{L}_S$, acting on $L^1(M)$, is defined via the equation

$$\int_M d\lambda\, \psi \circ S\, \phi = \int_M d\lambda\, \psi \mathcal{L}_S \phi \tag{18}$$

that, for given $\phi \in L^1(M)$, must hold for all $\psi \in L^\infty(M)$. The existence and uniqueness of $\mathcal{L}_S\phi \in L^1(M)$ is equivalent by the Radon–Nikodym theorem to the absolute continuity (w.r.t. $\lambda$) of the measure associated to the functional $\psi \mapsto \int_M d\lambda\, \psi \circ S\, \phi$ (the functional here is restricted to continuous functions $\psi$), and this follows from the non-singularity of $S$.

*Remark.* Setting $\psi \equiv 1$ in (18) we get that $\mathcal{L}_S$ preserves the integral:

$$\int_M d\lambda\, \mathcal{L}_S \phi = \int_M d\lambda\, \phi. \tag{19}$$

The normalized Lebesgue measure $\mu$ on $S^1$ is given by $d\mu(z) = (dz/2\pi i)(1/z)$ (this lifts w.r.t. the map $t \to e^{it}$ to the normalized Lebesgue measure $dt/2\pi$ on $[0, 2\pi)$) and the product measure $\mu^\Lambda$ on $(S^1)^\Lambda$ is given by

$$d\mu^\Lambda(\mathbf{z}) = \frac{d\mathbf{z}}{(2\pi i)^{|\Lambda|}} \frac{1}{\mathbf{z}} \stackrel{\text{def}}{=} \prod_{p\in\Lambda} \frac{dz_p}{2\pi i} \frac{1}{z_p}. \tag{20}$$

We also use $d\mu^\Lambda(\mathbf{z})$ as a shorthand notation for the right-hand side of (20) for $\mathbf{z} \in A_\delta^\Lambda$. The following representation of the Perron–Frobenius operator for finite-dimensional subsystems of our coupled map lattice by means of Cauchy kernels is essential for our analysis. Similar Cauchy kernels were used in [**17**].

PROPOSITION 3.3. *With $F^\Lambda$ and $T^{\Lambda,\epsilon}$ defined as above, set $S^\epsilon = F^\Lambda \circ T^{\Lambda,\epsilon}$ and let $S_p^\epsilon$ be the projection onto its pth component. Then the Perron–Frobenius operator (for $S^\epsilon$), acting on $\phi \in \mathcal{H}_\Lambda$, can be written in the following way:*

$$\mathcal{L}_{S^\epsilon}\phi(w) = \int_{\Gamma^\Lambda} d\mu^\Lambda(\mathbf{z})\phi(z) \prod_{p\in\Lambda} \left( \frac{1}{S_p^\epsilon(\mathbf{z}) - w_p} S_p^{\epsilon\prime}(\mathbf{z}) \right) \tag{21}$$

*where $\Gamma = \Gamma_+ \cup \Gamma_-$ is the positively-oriented boundary of $A_\delta$.*

## 4. *Further remarks on the infinite-dimensional system*

The subspace of complex-valued functions that depend only on finitely many variables is dense in $(\mathcal{C}(M), \|\cdot\|_\infty)$, and each such function (say depending on $\mathbf{z}_\Lambda$ only) can be uniformly approximated by (the restriction of) functions in $\mathcal{H}(A_\delta^\Lambda)$. The dual space of $\mathcal{C}(M)$ is $rca(M)$ (see e.g. [**7**]), the space of bounded, regular, countably additive, complex-valued set functions on $(M, \mathcal{B})$ where $\mathcal{B}$ is the Borel $\sigma$-algebra. The norm on $rca(M)$ is the total variation. For given $\vartheta$, $\Lambda$ we consider *rca* measures whose marginals have densities $\phi_{\Lambda|(S^1)^\Lambda}$ over $(S^1)^\Lambda$ (restriction of $\phi_\Lambda$ to $(S^1)^\Lambda$) s.t. $\phi = (\phi_\Lambda)_{\Lambda\in\mathcal{F}} \in \mathcal{H}_\vartheta$. We remark that not every $\phi \in \mathcal{H}_\vartheta$ with real-valued $\phi_{\Lambda|(S^1)^\Lambda}$ corresponds to an element in $rca(M)$ because its variation might not be bounded as $\int_\Lambda d\mu^\Lambda\, |\phi_\Lambda|$ might be unbounded with $\Lambda$. So we define for $\phi \in \mathcal{H}$

$$\|\phi\|_{\text{var}} \stackrel{\text{def}}{=} \lim_{\Lambda\to\mathbb{Z}^d} \int_{(S^1)^\Lambda} d\mu^\Lambda\, |\phi_\Lambda|. \tag{22}$$

We set $\mathcal{H}^{bv} \stackrel{\text{def}}{=} \{\phi \in \mathcal{H} : \|\phi\|_{\text{var}} < \infty\}$ and $\mathcal{H}^{bv}_\vartheta \stackrel{\text{def}}{=} \mathcal{H}^{bv} \cap \mathcal{H}_\vartheta$. In particular all real-analytic and non-negative $\phi \in \mathcal{H}$, i.e. $\phi_{\Lambda|(S^1)^\Lambda} \geq 0$ for all $\Lambda \in \mathcal{F}$, belong to this space.

We can view every $\phi \in \mathcal{H}^{bv}$ as an element of $rca(M)$: for $g \in \mathcal{C}(M)$ the net $(g_\Lambda)_{\Lambda\in\mathcal{F}}$ given by $g_\Lambda \stackrel{\text{def}}{=} \pi_\Lambda(g)$ converges uniformly to $g$. We set

$$\phi(g) \stackrel{\text{def}}{=} \lim_{\Lambda\to\mathbb{Z}^d} \int_{(S^1)^\Lambda} d\mu^\Lambda\, g_\Lambda \phi_\Lambda. \tag{23}$$

The limit exists because for $\Lambda_1 \subset \Lambda_2$

$$\begin{aligned}\left| \int_{(S^1)^{\Lambda_1}} d\mu^{\Lambda_1} g_{\Lambda_1}\phi_{\Lambda_1} - \int_{(S^1)^{\Lambda_2}} d\mu^{\Lambda_2} g_{\Lambda_2}\phi_{\Lambda_2} \right| &= \left| \int_{(S^1)^{\Lambda_2}} d\mu^{\Lambda_2} (g_{\Lambda_1} - g_{\Lambda_2})\phi_{\Lambda_2} \right| \\ &\leq \|g_{\Lambda_1} - g_{\Lambda_2}\|_{(S^1)^{\Lambda_2}} \|\phi\|_{\text{var}} \end{aligned} \tag{24}$$

gets arbitrarily small as $\Lambda_1 \to \mathbb{Z}^d$, i.e. the net has the Cauchy property.

We further see

$$\begin{aligned}\|\phi\|_{\text{var}} &= \sup_{\Lambda\in\mathcal{F}} \int_{(S^1)^\Lambda} d\mu^\Lambda\, |\phi_\Lambda| \\ &= \sup_{\Lambda\in\mathcal{F}} \sup_{\substack{g\in\mathcal{C}((S^1)^\Lambda) \\ \|g\|_\infty\leq 1}} \int_{(S^1)^\Lambda} d\mu^\Lambda\, g\phi_\Lambda \\ &= \sup_{\substack{g\in\mathcal{C}(M) \\ \|g\|_\infty\leq 1}} |\phi(g)|, \end{aligned} \tag{25}$$

so $\|\phi\|_{\text{var}}$ is in fact the total variation (the operator-norm, cf. [**7**]) of the corresponding linear functional on $\mathcal{C}(M)$.

Let $\mathcal{H}(\mathcal{F}) \stackrel{\text{def}}{=} \bigcup_{\Lambda\in\mathcal{F}} H(A^\Lambda_\delta)$ be the subspace of functions depending on only finitely many variables. We define the product $g^1\phi \in \mathcal{H}_\vartheta$ of $g^1 \in \mathcal{H}(A^{\Lambda_1}_\delta)$ and $\phi \in \mathcal{H}_\vartheta$ by

$$(g^1\phi)_\Lambda \stackrel{\text{def}}{=} \pi_\Lambda(g^1\phi_{\Lambda_1\cup\Lambda}). \tag{26}$$

LEMMA 4.1. *If $g^1 \in H(A^{\Lambda_1}_\delta)$, $g^2 \in H(A^{\Lambda_2}_\delta)$, $g \in \mathcal{C}(M)$ and $\phi \in \mathcal{H}_\vartheta$ the following hold:*

(1) *the product in (26) is well defined and $\|g^1\phi\|_\vartheta \leq \|g^1\|_{\Lambda_1} \vartheta^{-|\Lambda_1|} \|\phi\|_\vartheta$;*

(2) *$(g^1g^2)\phi = g^1(g^2\phi)$;*

(3) *$g^2$ can be considered as an element of $\mathcal{H}_\vartheta$ and the product $g^1g^2$ as defined in (26) is the same as the usual product between functions on $M$;*

(4) *$(g^1\phi)(g) = \phi(g^1g)$ where $(g^1\phi)$ and $\phi$ act as functionals in the sense of (23);*

(5) *$\mathcal{H}^{bv}_\vartheta$ is also a module over the ring $\mathcal{H}(\mathcal{F})$, i.e. in particular $\|g^1\phi\|_{\text{var}} \leq \|g^1\|_{\Lambda_1} \|\phi\|_{\text{var}}$.*

5. *Expansion of the Perron–Frobenius operator*

We split the integral kernel of the Perron–Frobenius operator for a finite-dimensional system. Recall that $T_p^\epsilon(\mathbf{z}) = z_p \exp\left(2\pi i\epsilon \sum_{k=1}^\infty g_{p,k}(\mathbf{z})\right) = z_p \prod_{k+1}^\infty \exp(2\pi i\epsilon g_{p,k}(\mathbf{z}))$ and that $S_p(\mathbf{z}) = f_p \circ T_p^\epsilon(\mathbf{z})$.

If we consider only finite-range interaction, say up to distance $l$, we have

$$T_{p,l}^\epsilon(\mathbf{z}) \stackrel{\text{def}}{=} z_p \exp\left(2\pi i\epsilon \sum_{k=1}^{l} g_{p,k}(\mathbf{z})\right). \tag{27}$$

For a finite-dimensional system (say on $(S^1)^{\Lambda_2}$) with fixed boundary conditions we have a special representation of $\mathcal{L}_{F^{\Lambda_2}\circ T^{\Lambda_2,\epsilon}}$ in terms of the integral kernel (Proposition 3.3).

LEMMA 5.1. *For the factors in the integral kernel in (21) we have the following splitting:*

$$\begin{aligned}\frac{1}{f_p\circ T_p^\epsilon(\mathbf{z}) - w_p} f_p\circ T_p^\epsilon(\mathbf{z}) &= \frac{1}{f_p(z_p) - w_p} f_p(z_p)\\ &\quad + w_p \sum_{k=1}^\infty \frac{f_p\circ T_{p,k-1}^\epsilon(\mathbf{z}) - f_p\circ T_{p,k}^\epsilon(\mathbf{z})}{(f_p\circ T_{p,k-1}^\epsilon(\mathbf{z}) - w_p)(f_p\circ T_{p,k}^\epsilon(\mathbf{z}) - w_p)}.\end{aligned} \tag{28}$$

*The sum in the right-hand side converges uniformly in* $\mathbf{z} \in \Gamma^\Lambda$ *and* $w_p \in A_\delta$.

5.1. *The unperturbed operator.* The first summand in (28) is just the one which appears in the uncoupled system (i.e. $T^{\epsilon=0} = \mathrm{id}$) and in this case each lattice site can be considered separately. We denote by $\mathcal{L}_{f_p}$ the restriction of the Perron–Frobenius operator to the Banach space of functions on $S^1$ that extend continuously on the closed annulus $A_\delta$ and holomorphically on the interior $A_\delta$. $\|\cdot\|_{A_\delta}$ denotes the uniform norm over $A_\delta$. The operator

$$\mathcal{L}_{f_p} : (\mathcal{H}(A_\delta), \|\cdot\|_{A_\delta}) \to (\mathcal{H}(A_\delta), \|\cdot\|_{A_\delta})$$

has 1 as simple eigenvalue and the rest of its spectrum is contained in a disc around 0 of radius strictly smaller than one. It splits into

$$\mathcal{L}_{f_p} = Q_p + R_p \tag{29}$$

with

$$R_p Q_p = Q_p R_p = 0 \tag{30}$$

and

$$\|R_p^n\|_{L(\mathcal{H}(A_\delta),\|\cdot\|_{A_\delta})} \le c_r \eta^n \tag{31}$$

with $c_r > 0$, $0 < \eta < 1$. For proofs of these statements see, for example, [**2**].

$Q_p$ is the projection onto the one-dimensional eigenspace spanned by $h_p \in \mathcal{H}(A_\delta)$, whose restriction to $S^1$ is positive and has integral $\int_{S^1} d\mu\, h_p = 1$.

We assume in Assumption 1 regarding the family $(f_p)_{p\in\mathbb{Z}^d}$ that

$$\|h_p\|_{A_\delta} \le c_h \tag{32}$$

and the exponential bound in (31) both hold uniformly in $p$. This is the case for example if the $f_p$ are uniformly close to each other as is shown using analytic perturbation theory.

$\mathcal{L}_{f_p}$ preserves the integral (cf. (19)) and so does $Q_p$, as follows e.g. from (29)–(31). Since $\Gamma_+$ is homologous to $S^1$ we can write $Q_p$ as

$$Q_p g(w) = h_p(w) \int_{S^1} d\mu \, g \tag{33}$$

$$= h_p(w) \int_{\Gamma_+} \frac{dz}{2\pi i} \frac{1}{z} g(z)$$

$$= \int_{\Gamma} \frac{dz}{2\pi i} \frac{1}{z} h_p(w,z) g(z) \tag{34}$$

where we have used that $g$ is holomorphic in $A_\delta$ and defined as

$$h_p(w_p, z_p) \stackrel{\text{def}}{=} \begin{cases} h_p(w_p) & \text{for } z_p \in \Gamma_+ \\ 0 & \text{for } z_p \in \Gamma_-. \end{cases} \tag{35}$$

The idempotency $Q_p^2 = Q_p$ results in the integral representation

$$\int_{\Gamma} \frac{dz_p^2}{2\pi i} \frac{1}{z_p^2} \int_{\Gamma} \frac{dz_p^1}{2\pi i} \frac{1}{z_p^1} h_p(w_p, z_p^2) h_p(z_p^2, z_p^1) g(z_p^1) = \int_{\Gamma} \frac{dz_p^1}{2\pi i} \frac{1}{z_p^1} h_p(w_p, z_p^1) g(z_p^1). \tag{36}$$

Here and throughout the section the upper indices in $z_p^1$, $z_p^2$, etc. refer to the temporal and the lower ones to the spatial coordinate in the space–time lattice $\mathbb{Z} \times \mathbb{Z}^d$.

According to Proposition 3.3 the operator $R_p$ can be written

$$R_p g(w_p) = \int_{\Gamma} \frac{dz}{2\pi i} \frac{1}{z} r_p(w_p, z_p) g(z_p) \tag{37}$$

with

$$r_p(w_p, z_p) = \frac{1}{f_p(z) - w_p} f_p(z_p) - h_p(w_p, z_p). \tag{38}$$

Then equation (30) results in the integral representation

$$\int_{\Gamma} \frac{dz_p^2}{2\pi i} \frac{1}{z_p^2} \int_{S^1} \frac{dz_p^1}{2\pi i} \frac{1}{z_p^1} r_p(w_p, z_p^2) h_p(z_p^2, z_p^1) g(z_p^1) = 0, \tag{39}$$

$$\int_{S^1} \frac{dz_p^2}{2\pi i} \frac{1}{z_p^2} \int_{\Gamma} \frac{dz_p^1}{2\pi i} \frac{1}{z_p^1} r_p(z_p^2, z_p^1) g(z_p^1) = 0. \tag{40}$$

5.2. *The perturbed operator.* In view of (28) we set

$$\beta_{p,k}(w_p, \mathbf{z}) \stackrel{\text{def}}{=} w_p \frac{f_p \circ T^\epsilon_{p,k-1}(\mathbf{z}) - f_p \circ T^\epsilon_{p,k}(\mathbf{z})}{(f_p \circ T^\epsilon_{p,k-1}(\mathbf{z}) - w_p)(f_p \circ T^\epsilon_{p,k}(\mathbf{z}) - w_p)}. \tag{41}$$

This corresponds to the difference between the operators for systems with interaction of finite-range of order $k$ and $k-1$, respectively. Using (1) we have the estimate

$$\begin{aligned} |\beta_{p,k}(w_p, \mathbf{z})| &\leq |w_p| |f_p \circ T^\epsilon_{p,k-1}(\mathbf{z}) - w_p|^{-1} |f_p \circ T^\epsilon_{p,k}(\mathbf{z}) - w_p|^{-1} \\ &\quad \times |f_p \circ T^\epsilon_{p,k-1}(\mathbf{z}) - f_p \circ T^\epsilon_{p,k}(\mathbf{z})| \\ &\leq (1+\delta) |c_7 \lambda - 1|^{-1} |c_7 \lambda - 1|^{-1} \|f_p'\|_{\{p\}} c_1 \epsilon \exp(-c_2 k^d) \\ &\leq \tilde{c_8} \epsilon \exp(-c_2 k^d). \end{aligned} \tag{42}$$

This estimate is uniform in $p \in \mathbb{Z}^d$, $w_p \in A_\delta$ and $\mathbf{z} \in \Gamma^\Lambda$.

5.3. *Time N Step.* Now we want to estimate the norm of (6) or equivalently that of

$$\pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_2} \circ T^{\Lambda_2,\epsilon}} : (\mathcal{H}_{\Lambda_2,\vartheta}, \|\cdot\|_{\Lambda_2,\vartheta}) \to (\mathcal{H}_{\Lambda_1,\vartheta}, \|\cdot\|_{\Lambda_1,\vartheta}) \tag{43}$$

$$\mathcal{L}^N_{F^{\Lambda_2} \circ T^{\Lambda_2,\epsilon}} \phi(\mathbf{z}^0) = \int_{\Gamma^{\Lambda_2}} d\mu^{\Lambda_2}(\mathbf{z}^{-1}) \cdots \int_{\Gamma^{\Lambda_2}} d\mu^{\Lambda_2}(\mathbf{z}^{-N}) \prod_{t=-N}^{-1} \prod_{p \in \Lambda_2} \times \left( h_p(z_p^{t+1}, z_p^t) + r_p(z_p^{t+1}, z_p^t) + \sum_{k=1}^{\infty} \beta_{p,k}(z_p^{t+1}, \mathbf{z}^t) \right) \phi(\mathbf{z}^{-N}) \tag{44}$$

(cf. also the beginning of §3.)

Distributing the product we get infinitely many summands. In each factor there is for each $-N \le m \le -1$, $p \in \Lambda_2$ a choice between $h_p$, $r_p$ and $\beta_{p,k}$ $(1 \le k < \infty)$ and we can interpret such a choice graphically as a *configuration* (similar objects were introduced in [**15**] where they were named polymers).

On $\Lambda_2 \times \{-N, \dots, 0\}$ we represent (see Figure 1):

- $h_p(z_p^{t+1}, z_p^t)$ by an *h-line* from $(p, t)$ to $(p, t+1)$;
- $r_p(z_p^{t+1}, z_p^t)$ by an *r-line* from $(p, t)$ to $(p, t+1)$;


FIGURE 1. The h-line and the r-line.

- $\beta_{p,k}(z_p^{t+1}, \mathbf{z}^t)$ by a *k-triangle* (actually rather a cone or pyramid, but in our pictures for $d = 1$ it is a triangle (see Figure 2)) with apex $(p, t+1)$ and base points $(q, t)$ with $\|p - q\| \le k$. (So some of the base points might not lie in $\Lambda_2 \times \{-N, \dots, -1\}$, but all the apices lie in $\Lambda_2 \times \{-N+1, \dots, 0\}$.)

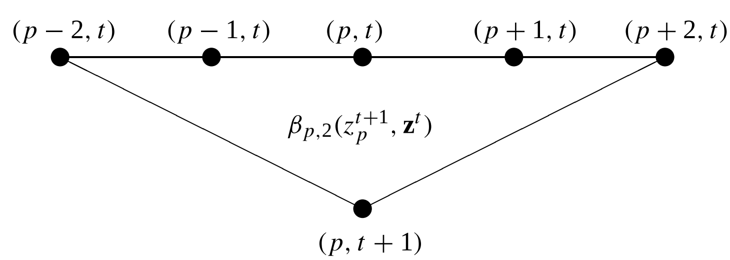


FIGURE 2. The 2-triangle.

Note that if

$$v(k) \stackrel{\text{def}}{=} |B_k(0)| \tag{45}$$

denotes the number of base points of a $k$-triangle, we have the estimate $v(k) \leq (3k)^d$. Each summand, that we get by distributing the product in (44), corresponds to a configuration and for each configuration $\mathcal{C}$ we have an operator $\mathcal{L}_{\mathcal{C}}$. So we can write

$$\mathcal{L}^N_{F^{\Lambda_2} \circ T^{\Lambda_2,\epsilon}} = \sum_{\mathcal{C}} \mathcal{L}_{\mathcal{C}}. \tag{46}$$

Some of these summands are zero, namely, if:

- a factor $h_p(z_p^{t+2}, z_p^{t+1}) r_p(z_p^{t+1}, z_p^t)$ or $r_p(z_p^{t+2}, z_p^{t+1}) h_p(z_p^{t+1}, z_p^t)$ appears, but no factor $\beta_{q,k}(z_q^{t+2}, \mathbf{z}^{t+1})$ with $\|p - q\| \leq k$ (i.e. an h-line follows or is followed by an r-line and, at their common endpoint, no triangle is attached with any of its base points, cf. Figure 3). This follows since, by Fubini's theorem, one can first perform the $dz_p^{t+1}\, dz_p^t$ integration and get zero by (39) or (40). (Note that the other factors in the integrand do not depend on $z_p^{t+1}$; so they can be considered as the function $g(z_p^1)$ in (39) or (40).)

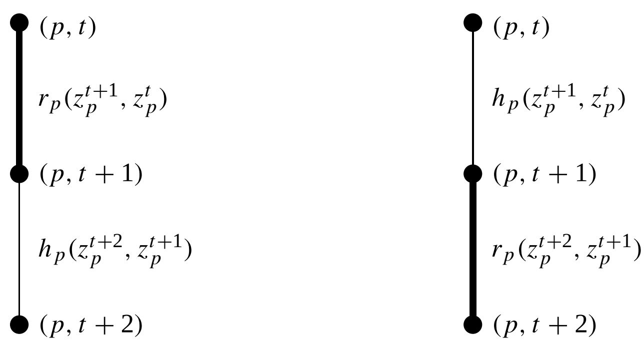


FIGURE 3. Consecutive r-line and h-line.

- a term $h_p(z_p^{t+2}, z_p^{t+1}) \beta_{p,k}(z_p^{t+1}, \mathbf{z}^t)$ appears but no $\beta_{q,l}(z_q^{t+2}, \mathbf{z}^{t+1})$ with $\|p - q\| \leq l$ (i.e. a triangle is followed by an h-line and at their common endpoint (the apex of the triangle) no other triangle is attached with any of its base points. Cf. Figure 4.) Indeed:

$$\beta_{p,k}(w_p, \mathbf{z}) = w_p \left[ \frac{1}{f_p \circ T^\epsilon_{p,k}(\mathbf{z}) - w_p} - \frac{1}{f_p \circ T^\epsilon_{p,k-1}(\mathbf{z}) - w_p} \right] \tag{47}$$

by the residue theorem:

$$\int_{S^1} \frac{dw_p}{2\pi i} \frac{1}{w_p} \beta_{p,k}(w_p, \mathbf{z}) = 0 \tag{48}$$

since the poles at $w_p = f_p \circ T^\epsilon_{p,k}(\mathbf{z})$ and $w_p = f_p \circ T^\epsilon_{p,k-1}(\mathbf{z})$ (with $\mathbf{z} \in \Gamma^N$, in particular $z_p \in \Gamma_+$ or $\Gamma_-$) both lie either outside $\Gamma_+$ or inside $\Gamma_-$ as $f_p$ is expanding, $T^\epsilon_{p,k}$ is close to $T^\epsilon_{p,k-1}$, and the two summands have residue $-1$ and 1, respectively.

Identity (48) is a consequence of the fact that $\beta_{p,k}$ is the kernel of a difference between two transfer operators (for the systems with interaction of range $k$ and $k-1$) both preserving the Lebesgue integral in the sense of (19). So the range of this operator difference consists of functions with integral zero and these are annihilated by the operator corresponding to $h_p$ (cf. (33) and (34)).

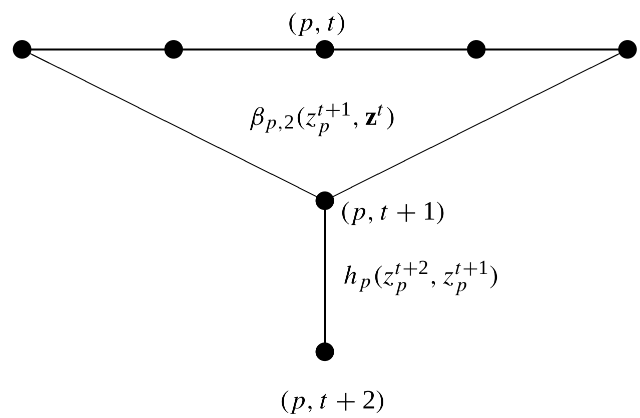


FIGURE 4. Combination 2-triangle and h-line.

Furthermore, we note that in

$$\pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_2} \circ T^{\Lambda_2,\epsilon}} = \sum_{\mathcal{C}} \pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}} \tag{49}$$

we get $\pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}} = 0$ unless $\mathcal{C}$ ends with h-lines in all points of $(\Lambda_2 \setminus \Lambda_1) \times \{0\}$ because of (40), (48) and the fact that $\pi_{\Lambda_1}$ means integration over $(S^1)^{\Lambda_2 \setminus \Lambda_1}$.

*Definition 5.1.* We call a configuration $\mathcal{L}_{\mathcal{C}}$ in the expansion (49) a *zero configuration* if it does not end with h-lines in all points of $(\Lambda_2 \setminus \Lambda_1) \times \{0\}$ or contains one of the constellations (consecutive r-line and h-line or $k$-triangle and h-line) mentioned above. Otherwise we call it a *non-zero configuration*.

*Remark.* For a zero configuration $\mathcal{C}$ we have just shown that its corresponding summand in (49) is 0. So we just have to sum over non-zero configurations. We note that the notion non-zero configuration does not exclude that $\mathcal{L}_{\mathcal{C}} = 0$.

We have to find an upper bound for the norm of each $\mathcal{L}_{\mathcal{C}}$. We do so by collecting r- and h-lines into chains and estimating the contributions of integrating the factors corresponding to these parts of the configuration.

*Definition 5.2.*

- Let $\mathcal{C}$ be a non-zero configuration with exactly $n_{\beta,k}$ $k$-triangles for $1 \leq k < \infty$. We define

$$n_\beta \stackrel{\text{def}}{=} (n_{\beta,1}, n_{\beta,2}, \ldots) \tag{50}$$

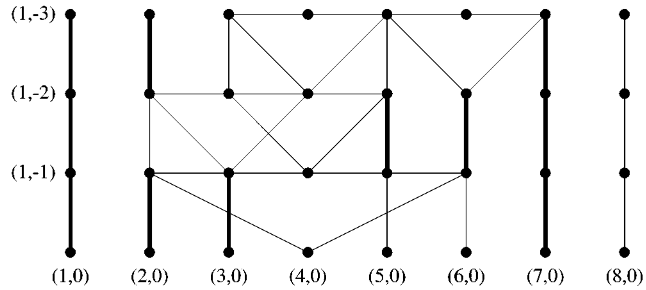


FIGURE 5. An example of a configuration.

and

$$|n_\beta| \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} n_{\beta,k} < \infty. \tag{51}$$

- A sequence of h-lines from $(p,t)$ to $(p,t+1),\ldots,(p,t+k-1)$ to $(p,t+k)$ with $p \in \Lambda_2$ and $-N \leq t \leq t+k \leq 0$ such that to the points $(p,t+1)\ldots(p,t+k-1)$ no triangles are attached is called an *h-chain of length k*.
- If such an h-chain is not contained in a longer chain it is called a *maximal h-chain*. Then $(p,t)$ and $(p,t+k)$ are denoted its *endpoints*.
- The definitions for a *maximal r-chain* and its *endpoints* are analogous.
- $\tilde{\Lambda}_{\mathcal{C}}$ denotes the set of points $p \in \Lambda_2$ that appear as the $\mathbb{Z}^d$-coordinate of a base point $(p,t)$ of a triangle in $\mathcal{C}$, and $\Lambda_{\mathcal{C}}$ denotes the set of those points $p \in \mathbb{Z}^d$ that appear as the $\mathbb{Z}^d$-coordinate of an apex $(p,t)$ that does not lie above (i.e. having the same spatial coordinate) any other triangle.
- $\Lambda_r$ is the set of $p \in \mathbb{Z}^d \setminus \tilde{\Lambda}_{\mathcal{C}}$ that appear as the $\mathbb{Z}^d$-coordinate of an r-line (this implies that there is an r-chain from $(p,-N)$ to $(p,0)$ for otherwise an r-line would have a common endpoint $(p,t)$ with an h-line and $\mathcal{C}$ would be a zero configuration.)
- We write $\Lambda(\mathcal{C}) \stackrel{\text{def}}{=} \tilde{\Lambda}_{\mathcal{C}} \cup \Lambda_r$.

In Figure 5 there are, for example, maximal r-chains from $(1,-3)$ to $(1,0)$ or from $(2,-3)$ to $(2,-2)$. $\Lambda_2 = \{1,\ldots,8\}$, $\tilde{\Lambda}_{\mathcal{C}} = \{2,\ldots,7\}$, $\Lambda_{\mathcal{C}} = \{4\}$ and $\Lambda_r = \{1\}$. As each $k$-triangle has $v(k) \leq (3k)^d$ base points we have

$$|\tilde{\Lambda}_{\mathcal{C}}| \leq \sum_{k=1}^{\infty} (3k)^d n_{\beta,k}. \tag{52}$$

To get the estimate for the norm of (43) we proceed in the following order.

(1) We integrate in $|\pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}} \phi(\mathbf{z}^0_{\Lambda_1})|$ over all $dz^t_p$ for which a factor $r_p(z^{t+1}_p, z^t_p)$ appears. For each maximal r-chain of length $l$ we get, according to (31), a factor not greater than $c_r \eta^l$.

(2) For each maximal h-chain starting at $(p,t)$ and ending at $(p,t+l)$ we perform the integration

$$\int_\Gamma d\mu(z^{t+l-1}_p) \cdots \int_\Gamma d\mu(z^{t+1}_p)\, h_p(z^{t+l}_p, z^{t+l-1}_p) \cdots h_p(z^{t+1}_p, z^t_p) = h_p(z^{t+l}_p). \tag{53}$$

(3) We perform the integration corresponding to $\pi_{\Lambda_1}$

$$\prod_{p\in\Lambda_2\setminus\Lambda_1}\int_{S^1} d\mu(z_p^0)\, h_p(z_p^0) = 1 \tag{54}$$

(4) We estimate the contribution of each (from step (2) and (3) remaining) factor $h_p(z_p^t)$ by $\|h_p\|_{A_\delta} \le c_h$ and, using (42), the contribution of each factor $\beta_{p,k}(z_p^{t+1}, \mathbf{z}^t)$ via

$$\left|\int_\Gamma \frac{dz_p^t}{2\pi i}\frac{1}{z_p^t}\,\beta_{p,k}(z_p^{t+1},\mathbf{z}^t)\psi(z_p^t)\right| \le \frac{|\Gamma|}{2\pi}\frac{1}{1-\delta}\tilde{c_8}\epsilon\,\exp(-c_2k^d)\|\psi\|$$
$$\le c_8\epsilon\,\exp(-c_2k^d)\|\psi\|. \tag{55}$$

Here $|\Gamma|$ denotes the euclidean length of $\Gamma$ and $\psi$ the remaining factors, containing other integrals. Finally, the contribution of the factors $|\phi(\mathbf{z}^{-N})|$ is estimated by $\|\phi_{\tilde\Lambda_{\mathcal{C}}\cup\Lambda_r}\|_{\tilde\Lambda_{\mathcal{C}}\cup\Lambda_r}$ (cf. remark below).

*Remark.* For all points $q \notin \tilde\Lambda_{\mathcal{C}} \cup \Lambda_r$ we must have h-chains in $\mathcal{C}$ from $(q, -N)$ to $(q, 0)$. Therefore we have

$$\pi_{\Lambda_1}\circ\mathcal{L}_{\mathcal{C}}\phi_{\Lambda_2}(\mathbf{z}^0_{\Lambda_1}) = \pi_{\Lambda_1}\circ\mathcal{L}_{\mathcal{C}}\phi_{\tilde\Lambda_{\mathcal{C}}\cup\Lambda_r}(\mathbf{z}^0_{\Lambda_1}) \tag{56}$$

where on the right-hand side we use the same notation '$\mathcal{L}_{\mathcal{C}}$' for the operator on $H_{\tilde\Lambda_{\mathcal{C}}\cup\Lambda_r,\vartheta}$.

So if $n_r$ denotes the number of r-lines, $\tilde n_r$ the number of maximal r-chains and $\tilde n_h$ the number of maximal h-chains having spatial coordinates in $\tilde\Lambda_{\mathcal{C}}\cup\Lambda_1$ (for otherwise they are 'integrated away' giving a factor of one) we get, using (31) and (55),

$$\|\pi_{\Lambda_1}\circ\mathcal{L}_{\mathcal{C}}\phi\|_{\Lambda_1} \le (c_8\epsilon)^{|n_\beta|}\exp\left(-c_2\sum_{k=1}^{\infty}k^d n_{\beta,k}\right)c_h^{\tilde n_h}c_r^{\tilde n_r}\eta^{n_r}\|\phi_{\tilde\Lambda_{\mathcal{C}}\cup\Lambda_r}\|_{\tilde\Lambda_{\mathcal{C}}\cup\Lambda_r} \tag{57}$$

and, using (52),

$$\|\phi_{\tilde\Lambda_{\mathcal{C}}\cup\Lambda_r}\|_{\tilde\Lambda_{\mathcal{C}}\cup\Lambda_r} \le \vartheta^{-|\Lambda_r|-\sum_{k=1}^{\infty}(3k)^d n_{\beta,k}}\|\phi\|_{\Lambda_2,\vartheta}$$
$$\le \vartheta^{-|\Lambda_r|}\prod_{k=1}^{\infty}\vartheta^{-(3k)^d n_{\beta,k}}\|\phi\|_{\Lambda_2,\vartheta} \tag{58}$$

for all $\Lambda_2 \in \mathcal{F}$ and with $\|\cdot\|_{\Lambda_2,\vartheta}$ defined in (5). Inequalities (57) and (58) are the basic estimates for a single configuration. We use refined versions of them throughout the paper. In particular the idea of taking the norm of $\phi_{\tilde\Lambda_{\mathcal{C}}\cup\Lambda_r}$ rather than that of $\phi_{\Lambda_2}$ which grows with the size of $\Lambda_2$, is the key point in our analysis.

## 6. *Operators for the infinite-dimensional system*

Estimates (57) and (58) bound the particular summands in an expansion such as (49). We see that triangles and maximal r-chains in a configuration $\mathcal{C}$ lead to small factors on the right-hand side of (57). (A maximal r-chain consisting of $n$ r-lines contributes a factor $c_r\eta^n$. The factor $c_r$ is greater than one in general, but either it will be compensated for by a small factor due to a triangle, e.g. as in (99), or $n$ will be large, cf., for example, (103).) This motivates the following definition of the length of a configuration. The length gives rise to a lower bound for the number of triangles or r-lines, i.e. a long configuration will lead to a small contribution in the total sum in (49).

*Definition 6.1.*

- The *length*, length($\mathcal{C}$), of a non-zero configuration $\mathcal{C}$ (that we obtained in an expansion such as (46)) is the maximal difference $0 - t$ such that there are points $(p, t)$ and $(q, 0)$ being end-points of r-lines or base points or apices of triangles. (Note that if there are any triangles or r-lines, there is also a triangle or an r-line ending at $\Lambda \times \{0\}$.) If there are no triangles or r-lines in $\mathcal{C}$ its length is zero.
- We *identify* two non-zero configurations $\mathcal{C}_1$ and $\mathcal{C}_2$ if they agree in their triangles, r-lines and their number of maximal h-chains that go upwards from the base points of triangles (but might be defined on space time boxes $\Lambda_2 \times \{-t_0, \dots, 0\}$ of different sizes, i.e. with different $\Lambda_2$ and $t_0$). We still speak of configurations rather than equivalence classes. For a configuration $\mathcal{C}$ length($\mathcal{C}$), $\tilde{\Lambda}_{\mathcal{C}}$, $\Lambda(\mathcal{C})$ (as in the Definition 5.2) and the operator $\pi_\Lambda \circ \mathcal{L}_{\mathcal{C}} \in L((\mathcal{H}(A_\delta^{\Lambda(\mathcal{C})}), \|\cdot\|_{\Lambda(\mathcal{C})}), (\mathcal{H}(A_\delta^{\Lambda}), \|\cdot\|_\Lambda))$ are well-defined.
- For $\Lambda_1 \in \mathcal{F}$ we define $E(\Lambda_1)$ as the set of all non-zero configurations $\mathcal{C}$ in some $\Lambda_2 \times \{-t_0, \dots, 0\}$ with $\Lambda_1 \subset \Lambda_2 \in \mathcal{F}$, $t_0 \in N$ and $t_0 >$ length($\mathcal{C}$), and that do not end in $\Lambda_1 \times \{0\}$ with triangles or r-lines.
- $E_N(\Lambda_1)$ is the set of non-zero configurations $\mathcal{C}$ in $\Lambda_2 \times \{-N, \dots, 0\}$ with $\Lambda_1 \subset \Lambda_2 \in \mathcal{F}$ and $\Lambda(\mathcal{C}) \subseteq \Lambda_2$.

We define

$$\nu_\Lambda \stackrel{\text{def}}{=} \sum_{\mathcal{C} \in E(\Lambda)} \pi_\Lambda \circ \mathcal{L}_{\mathcal{C}} h_{\Lambda(\mathcal{C})}. \tag{59}$$

The convergence of this infinite sum and other properties of $\nu$ are stated in the following proposition additional to Theorem 2.1.

PROPOSITION 6.1. *Let $\vartheta$, the sequence of $\vartheta_i$, $\epsilon$, $c_2$, $N_0$ and $\Lambda_1$ be as in Theorem 2.1 and $N \geq N_0$.*

(1)

$$\pi_{\Lambda_1} \circ \mathcal{L}^N_{F \circ T^\epsilon} = \sum_{\mathcal{C} \in E_N(\Lambda_1)} \pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}}. \tag{60}$$

(2)

$$\|\mathcal{L}^N_{F \circ T^\epsilon} - \mathcal{L}^{N+1}_{F \circ T^\epsilon}\|_{L((\mathcal{H}_\vartheta, \|\cdot\|_\vartheta))} \leq c_9 \tilde{\eta}^N. \tag{61}$$

(3) *For $N_1, N_2 \in \mathbb{N}$ the operator $\mathcal{L}^{N_2}_{F \circ T^\epsilon}$ is defined on $\mathcal{L}^{N_1}_{F \circ T^\epsilon}(\mathcal{H}_\vartheta) \subset \mathcal{H}_{\vartheta_{N_1}}$. It maps this space to $\mathcal{H}_{\vartheta_{N_1+N_2}}$ and*

$$\mathcal{L}^{N_2}_{F \circ T^\epsilon} \circ \mathcal{L}^{N_1}_{F \circ T^\epsilon} = \mathcal{L}^{N_1+N_2}_{F \circ T^\epsilon}. \tag{62}$$

(4) *For $\phi \in \mathcal{H}^{bv}_\vartheta$ we have the estimate*

$$\|\mathcal{L}_{F \circ T^\epsilon} \phi\|_{\text{var}} \leq \|\phi\|_{\text{var}}. \tag{63}$$

*For $g \in \mathcal{C}(M)$ and $\phi \in \mathcal{H}^{bv}_\vartheta$ we have the identity*

$$\int_M d\mu\, g \circ S\phi = \int_M d\mu\, g \mathcal{L}_{F \circ T^\epsilon} \phi \tag{64}$$

*and in particular*

$$\mu(\phi) = \mu(\mathcal{L}_{F\circ T^{\epsilon}}\phi). \tag{65}$$

*For finite-range interactions the inequality and both equations also hold for* $\phi \in \mathcal{H}^{bv}$.

(5) $\mathcal{L}_{F\circ T^{\epsilon}}$ *is non-negative, i.e.* $\phi \geq 0$ *implies* $\mathcal{L}_{F\circ T^{\epsilon}}\phi \geq 0$. ($\phi \geq 0$ *means* $\phi_{\Lambda\,|(S^1)^{\Lambda}} \geq 0$ *for all* $\Lambda \in \mathcal{F}$.)

## 7. *Decay of correlations*

We have found the unique invariant $\nu \in \mathcal{H}_{\vartheta}$ with $\mu(\nu) = 1$. This corresponds to a non-negative measure on $(M, \mathcal{B})$ whose marginal on $(S^1)^{\Lambda}$ has density $\nu^{\Lambda}_{|(S^1)^{\Lambda}}$ w.r.t. $\mu^{\Lambda}$. In the next theorem we state the decay of correlation for $\nu$ in terms of the weighted norms. We will use these results in the proof of Theorem 2.2.

THEOREM 7.1. *For sufficiently small* $\vartheta$ *and* $\epsilon$, *large* $c_2$, *finite disjoint* $\Lambda_1, \Lambda_2$ *and* $f \in H(A_{\delta}^{\Lambda_2})$ *there are a* $\kappa \in (0, 1)$ *and a* $\tilde{\vartheta} \in (0, 1)$ *such that:*

(1) $\|\nu_{\Lambda_1\cup\Lambda_2} - \nu_{\Lambda_1}\nu_{\Lambda_2}\|_{\Lambda_1\cup\Lambda_2,\vartheta} \leq c_{10}\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)}$;

(2) $\|\pi_{\Lambda_1}(f\nu) - \nu(f)\nu_{\Lambda_1}\|_{\Lambda_1,\vartheta} \leq c_{11}\vartheta^{-|\Lambda_2|}\|f\|_{\Lambda_2}\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)}$;

(3) $\|\pi_{\Lambda_1} \circ \mathcal{L}^N_{F\circ T^{\epsilon}}(f\nu) - \nu(f)\nu_{\Lambda_1}\|_{\Lambda_1,\tilde{\vartheta}} \leq c_{12}\vartheta^{-|\Lambda_2|}\|f\|_{\Lambda_2}\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)}\tilde{\eta}^N$ *for every* $N \geq 0$.

*Remark.* As in Theorem 2.2 we can choose the rate of decay $\kappa$ first and then the other parameters.

## 8. *Proofs*

In the proof of Proposition 3.1 we use the following lemma which is rather standard in real analysis. Here we formulate it in the setting of holomorphic functions.

LEMMA 8.1. *If* $T : U \to \mathbb{C}^n$ *is a holomorphic map on a convex set* $U \subset \mathbb{C}^n$ *and satisfies the estimate* $\|DT(z) - \mathrm{id}\| \leq c_{18} < 1$ *then* $T$ *is biholomorphic onto its image (in this lemma the chosen norm on* $\mathbb{C}^n$ *and the corresponding operator norm are both denoted by* $\|\cdot\|$*).*

*Proof.* $T$ is locally biholomorphic by the inverse function theorem. So we only have to show injectivity. Let $z^0, z^1 \in U$ with $T(z^0) = T(z^1)$ and $\gamma : [0, 1] \to U$, $\gamma(t) = z^0 + t(z^1 - z^0)$. Then

$$\begin{aligned}
|z^1 - z^0| &= \|T(z^1) - z^1 - T(z^0) + z^0\| \\
&= \|T\circ\gamma(1) - \gamma(1) - T\circ\gamma(0) + \gamma(0)\| \\
&= \left\| \int_0^1 (DT(\gamma(t)) - \mathrm{id})(z^1 - z^0)\, dt \right\| \\
&\leq \|z^1 - z^0\| \int_0^1 \|DT(\gamma(t)) - \mathrm{id}\|\, dt \\
&\leq \|z^1 - z^0\| c_{18}
\end{aligned} \tag{66}$$

which implies $z^1 = z^0$. □

*Proof of Proposition 3.1.* We have a Cauchy estimate for the partial derivatives of the functions $g_{p,k} : A_\delta^{B_k(p)} \to \mathbb{C}$ on a smaller polyannulus. Let $q \in B_k(p)$, Then

$$\left\| \frac{\partial}{\partial z_q} g_{p,k} \right\|_{A_\delta^{B_k(p)}} \leq \frac{1}{|e^\delta - e^{\delta_1}|} c_1 \exp(-c_2 k^d) \tag{67}$$

$$= c_{13} \exp(-c_2 k^d). \tag{68}$$

Also note that $(\partial/\partial z_q) g_{p,k} = 0$ for $q \notin B_k(p)$. Therefore

$$\begin{aligned}\left\| \frac{\partial}{\partial z_q} g_p \right\|_{A_\delta^{\mathbb{Z}^d}} &= \left\| \frac{\partial}{\partial z_q} \sum_{k=\|p-q\|}^{\infty} g_{p,k} \right\|_{A_{\delta_1}^{\mathbb{Z}^d}} \\ &\leq c_{13} \sum_{k=\|p-q\|}^{\infty} \exp(-c_2 k^d) \\ &\leq c_{13} \frac{1}{1-\exp(-c_2)} \exp(-c_2 \|p-q\|^d) \\ &= c_{14} \exp(-c_2 \|p-q\|^d). \end{aligned}\tag{69}$$

Now we consider the lift given by $pr : \mathbb{C}_\delta^\Lambda \to A_\delta^\Lambda$, $(\tilde{z_p})_{p\in\Lambda} \mapsto (e^{i\tilde{z_p}})_{p\in\Lambda}$, where $\mathbb{C}_\delta \stackrel{\text{def}}{=} \{w \in \mathbb{C} \mid \operatorname{Im} w \in [-\delta, \delta]\}$.

Then we have for the lifted functions $(\widetilde{T^{\Lambda,\epsilon}}(\tilde{\mathbf{z}}))_p = \tilde{z}_p + 2\pi\epsilon \tilde{g_p}(\tilde{\mathbf{z}})$. The function $\tilde{g_p}(\tilde{\mathbf{z}}) = g_p(pr(\tilde{\mathbf{z}}))$ satisfies the same estimate (1) with a different constant $\tilde{c_1}$ for $\delta < \delta_1$ sufficiently small since $pr$ and its partial derivatives are uniformly bounded on $\mathbb{C}_\delta^\Lambda$.

Then we have

$$|D(\widetilde{T^{\Lambda,\epsilon}}(\tilde{\mathbf{z}}))_{p,q} - \delta_{p,q}| \leq 2\pi\epsilon\tilde{c_1} \exp(-c_{13}\|p-q\|^d).$$

In particular the row sum norm (the operator-norm induced by the $l^\infty$-norm on $\mathbb{C}^\Lambda$) of $(D\widetilde{T^{\Lambda,\epsilon}} - \text{id})$ is smaller than one for $\epsilon$ small enough, independent of $\Lambda$. According to Lemma 8.1 (note that $\mathbb{C}_\delta$ is convex), $\widetilde{T^{\Lambda,\epsilon}}$ is a biholomorphic map onto its image and so is $T^{\Lambda,\epsilon}$.

Now fix $\delta < \delta_1$ according to the first part of the proof. If $\mathbf{z} \in \partial A_\delta^\Lambda$ we have $z_p \in \partial A_\delta$ for at least one $p \in \Lambda$. From the formula $z_p' \stackrel{\text{def}}{=} T_p^{\Lambda,\epsilon}(\mathbf{z}) = z_p \exp(2\pi i \epsilon g_p(\mathbf{z}))$ and the assumption that $g_p$ is uniformly bounded on $A_{\delta_1}$ we see that

$$|\ln|z_p'|| \geq \delta - c_{16}\delta\epsilon > c_7\delta \tag{70}$$

for sufficiently small $\epsilon$.

Now assume $\emptyset \neq A_{c_7\delta} \setminus T^{\Lambda,\epsilon}(A_\delta) \ni \mathbf{z}$. Let $s$ be the line-segment between $\mathbf{z}$ and its nearest point $\mathbf{w}$ on $(S^1)^\Lambda$ (w.r.t. the metric $d_\Lambda$). For each point $\mathbf{y}$ on $s$ the inequality $\ln d_\Lambda(\mathbf{w}, \mathbf{y}) \leq \ln d_\Lambda(\mathbf{w}, \mathbf{z}) \leq c_{17}\delta$ holds.

In particular there is a $\mathbf{y} \in T^{\Lambda,\epsilon}(\partial A_\delta^\Lambda)$ on $s$ with $|y_p| \leq c_7\delta$ for all $p \in \Lambda$, but this contradicts the estimate (70) above. □

*Proof of Proposition 3.2.* As $F$ acts on each coordinate separately by an $f_p$ we have (in view of the chosen metric (15)) to show the statement just for the map $f$ (we drop the index $p$), i.e. the case when $\Lambda$ contains just one element.

Consider the lift $\mathbb{R}_\delta \times \mathbb{R} \ni (r, \phi) \mapsto re^{i\phi}$ where $\mathbb{R}_\delta \stackrel{\text{def}}{=} [1 - \ln\delta, 1 + \ln\delta]$. This defines (modulo $(0, 2\pi)$) a $(0, 2\pi)$-periodic map $\tilde{f} = (\tilde{f}_r, \tilde{f}_\phi)$ via $f(re^{i\phi}) = \tilde{f}_r(r, \phi)e^{i\tilde{f}_\phi(r,\phi)}$. On $\{1\} \times \mathbb{R}$ one has $(\partial/\partial r)\tilde{f}_r \geq \lambda_0$ and so because of periodicity and a compactness argument, $(\partial/\partial r)\tilde{f}_r \geq \lambda$ on a thin ($0 < \delta < \delta_0$ small) strip $\mathbb{R}_\delta \times \mathbb{R}$. It follows similarly, as in the proof of Proposition 3.1, that $\tilde{f}(\mathbb{R}_\delta \times \mathbb{R}) \supset \mathbb{R}_{\lambda\delta} \times \mathbb{R}$, $\tilde{f}$ is diffeomorphic onto its image and each point in $\mathbb{R}_\delta \times \mathbb{R}$ has the same number of preimages (which is equal to $(\tilde{f}(1, 2\pi) - \tilde{f}(1, 0))/2\pi$). From this, the claim about $f$ follows. □

*Proof of Proposition 3.3.* We substitute the expression (21) into the right-hand side of equation (18) and get

$$\int_{(S^1)^\Lambda} \frac{d\mathbf{w}}{(2\pi i)^{|\Lambda|}} \frac{1}{\mathbf{w}} \psi(\mathbf{w}) \int_{\Gamma^\Lambda} \frac{d\mathbf{z}}{(2\pi i)^{|\Lambda|}} \phi(\mathbf{z}) \prod_{p \in \Lambda} \left( \frac{1}{S_p^\epsilon(\mathbf{z}) - w_p} \frac{S_p^\epsilon(\mathbf{z})}{z_p} \right). \tag{71}$$

To simplify notation we assume that $\Lambda = \{1, \ldots, N\}$. As (18) is linear in $\psi$ we can assume (by using a continuous partition of unity) that $\psi$ vanishes outside a small set $K \subset (S^1)^N$ having distinct preimages under $S^t$ (for all $0 \leq t \leq \epsilon$) contained in $K_\alpha = K_{\alpha_1} \times \cdots \times K_{\alpha_N}$ such that each $K_\alpha$ is contained in a polydisc $D_\alpha = D_{\alpha_1} \times \cdots \times D_{\alpha_N}$. These are mutually disjoint and $S_\alpha^t \stackrel{\text{def}}{=} S^t_{|D_\alpha}$ is biholomorphic onto its image (for all $0 \leq t \leq \epsilon$). (To make this more precise we note that for $t = 0$ the map $S^0$ is the product of maps $f_i$ ($1 \leq i \leq N$) and each $f_i$ gives rise to an $M_i$-fold covering map of $A_\delta$. So locally we can index the disjoint pre-images of $K$ under $S^0$ by $\alpha = (\alpha_1, \ldots, \alpha_N)$ where $1 \leq \alpha_i \leq M_i$. If we take the set $K$ small enough this is still true under small ($0 \leq t \leq \epsilon$) perturbations.)

For given $\mathbf{w} \in K$, index $\alpha$ as above, $k \in \{1, \ldots, N\}$ and fixed $z_l \in A_{\delta_1}$ ($l \neq k$) the function $z_k \mapsto (S_k^\epsilon(z_1, \ldots, z_k, \ldots, z_N) - w_k)^{-1}$ has exactly one simple pole in each $D_{\alpha_k}$ and is holomorphic in $A_{\delta_1}^\Lambda$ away from these poles. Therefore we get the same if we just integrate around these poles:

$$\int_K \frac{d\mathbf{w}}{(2\pi i)^N} \frac{1}{\mathbf{w}} \psi(\mathbf{w}) \sum_\alpha \left( \prod_{k=1}^N \int_{\partial D_{\alpha_k}} \frac{dz_k}{2\pi i} \right) \phi(\mathbf{z}) \prod_{k=1}^N \frac{S_{\alpha,k}^\epsilon(\mathbf{z})}{z_k} \prod_{k=1}^N \frac{1}{S_{\alpha,k}^\epsilon(\mathbf{z}) - w_k}. \tag{72}$$

For each $\alpha$ we can write each of the inner integrals as an integral of a differential form over the distinguished boundary $b_0 D_\alpha \stackrel{\text{def}}{=} \partial D_{\alpha_1} \times \cdots \times \partial D_{\alpha_N}$, parameterized by $[0, 1)^N \ni t \mapsto (e^{2\pi i t_1}, \ldots, e^{2\pi i t_N})$, whence

$$\int_{b_0 D_\alpha} \phi(\mathbf{z}) \prod_{k=1}^N \frac{S_{\alpha,k}^\epsilon(\mathbf{z})}{z_k} \prod_{k=1}^N \frac{1}{S_{\alpha,k}^\epsilon(\mathbf{z}) - w_k} \, dz_1 \wedge \cdots \wedge dz_N. \tag{73}$$

We want to split the singular factor into a product of single poles in each variable. So we apply the transformation $\mathbf{u} = S_\epsilon(\mathbf{z}) \stackrel{\text{def}}{=} S_\alpha^\epsilon(\mathbf{z})$ to get

$$\int_{S_\epsilon(b_0 D_\alpha)} \phi \circ S_\epsilon^{-1}(\mathbf{u}) \prod_{k=1}^N \frac{u_k}{(S_\epsilon^{-1}(\mathbf{u}))_k} \det(S_\epsilon^{-1})'(\mathbf{u}) \prod_{k=1}^N \frac{1}{u_k - w_k} \, du_1 \wedge \cdots \wedge du_N \tag{74}$$

where $(S_\epsilon^{-1})'$ is the complex derivative and so is holomorphic in $\mathbf{u}$. To apply Cauchy's formula we have to integrate over a product of cycles (each lying in $\mathbb{C}$). For example $b_0 D$

or $S_0(b_0 D)$ are such products of cycles, but $S_\epsilon(b_0 D)$ in general is not. So first we have to deform $S_\epsilon(b_0 D)$ into $S_0(b_0 D)$. The map $t \mapsto S_t \stackrel{\text{def}}{=} S_\alpha^t$ is a smooth homotopy between $S_\epsilon$ and the product map $S_0$ and avoids singularities of the integrand in (74) since for $\epsilon$ small enough the set $\{S_t(b_0 D_\alpha) \mid 0 \le t \le \epsilon\}$ has positive distance (uniformly in $\Lambda$) from the set of singularities $\bigcup_{k=1}^N \{u \in D_\alpha : u_k = w_k\}$. $S_0(b_0 D_\alpha) = S_{0,1}(\partial D_{\alpha_1}) \times \cdots \times S_{0,N}(\partial D_{\alpha_N})$ is a product of cycles and hence a cycle. The differential n-form in (74) is a cocycle because its coefficient is holomorphic. So we get by Stokes' theorem

$$\int_{S_0(b_0 D_\alpha)} \phi \circ S_\epsilon^{-1}(\mathbf{u}) \prod_{k=1}^N \frac{u_k}{(S_\epsilon^{-1}(\mathbf{u}))_k} \det(S_\epsilon^{-1})'(\mathbf{u}) \prod_{k=1}^N \frac{1}{u_k - w_k} \, du_1 \wedge \cdots \wedge du_N \tag{75}$$

and by Cauchy's formula

$$\phi \circ S_\epsilon^{-1}(\mathbf{w}) \prod_{k=1}^N \frac{w_k}{(S_\epsilon^{-1}(\mathbf{w}))_k} \frac{1}{\det(S_\epsilon'(S_\epsilon^{-1}(\mathbf{w})))}. \tag{76}$$

So (72) is equal to

$$\sum_\alpha \int_K \frac{d\mathbf{w}}{(2\pi i)^N} \frac{1}{\mathbf{w}} \psi(\mathbf{w}) \phi \circ (S_\alpha^\epsilon)^{-1}(\mathbf{w}) \frac{1}{\det(S^\epsilon)'((S_\alpha^\epsilon)^{-1}(\mathbf{w}))} \prod_{k=1}^N \frac{w_k}{((S_\alpha^\epsilon)^{-1}(\mathbf{w}))_k}. \tag{77}$$

For each index $\alpha$, the coordinate transformation $\mathbf{u} = (S_\alpha^\epsilon)^{-1}(\mathbf{w})$ yields

$$\sum_\alpha \int_{K_\alpha} \frac{d\mathbf{u}}{(2\pi i)^N} \frac{1}{\mathbf{u}} \psi \circ S_\alpha^\epsilon(\mathbf{u}) \phi(\mathbf{u}). \tag{78}$$

As $\psi \circ F = 0$ outside $\bigcup_\alpha K_\alpha$ and the $K_\alpha$ are mutually disjoint this equals

$$\int_{(S^1)^N} \frac{d\mathbf{u}}{(2\pi i)^N} \frac{1}{\mathbf{u}} \psi \circ S(\mathbf{u}) \phi(\mathbf{u}) \tag{79}$$

which equals

$$\int_{(S^1)^N} d\mu^N \, \psi \circ S\phi \tag{80}$$

as was to be shown. □

*Proof of Lemma 4.1.* Consistency follows from

$$\begin{aligned} \pi_{\Lambda_3}(g^1\phi)_{\Lambda_4} &= \pi_{\Lambda_3} \circ \pi_{\Lambda_4}(g^1\phi_{\Lambda_1 \cup \Lambda_4}) \\ &= \pi_{\Lambda_3}(g^1\phi_{\Lambda_1 \cup \Lambda_4}) \\ &= \pi_{\Lambda_3}(g^1\phi_{\Lambda_1 \cup \Lambda_3}) \\ &= (g^1\phi)_{\Lambda_3} \end{aligned} \tag{81}$$

for all $\Lambda_3 \subset \Lambda_4 \in \mathcal{F}$.

As $g^1$ depends only on the $\Lambda_1$-coordinates we have

$$\begin{aligned} \|(g^1\phi)_{\Lambda_1 \cup \Lambda}\|_{\Lambda_1 \cup \Lambda} &= \|g^1\phi_{\Lambda_1 \cup \Lambda}\|_{\Lambda_1 \cup \Lambda} \\ &\le \|g^1\|_{\Lambda_1} \|\phi_{\Lambda_1 \cup \Lambda}\|_{\Lambda_1 \cup \Lambda} \\ &\le \|g^1\|_{\Lambda_1} \vartheta^{-|\Lambda_1| - |\Lambda|} \|\phi\|_\vartheta \end{aligned} \tag{82}$$

and so

$$\vartheta^{|\Lambda|}\|(g^1\phi)_\Lambda\|_\Lambda \le \|g^1\|_{\Lambda_1}\vartheta^{-|\Lambda_1|}\|\phi\|_\vartheta \tag{83}$$

and

$$\|g\phi\|_\vartheta \le \|g^1\|_{\Lambda_1}\vartheta^{-|\Lambda_1|}\|\phi\|_\vartheta. \tag{84}$$

For $\Lambda_1$ fixed, the product is continuous in both factors. Lemma 4.1(2) follows from

$$\begin{aligned}((g^1g^2)\phi)_\Lambda &= \pi_\Lambda(g^1_{\Lambda_1}g^2_{\Lambda_2}\phi_{\Lambda\cup\Lambda_1\cup\Lambda_2})\\ &= \pi_\Lambda(g^1_{\Lambda_1}\pi_{\Lambda\cup\Lambda_1}(g^2_{\Lambda_2}\phi_{\Lambda\cup\Lambda_1\cup\Lambda_2}))\\ &= \pi_\Lambda(g^1_{\Lambda_1}\pi_{\Lambda\cup\Lambda_1}(g^2\phi))\\ &= (g^1(g^2\phi))_\Lambda.\end{aligned} \tag{85}$$

To see Lemma 4.1(3) we note that the projection of the product of $g^1$ and $g^2$ is

$$\pi_\Lambda(g^1g^2) = \pi_\Lambda(g^1_{\Lambda_1}g^2_{\Lambda_2}) \tag{86}$$

and the product in the sense of (26) projects to

$$\begin{aligned}\pi_\Lambda(g^1g^2) &= \pi_\Lambda(g^1_{\Lambda_1}g^2_{\Lambda\cup\Lambda_2})\\ &= \pi_\Lambda(g^1_{\Lambda_1}g^2_{\Lambda_2})\end{aligned} \tag{87}$$

as $g^2$ does not depend on $\Lambda\setminus\Lambda_2$-coordinates.

If $\Lambda_1 \subseteq \Lambda_2$ then

$$\begin{aligned}g_{\Lambda_2}(g^1\phi)_{\Lambda_2} &= g_{\Lambda_2}g^1\phi_{\Lambda_2}\\ &= (g^1g)_{\Lambda_2}\phi_{\Lambda_2}\end{aligned} \tag{88}$$

and so Lemma 4.1(4) follows from

$$\begin{aligned}(g^1\phi)(g) &= \lim_{\Lambda_2\to\mathbb{Z}^d}\int_{(S^1)^{\Lambda_2}} d\mu^{\Lambda_2}\, g_{\Lambda_2}(g^1\phi)_{\Lambda_2}\\ &= \lim_{\Lambda_2\to\mathbb{Z}^d}\int_{(S^1)^{\Lambda_2}} d\mu^{\Lambda_2}\, (g^1g)_{\Lambda_2}\phi_{\Lambda_2}\\ &= \phi(g^1g)\end{aligned} \tag{89}$$

and Lemma 4.1(5) follows from

$$\begin{aligned}\|g^1\phi\|_{\mathrm{var}} &= \lim_{\Lambda\to\mathbb{Z}^d}\int_{(S^1)^\Lambda} d\mu^\Lambda\, |(g^1\phi)_\Lambda|\\ &= \lim_{\substack{\Lambda\to\mathbb{Z}^d\\ \Lambda\supset\Lambda_1}}\int_{(S^1)^\Lambda} d\mu^\Lambda\, |g^1||\phi_\Lambda|\\ &\le \|g^1\|_{\Lambda_1}\|\phi\|_{\mathrm{var}}.\end{aligned} \tag{90}$$

□

*Proof of Lemma 5.1.* We get, recursively,

$$\begin{aligned}
&\frac{1}{f_p \circ T^{\epsilon}_{p,l}(\mathbf{z}) - w_p} f_p \circ T^{\epsilon}_{p,l}(\mathbf{z}) \\
&\quad = \frac{1}{f_p \circ T^{\epsilon}_{p,l-1}(\mathbf{z}) - w_p} f_p \circ T^{\epsilon}_{p,l-1}(\mathbf{z}) \\
&\qquad + w_p \frac{f_p \circ T^{\epsilon}_{p,l-1}(\mathbf{z}) - f_p \circ T^{\epsilon}_{p,l}(\mathbf{z})}{(f_p \circ T^{\epsilon}_{p,l-1}(\mathbf{z}) - w_p)(f_p \circ T^{\epsilon}_{p,l}(\mathbf{z}) - w_p)} \\
&\quad = \frac{1}{f_p(z_p) - w_p} f_p(\mathbf{z}) + w_p \sum_{k=1}^{l} \frac{f_p \circ T^{\epsilon}_{p,k-1}(\mathbf{z}) - f_p \circ T^{\epsilon}_{p,k}(\mathbf{z})}{(f_p \circ T^{\epsilon}_{p,k-1}(\mathbf{z}) - w_p)(f_p \circ T^{\epsilon}_{p,k}(\mathbf{z}) - w_p)}. \qquad (91)
\end{aligned}$$

The estimate (42) yields uniform convergence of this sum as $l \to \infty$. So we get (28). □

In (57) we estimate the norm of the operator corresponding to one particular configuration in terms of the lines and triangles it contains. Now we have to bound sums over all such configurations as they arise in the expansions for the full operators. For this we use our analysis and some combinatorics at the same time. The idea is that a configuration of a given length must have at least a certain number of triangles and r-chains that lead to small factors in the estimates. In fact, certain r-chains could not be replaced by h-chains in the configuration as we would get a zero configuration.

*Definition 8.1.*

- A maximal r-chain going from an apex of a triangle downwards to the next base point of a triangle or to a bottom point is called an *(a-r)-chain*. (If the apex coincides with a base or bottom point the (a-r)-chain has length zero.)
- The *(a-r)-length* of a configuration $\mathcal{C}$ is the sum of the lengths of all its (a-r)-chains plus the number of its triangles, i.e. if $\mathcal{C}$ has $|n_\beta|$ triangles with corresponding (a-r)-chains of length $l_1, \ldots, l_{|n_\beta|}$ then

$$\begin{aligned}
\text{(a-r)-length}(\mathcal{C}) &\stackrel{\text{def}}{=} l_1 + \cdots + l_{|n_\beta|} + |n_\beta| \\
&= (l_1 + 1) + \cdots + (l_{|n_\beta|} + 1). \qquad (92)
\end{aligned}$$

  (In particular (a-r)-length$(\mathcal{C}) \geq |n_\beta|$.)
- We call a maximal r-chain going from a base point $(p, t)$ of a triangle to $(p, -N)$ (such that $(p, -N)$ is not a base point of another triangle) a *(u-r)-chain* (upwards going r-chain), a maximal r-chain going downwards from a base point a *(d-r)-chain* (*(d-h)-chains* are defined analogously).
- A maximal r-chain going from a bottom point $(p, 0)$ to $(p, -N)$ is called an *(l-r)-chain* (long r-chain). We denote the number of (l-r)-chains of $\mathcal{C}$ by $l(\mathcal{C})$.

The configuration in figure 5 has length three, (a-r)-length six, only one (a-r)-chain of positive length from $(6, -2)$ to $(6, -1)$, only one (u-r)-chain of positive length from $(2, -3)$ to $(2, -2)$, and only one (l-r)-chain from $(1, -3)$ to $(1, 0)$.

We prepare the proofs of Theorem 2.1 and Proposition 6.1 in the following technical proposition that provides the key bounds and basic analysis and combinatorics for the other proofs.

PROPOSITION 8.1. *For sufficiently small* $\vartheta$, $\epsilon$ *and large* $c_2$ *and* $N$ *we have for all* $\Lambda_1 \subseteq \Lambda_2 \in \mathcal{F}$ *the following bound for the terms in the expansion of (49) for* $\pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_2} \circ T^{\Lambda_2,\epsilon}}$ *with constants* $c_{19}$, $c_{20}$:

(1)

$$\sum_{\mathcal{C}:\text{length}(\mathcal{C})=N} \|\pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}}\|_{L((\mathcal{H}_{\Lambda_2,\vartheta},\|\cdot\|_{\Lambda_2,\vartheta}),(\mathcal{H}_{\Lambda_1,\vartheta},\|\cdot\|_{\Lambda_1,\vartheta}))} \leq c_{19}\tilde{\eta}^N \tag{93}$$

with $\tilde{\eta} \stackrel{\text{def}}{=} \sqrt{\eta} < 1$

(2)

$$\|\pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_2} \circ T^{\Lambda_2,\epsilon}}\|_{L((\mathcal{H}_{\Lambda_2,\vartheta},\|\cdot\|_{\Lambda_2,\vartheta}),(\mathcal{H}_{\Lambda_1,\vartheta},\|\cdot\|_{\Lambda_1,\vartheta}))} \leq c_{20} \tag{94}$$

For the proof of Proposition 8.1 we need a graph-theoretical lemma. We consider labelled tree graphs that are constructed in the following way (cf. Figure 6). We start with a star graph with a *root*-vertex, labelled (0), to which $K$ edges are attached, each connecting to one *leaf*. The leaves are labelled by $(0, 1), \dots, (0, K)$. Then we successively add star graphs (each of these has a certain finite number $v(k)$ of leaves. These numbers are defined in (45)) to the already built-up tree. We identify one of the leaves of the tree, say labelled by $s = (s_1, \dots, s_n)$, with the root of the added star and label the new leaves by $(s_1, \dots, s_n, 1), \dots, (s_1, \dots, s_n, v(k))$. In total we use, in addition to the star graph with $K$ leaves, exactly $n_{\beta,k}$ star graphs with exactly $v(k)$ leaves. We say *the tree has parameters* $K$ and $n_\beta = (n_{\beta,1}, n_{\beta,2}, \dots)$.

We also introduce a linear order on the set of tuples (and so on the set of vertices of the labelled graph): we say $s = (s_1, \dots, s_n) \prec t = (t_1, \dots, t_m)$ if $n < m$ and $s_i = t_i$ for $1 \leq i \leq n$ or if $s_i = t_i$ $(1 \leq i < k)$ and $s_k < t_k$ for some $k$.

LEMMA 8.2.

(1) *The number of labelled tree graphs with exactly n edges is not greater than* $2^{2n}$.

(2) *Given* $K, n_{\beta,1}, n_{\beta,2}, \dots$ *with* $K + \sum_{k=1}^{\infty} n_{\beta,k} < \infty$. *The number of labelled tree graphs with parameters* $K$ *and* $n_\beta$ *is bounded from above by* $4^K \prod_{k=1}^{\infty} c_{21}^{k^d n_{\beta,k}}$ *with* $c_{21} = 4^{3^d}$.

*Proof of Lemma 8.2.* We first prove (1) For every labelled tree graph in question we can define a path starting and ending at the root point (0) and running through each edge exactly twice in the following way. From a (labelled) vertex $t = (t_1, \dots, t_k)$ we go to the next greater (w.r.t. $\prec$) vertex where we have not yet been (*going up*), or if this is not possible (i.e. $t$ is a leaf or we have already been at all vertices $(t_1, \dots, t_{k+1})$) back to $(t_1, \dots, t_{k-1})$ (*going down*). So we return to (0) after $2n$ steps. We encode the path in a word $(a_1, \dots, a_{2n})$ with $a_i = 1$ if we go up in the $i$th step and $a_i = 0$ otherwise. Obviously the labelled graph is uniquely determined by its word. (Note that not every word of length $2n$ with symbols '0' and '1' corresponds to such a labelled graph, but the map between these two data is injective.) As there are $2^{2n}$ words of length $2n$ with at most two different symbols this is also an upper bound for the number of graphs in question, so (1) is proved.

To see (2) we note, using the estimate for $v(k)$ that we obtain after (45), that the number of edges in such a tree graph is not greater than $K + \sum_{k=1}^{\infty} (3k)^d n_{\beta,k}$. □

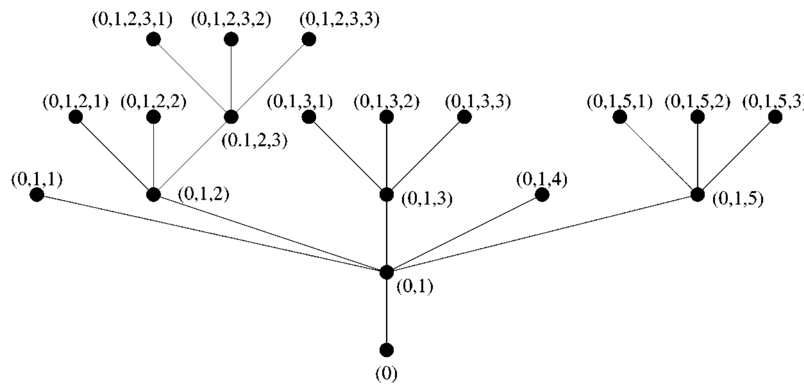


FIGURE 6. The labelled graph for the configuration in Figure 5.

*Proof of Proposition 8.1.* We estimate the norm of each $\mathcal{L}_{\mathcal{C}}$ in (93) in terms of the number of particular triangles, r-chains etc. of $\mathcal{C}$ as we do in (57). We also have to bound the number of configurations in (93) that have the same set of triangles. We do so by assigning (in (i)–(iv)) to each configuration a labelled tree graph and estimating the number of graphs that have certain properties.

(i) We fix $0 \leq K \leq |\Lambda_1|$ and $\Lambda_3 \subseteq \Lambda_1$ with $|\Lambda_3| = K$ (so there are $\binom{|\Lambda_1|}{K}$ possible choices for $\Lambda_3$) and want to estimate the number of configurations $\mathcal{C}$ such that $\Lambda_{\mathcal{C}} = \Lambda_3$. So let us consider such a configuration. We call the triangles whose apex lies at, or whose (a-r)-chain ends in, $\Lambda_3 \times \{0\}$, *root triangles*. We can assign to $\mathcal{C}$ a graph of the type we consider in Lemma 8.2 as follows. We start with a star graph with a star point labelled (0) and $K$ leaves, labelled $(0, 1), \dots, (0, K)$. These leaves are in bijection with $\Lambda_3 \times \{0\}$. Now we add successively for each $l$-triangle (as introduced in §5.3) in $\mathcal{C}$ a star graph with one star point and $v(l)$ leaves (cf. definition of $v(l)$ in (45)) to the graph and label the new vertices: if an $l$-triangle lies with its apex or ends with its (a-r)-chain on a base point of another triangle (for which we have already assigned a small tree) or on a point in $\Lambda_3 \times \{0\}$ (this point is labelled say $s = (s_1, \dots, s_n)$) we add a small $l$-tree to the graph by identifying its star point with $s$ and label the $v(l)$ new leaves in the graph $(s_1, \dots, s_n, 1), \dots, (s_1, \dots, s_n, v(l))$. Since, for example, an apex could coincide with more than one other triangle's base point we use the linear order $\prec$ (defined above Lemma 8.2) to define an order in our successive assignment of triangles to star graphs. We always choose the next triangle such that the corresponding star graph is attached to the smallest (w.r.t. $\prec$) labelled leaf in the graph. This also defines a unique choice of the triangle and the leaf where we attach the star graph. So the position of triangles and the (a-r)-chains of $\mathcal{C}$ are completely determined by the datum consisting of the corresponding labelled graph *and* the lengths of its (a-r)-chains. Note that it is not the case that for every graph together with a choice of lengths for the particular (a-r)-chains there was a corresponding configuration.

For the configuration in Figure 5, for example, we get the labelled graph in Figure 6.

Let $n_{\beta,k}$ be, as in Definition 5.2, the total number of $k$-triangles. The number of graphs with parameters $K$ and $n_\beta$ is bounded by $4^K \prod_{k=1}^{\infty} c_{21}^{k^d n_{\beta,k}}$ (by Lemma 8.2). As mentioned above we have for each of the $|n_\beta|$ (a-r)-chains a length $0 \le l_i < \infty$. The (a-r)-length is

$$L = (l_1 + 1) + \cdots + (l_{|n_\beta|} + 1). \tag{95}$$

So $L \ge |n_\beta|$. For a given $n_\beta$ with $|n_\beta| \ge 1$ and $L \ge 1$ there are $\binom{L-1}{|n_\beta|-1}$ different choices of $(l_1, \ldots, l_{|n_\beta|})$ that satisfy (95). For $|n_\beta| = 0$ we have $L = 0$ and the (unique) configuration without triangles or r-lines. So, in any case, the number of choices is bounded from above by $\binom{L}{|n_\beta|}$. The integration over these $|n_\beta|$ (a-r)-chains leads to a factor $c_r^{|n_\beta|} \eta^L$ in our estimates (cf. (57)) and each $k$-triangle contributes by (55) a factor $c_8 \epsilon \exp(-c_2 k^d)$.

(ii) There are choices between (d-r)-chains and (d-h)-chains in the configuration. There are not more than $\sum_{k=1}^{\infty} (3k)^d n_{\beta,k}$ base points for which we can choose between a (d-h)-chain (giving factor $c_h$ in our estimates) and a (d-r)-chain (giving factor at most $c_r \eta$). So the total sum over these combinations is bounded from above by

$$(c_h + c_r \eta)^{\sum_{k=1}^{\infty} (3k)^d n_{\beta,k}} \le \prod_{k=1}^{\infty} (\exp(c_{22} k^d))^{n_{\beta,k}}.$$

(iii) There are choices between (u-r)-chains and (u-h)-chains in the configuration. There are not more than $\sum_{k=1}^{\infty} (3k)^d n_{\beta,k}$ base points. To each of them we can attach either a (u-h)-chain, giving a factor $c_h$, or a (u-r)-chain, giving a factor $c_r \eta^{\max\{0, N-L\}}$, because if $N - L > 0$, such a (u-r)-chain cannot have length smaller than $N - L$, for otherwise it would not end in $\Lambda_2 \times \{-N\}$. We get in total a factor not greater than

$$(c_h + c_r)^{\sum_{k=1}^{\infty} (3k)^d n_{\beta,k}} = \prod_{k=1}^{\infty} (\exp(c_{23} k^d))^{n_{\beta,k}}. \tag{96}$$

(iv) There are choices left between (l-h)-chains and (l-r)-chains in $(\Lambda_1 \setminus \tilde{\Lambda}_{\mathcal{C}}) \times \{-N, \ldots, 0\}$, giving factor $c_h$ or $c_r \eta^N$, respectively. Let $l$ $(0 \le l \le |\Lambda_1 \setminus \tilde{\Lambda}_{\mathcal{C}}| \le |\Lambda_1| - K)$ denote the number of (l-r)-chains in such a choice. For given $l$ there are $|\Lambda_1 \setminus \tilde{\Lambda}_{\binom{\mathcal{C}}{|}} l \le \binom{|\Lambda_1|-K}{l}$ different subsets $\Lambda_r$ of $\Lambda_1 \setminus \tilde{\Lambda}_{\mathcal{C}}$ of cardinality $l$ (that correspond to a particular choice of exactly $l$ (l-r)-chains.) The configuration $\mathcal{C}$ is determined by all the choices mentioned up to now.

Consider now a $\mathcal{C}$ with length$(\mathcal{C}) = N$. If $N - L > 0$ then there must be at least one (u-r)-chain giving rise to an extra factor $\eta^{\max\{0, N-L\}}$ or an (l-r)-chain giving rise to a factor $\eta^n$. To get (98) we split

$$\eta^{\max\{0, N-L\}} = \tilde{\eta}^{\max\{0, N-L\}} \tilde{\eta}^{\max\{0, N-L\}}$$

or

$$\eta^N = \tilde{\eta}^N \tilde{\eta}^N$$

with $\tilde{\eta} \stackrel{\text{def}}{=} \sqrt{\eta}$. Therefore we get the factor $\tilde{\eta}^{\max\{0, N-L\}}$.

In the configuration $\mathcal{C}$ there are h-chains at points with $\mathbb{Z}^d$-coordinate in $\Lambda_1 \setminus (\tilde{\Lambda}_{\mathcal{C}} \cup \Lambda_r)$. The operator $\mathcal{L}_{\mathcal{C}}$ acts on $\phi_{\Lambda_2}$ by integration over these coordinates. So for the uniform estimate of $\mathcal{L}_{\mathcal{C}}\phi_{\tilde{\Lambda}(\mathcal{C})}$ we use (58).

First we estimate in (97)–(104) the sum over $\mathcal{C}$ with length$(\mathcal{C}) = N$ and then in (105)–(107) the sum over $\mathcal{C}$ with length$(\mathcal{C}) = m < N$. We do these separately because in the second case $\mathcal{C}$ has no (l-r)-chains, while in the first case every (l-r)-chain leads to a small factor $c_r \eta^N$. The idea of making this distinction is similar to the idea of 'vacuum polymers' in other papers (cf. [**2, 15, 1**]).

$$\vartheta^{|\Lambda_1|} \sum_{\mathcal{C}:\text{length}(\mathcal{C})=N} \|\pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}} \phi_{\Lambda_2}\|_{\Lambda_1} \tag{97}$$

$$\begin{aligned}
&\leq \vartheta^{|\Lambda_1|} \sum_{K=0}^{|\Lambda_1|} \binom{|\Lambda_1|}{K} \sum_{\substack{n_\beta \\ K \leq |n_\beta| < \infty}} 4^K \prod_{k=1}^{\infty} (\exp(c_{21} k^d))^{n_{\beta,k}} (c_1 \epsilon)^{|n_\beta|} \\
&\quad \times \prod_{k=1}^{\infty} (\exp(-c_2 k^d))^{n_{\beta,k}} \sum_{L=|n_\beta|}^{\infty} \binom{L}{|n_\beta|} c_r^{|n_\beta|} \tilde{\eta}^L \prod_{k=1}^{\infty} (\exp(c_{22} k^d))^{n_{\beta,k}} \\
&\quad \times \prod_{k=1}^{\infty} (\exp(c_{23} k^d))^{n_{\beta,k}} \tilde{\eta}^{\max\{0,N-L\}} \sum_{l=0}^{|\Lambda_1|-K} \binom{|\Lambda_1| - K}{l} (c_r \tilde{\eta}^N)^l \\
&\quad \times c_h^{|\Lambda_1|-K-l} \vartheta^{-l} \prod_{k=1}^{\infty} \vartheta^{-(3k)^d n_{\beta,k}} \|\phi\|_{\Lambda_2,\vartheta}
\end{aligned} \tag{98}$$

$$\begin{aligned}
&= \vartheta^{|\Lambda_1|} \sum_{K=0}^{|\Lambda_1|} \binom{|\Lambda_1|}{K} \sum_{\substack{n_\beta \\ K \leq |n_\beta| < \infty}} 4^K (c_1 \epsilon c_r)^{|n_\beta|} \\
&\quad \times \prod_{k=1}^{\infty} \exp((c_{21} - c_2 + c_{22} + c_{23} - 3^d \ln \vartheta) k^d)^{n_{\beta,k}} \\
&\quad \times \sum_{L=|n_\beta|}^{\infty} \binom{L}{|n_\beta|} \tilde{\eta}^{\max\{N,L\}} (\vartheta^{-1} c_r \tilde{\eta}^N + c_h)^{|\Lambda_1|-K} \|\phi\|_{\Lambda_2,\vartheta} \tilde{\eta}^N.
\end{aligned} \tag{99}$$

We assume $\epsilon < 1$. We set $\epsilon_1 \stackrel{\text{def}}{=} 4\epsilon c_1 c_r$ and $\epsilon_2 \stackrel{\text{def}}{=} \sqrt{\epsilon_1}$. Then we have $\epsilon_1^{|n_\beta|} \leq \epsilon_2^K \epsilon_2^{|n_\beta|}$. We set $\tilde{c_2} \stackrel{\text{def}}{=} c_2 - c_{21} - c_{22} - c_{23} + 3^d \ln \vartheta$. Then $\tilde{c_2} > 0$ if

$$c_2 > c_{21} + c_{22} + c_{23} - 3^d \ln \vartheta. \tag{100}$$

(We assume this condition on the decay of the coupling. Note that we first have to choose $\vartheta$ below, after (104), depending on the other parameters of the system (but not on $c_2$) and then condition (100) is well defined.) Then (99) can be bounded as follows:

$$\begin{aligned}
&\leq \sum_{K=0}^{|\Lambda_1|} \binom{|\Lambda_1|}{K} (c_r \tilde{\eta}^N + \vartheta c_h)^{|\Lambda_1|-K} \epsilon_2^K \sum_{\substack{n_\beta \\ K \leq |n_\beta| < \infty}} \sum_{L=|n_\beta|}^{\infty} \binom{L}{|n_\beta|} \tilde{\eta}^L \epsilon_2^{|n_\beta|} \\
&\quad \times \prod_{k=1}^{\infty} (\exp(-\tilde{c_2} k^d))^{n_{\beta,k}} \|\phi\|_{\Lambda_2,\vartheta} \tilde{\eta}^N
\end{aligned}$$

$$\leq (c_r \tilde{\eta}^N + \vartheta c_h + \epsilon_2)^{|\Lambda_1|} \sum_{L=0}^{\infty} \sum_{n=0}^{L} \binom{L}{n} \tilde{\eta}^L \epsilon_2^n \sum_{\substack{n_\beta \\ |n_\beta|=n}} \prod_{k=1}^{\infty} (\exp(-\tilde{c_2} k^d))^{n_{\beta,k}}$$
$$\times \|\phi\|_{\Lambda_2,\vartheta} \tilde{\eta}^N. \tag{101}$$

We have

$$\sum_{\substack{n_\beta \\ |n_\beta|=n}} \prod_{k=1}^{\infty} (\exp(-\tilde{c_2} k^d))^{n_{\beta,k}} \leq \prod_{k=1}^{\infty} \sum_{n_{\beta,k}=0}^{\infty} (\exp(-\tilde{c_2} k^d))^{n_{\beta,k}} \tag{102}$$

and the last infinite product converges (to $c_{24}$ say) since for $k$ sufficiently large $\exp(-\tilde{c_2} k^d) < \frac{1}{2}$ and $\sum_{n_{\beta,k}=0}^{\infty} (\exp(-\tilde{c_2} k^d))^{n_{\beta,k}} \leq 1 + 2\exp(-\tilde{c_2} k^d)$ and $\sum_{k=0}^{\infty} \exp(-\tilde{c_2} k^d) < \infty$. (Recall $\prod_{k=1}^{\infty}(1+u_k)$ *convergent* $\Leftarrow \sum_{k=1}^{\infty} |u_k| < \infty$.)

$$\leq (\epsilon_2 + c_r \tilde{\eta}^N + c_h \vartheta)^{|\Lambda_1|} c_{24} \sum_{L=0}^{\infty} (\epsilon_2 + \tilde{\eta})^L \|\phi\|_{\Lambda_2,\vartheta} \tilde{\eta}^N$$
$$= (\epsilon_2 + c_r \tilde{\eta}^N + c_h \vartheta)^{|\Lambda_1|} \frac{1}{1 - \epsilon_2 - \tilde{\eta}} c_{24} \|\phi\|_{\Lambda_2,\vartheta} \tilde{\eta}^N \tag{103}$$
$$\leq c_{19} \tilde{\eta}^N \|\phi\|_{\Lambda_2,\vartheta} \tag{104}$$

for $\vartheta$ and $\epsilon$ sufficiently small and $N$ sufficiently large. This also holds for $\Lambda \subset \Lambda_1$. So Proposition 8.1(1) is proved.

To show Proposition 8.1(2) we have to estimate, in addition to (93), the contribution of non-zero configurations $\mathcal{C}$ of length $0 \leq m < N$ in the expansion of $\pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_2} \circ T^{\Lambda_2,\epsilon}}$. These have no (l-r)-chains. So this time we have $l(\mathcal{C}) = 0$. Using the splitting $\eta^L \leq \tilde{\eta}^L \tilde{\eta}^m$ we get, in a similar way,

$$\vartheta^{|\Lambda_1|} \sum_{\substack{\mathcal{C}:\text{length}(\mathcal{C})=m, \\ l(\mathcal{C})=0}} \|\pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}} \phi_{\Lambda_2}\|_{\Lambda_1} \tag{105}$$
$$\leq \vartheta^{|\Lambda_1|} \sum_{K=0}^{|\Lambda_1|} \binom{|\Lambda_1|}{K} \sum_{\substack{n_\beta \\ K \leq |n_\beta| < \infty}} 4^K \prod_{k=1}^{\infty} (\exp(c_{21} k^d))^{n_{\beta,k}}$$
$$\times (c_1 \epsilon)^{|n_\beta|} \prod_{k=1}^{\infty} (\exp(-c_2 k^d))^{n_{\beta,k}} \sum_{L=|n_\beta|}^{\infty} \binom{L}{|n_\beta|} c_r^{|n_\beta|} \tilde{\eta}^L \prod_{k=1}^{\infty} (\exp(c_{22} k^d))^{n_{\beta,k}}$$
$$\times \prod_{k=1}^{\infty} (\exp(c_{23} k^d))^{n_{\beta,k}} c_h^{|\Lambda_1|-K} \prod_{k=1}^{\infty} \vartheta^{-(3k)^d n_{\beta,k}} \|\phi\|_{\Lambda_2,\vartheta} \tilde{\eta}^m$$
$$\leq \sum_{K=0}^{|\Lambda_1|} \binom{|\Lambda_1|}{K} (c_h \vartheta)^{|\Lambda_1|-K} \sum_{\substack{n_\beta \\ K \leq |n_\beta| < \infty}} (c_1 \epsilon 4 c_r)^{|n_\beta|} \prod_{k=1}^{\infty} (\exp(-\tilde{c_2} k^d))^{n_{\beta,k}}$$
$$\times \sum_{L=|n_\beta|}^{\infty} \binom{L}{|n_\beta|} \tilde{\eta}^L \tilde{\eta}^m \|\phi\|_{\Lambda_2,\vartheta}$$

$$
\begin{aligned}
&\leq \sum_{K=0}^{|\Lambda_1|} \binom{|\Lambda_1|}{K} (c_h \vartheta)^{|\Lambda_1|-K} \epsilon_2^K \sum_{\substack{n_\beta \\ K \leq |n_\beta| < \infty}} \sum_{L=|n_\beta|}^{\infty} \binom{L}{|n_\beta|} \tilde{\eta}^L \epsilon_2^{|n_\beta|} \\
&\quad \times \prod_{k=1}^{\infty} (\exp(-\tilde{c_2} k^d))^{n_{\beta,k}} \tilde{\eta}^m \|\phi\|_{\Lambda_2,\vartheta} \\
&\leq (\epsilon_2 + c_h \vartheta)^{|\Lambda_1|} \frac{1}{1 - \epsilon_2 - \tilde{\eta}} c_{25} \tilde{\eta}^m \|\phi\|_{\Lambda_2,\vartheta} \\
&\leq c_{26} \tilde{\eta}^m \|\phi\|_{\Lambda_2,\vartheta}. \qquad (106)
\end{aligned}
$$

Again this also holds for $\Lambda \subset \Lambda_1$ and so

$$
\vartheta^{|\Lambda_1|} \sum_{\substack{\mathcal{C}:\text{length}(\mathcal{C})=m, \\ l(\mathcal{C})=0}} \|\pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}} \phi_{\Lambda_2}\|_{\Lambda_1,\vartheta} \leq c_{26} \|\phi\|_{\Lambda_2,\vartheta} \tilde{\eta}^m. \qquad (107)
$$

Therefore

$$
\begin{aligned}
\|\pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_2} \circ T^{\Lambda_2,\epsilon}}\|_{L((\mathcal{H}_{\Lambda_2,\vartheta}, \|\cdot\|_{\Lambda_2,\vartheta}),(\mathcal{H}_{\Lambda_1,\vartheta} \|\cdot\|_{\Lambda_1,\vartheta}))} &\leq \sum_{m=0}^{N} c_{26} \tilde{\eta}^m \\
&\leq \sum_{m=0}^{\infty} c_{26} \tilde{\eta}^m \\
&\leq c_{20} \qquad (108)
\end{aligned}
$$

which was to be shown. □

*Proof of Theorem 2.1.* First we consider the case $N \geq N_0$. The difference between $\pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_2} \circ T^{\Lambda_2,\epsilon}} \circ \pi_{\Lambda_2}$ and $\pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_3} \circ T^{\Lambda_3,\epsilon}} \circ \pi_{\Lambda_3}$ for $\Lambda_1 \subseteq \Lambda_2 \subseteq \Lambda_3 \in \mathcal{F}$ is due to the summands involving configurations that do not lie completely (with all their triangles) in $\Lambda_2 \times \{0, -1, \ldots\}$. For those summands we have the lower bound for the spatial extension of the set of triangles:

$$
\begin{aligned}
b(\mathcal{C}) &\stackrel{\text{def}}{=} \sum_{k=1}^{\infty} k n_{\beta,k} \\
&\geq \text{dist}(\Lambda_1, \Lambda_2^C) \qquad (109)
\end{aligned}
$$

As the analysis in the proof of Proposition 8.1 shows we have in the estimate for each such configuration a factor

$$
\begin{aligned}
\prod_{k=1}^{\infty} (\exp(-\tilde{c_2} k^d))^{n_{\beta,k}} &\leq \prod_{k=1}^{\infty} [\exp(-(\tilde{c_2} - \xi) k^d)]^{n_{\beta,k}} \prod_{k=1}^{\infty} (\exp(-\xi k n_{\beta,k})) \\
&\leq \prod_{k=1}^{\infty} [\exp(-(\tilde{c_2} - \xi) k^d)]^{n_{\beta,k}} \exp(-\xi \, \text{dist}(\Lambda_1, \Lambda_2^C)). \qquad (110)
\end{aligned}
$$

If we take $\xi > 0$ small enough we can take out a factor $\exp(-\xi \, \text{dist}(\Lambda_1, \Lambda_2^C))$ and do the analysis with the remaining factor as before since $\tilde{c_2} - \xi > 0$. So we get

$$
\begin{aligned}
&\|\pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_2} \circ T^{\Lambda_2,\epsilon}} \circ \pi_{\Lambda_2} - \pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_3} \circ T^{\Lambda_3,\epsilon}} \circ \pi_{\Lambda_3}\|_{L((\mathcal{H}_\vartheta, \|\cdot\|_\vartheta),(\mathcal{H}_{\Lambda_1,\vartheta}, \|\cdot\|_{\Lambda_1,\vartheta}))} \\
&\quad \leq c_{27} \exp(-\xi \, \text{dist}(\Lambda_1, \Lambda_2^C)) \qquad (111)
\end{aligned}
$$

with some constant $c_{27}$ and the limit in (7) exists for $N \geq N_0$. The proof for the case $N < N_0$ is similar. We use the modified estimates that we get by replacing in (97) and (105) $\vartheta$ by a sufficiently small $\tilde{\vartheta}$. For example, (97) and (103) become

$$\tilde{\vartheta}^{|\Lambda_1|} \sum_{\mathcal{C}:\text{length}(\mathcal{C})=N} \|\pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}} \phi_{\Lambda_2}\|_{\Lambda_1} \leq c_{28} \left(\epsilon_2 + c_r \tilde{\eta}^N \frac{\tilde{\vartheta}}{\vartheta} + c_h \tilde{\vartheta}\right)^{|\Lambda_1|} \|\phi\|_{\Lambda_2,\vartheta} \tilde{\eta}^N \tag{112}$$

and the term in parentheses is smaller than one if $\tilde{\vartheta}$ and $\tilde{\vartheta}/\vartheta$ are small enough. The statement for systems with finite-range interaction follows from the fact that in that case all limits are already attained for some sufficiently large $\Lambda_2 \in \mathcal{F}$ and that all considered sums are finite.

For the proof of Theorem 2.1(2) we use results from Proposition 6.1 that we prove below. By (7) the operators $\mathcal{L}^N_{F\circ T^\epsilon} \in L(\mathcal{H}_\vartheta, \|\cdot\|_\vartheta)$ are well defined for $N \geq N_0$ and, by Proposition 6.1(2), give rise to a Cauchy sequence. With the same argument we see that the infinite sum in the definition of $\nu_\Lambda$ (cf. (59)) converges and $\nu \in \mathcal{H}_\vartheta$. $\nu \geq 0$ and so $\nu \in \mathcal{H}^{bv}$ follow from of Proposition 6.1(6).

The difference in (9) is only due to configurations of length $\geq N$ and we estimate it, using Proposition 6.1(2), by $c_3 \tilde{\eta}^N$. So $\nu = \lim_{N\to\infty} \mathcal{L}^N_{F\circ T^\epsilon} h$ and by (3) and (4) of Proposition 6.1, $\mathcal{L}_{F\circ T^\epsilon} \nu = \nu$ and $\mu(\nu) = 1$, respectively. For any $\phi \in \mathcal{H}_\vartheta$ with $\mathcal{L}_{F\circ T^\epsilon}\phi = \phi$ and $\mu(\phi) = 1$ we have by (9)

$$\phi = \lim_{N\to\infty} \mathcal{L}^N_{F\circ T^\epsilon} \phi = \mu(\phi)\nu = \nu. \tag{113}$$

This shows uniqueness of $\nu$ and so of $\nu^*$ and the proof of Theorem 2.1(2) is complete. □

*Proof of Proposition 6.1.* Using the same argument as in the proof of Theorem 2.1(1), we see that the right-hand side term in (60) differs from the operator in (49) only in the summands for $\mathcal{C}$ with $b(\mathcal{C}) \geq \text{dist}(\Lambda_1, \Lambda_2^C)$. So the difference is bounded by $c_{29} \exp(-\xi \,\text{dist}(\Lambda_1, \Lambda_2^C))$ for some $c_{29} > 0$ and (60) follows from taking the limit $\Lambda_2 \to \mathbb{Z}^d$.

In order to prove Proposition 6.1(2) we first observe that configurations $\mathcal{C} \in E_N(\Lambda_1)$ of length $\leq N-1$ extend canonically to $\mathcal{C}' \in E_{N+1}(\Lambda_1)$ with $\mathcal{L}_{\mathcal{C}} = \mathcal{L}_{\mathcal{C}'}$ because there are only h-lines in the step from time $-N$ to $-N+1$. So we can extend $\mathcal{C}$ to $\mathcal{C}'$ on $\Lambda_2 \times \{-N-1, \ldots, 0\}$ (where $\Lambda_2$ is so large that $\Lambda_2 \times \{-N-1, \ldots, 0\}$ contains all triangles of $\mathcal{C}$) by adding h-lines from $(p, -N-1)$ to $(p, -N)$ for all $p \in \Lambda_2$ and obviously $\mathcal{L}_{\mathcal{C}} = \mathcal{L}_{\mathcal{C}'}$.

Note that a configuration $\mathcal{C}'$ in $\Lambda_2 \times \{-N-1, \ldots, 0\}$ of length $\leq N-1$ is the extension in the above sense of a (uniquely defined) $\mathcal{C}$. So in the difference (61), all terms $\mathcal{L}_{\mathcal{C}}$ with length$(\mathcal{C}) \leq N-1$ are cancelled. Using Proposition 8.1(1), (107) and Proposition 6.1(1) we get for all $\Lambda_1 \in \mathcal{F}$

$$\begin{aligned}\|(\pi_{\Lambda_1} \circ \mathcal{L}^N_{F\circ T^\epsilon} - \pi_{\Lambda_1} \circ \mathcal{L}^{N+1}_{F\circ T^\epsilon})\phi\|_{\Lambda_1,\vartheta} &\leq (c_{19}\tilde{\eta}^N + c_{20}\tilde{\eta}^N + c_{19}\tilde{\eta}^{N+1})\|\phi\|_\vartheta \\ &\leq c_{30}\tilde{\eta}^N \|\phi\|_\vartheta \end{aligned} \tag{114}$$

with $c_{30}$ independent of $\Lambda_1$. This proves Proposition 6.1(2); next we prove Proposition 6.1(3). For $\Lambda_1 \in \mathcal{F}$,

$$\begin{aligned}
\pi_{\Lambda_1} \circ \mathcal{L}^{N_2}_{F\circ T^\epsilon} \circ \mathcal{L}^{N_1}_{F\circ T^\epsilon}\phi &= \sum_{\mathcal{C}_2 \in E_{N_2}(\Lambda_1)} \pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}_2}(\mathcal{L}^{N_1}_{F\circ T^\epsilon}\phi) \\
&= \sum_{\mathcal{C}_2 \in E_{N_2}(\Lambda_1)} \left( \pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}_2} \circ \sum_{\mathcal{C}_1 \in E_{N_2}(\Lambda(\mathcal{C}_2))} \pi_{\Lambda(\mathcal{C}_2)} \circ \mathcal{L}_{\mathcal{C}_1}\phi_{\Lambda(\mathcal{C}_1)} \right) \\
&= \sum_{\substack{\mathcal{C}_2 \in E_{N_2}(\Lambda_1) \\ \mathcal{C}_1 \in E_{N_2}(\Lambda(\mathcal{C}_2))}} \pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}_2 \circ \mathcal{C}_1}\phi_{\Lambda(\mathcal{C}_1)} \\
&= \sum_{\mathcal{C}_3 \in E_{N_1+N_2}(\Lambda_1)} \pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}_3}\phi_{\Lambda(\mathcal{C}_3)} \\
&= \pi_{\Lambda_1} \circ \mathcal{L}^{N_1+N_2}_{F\circ T^\epsilon}\phi. \qquad (115)
\end{aligned}$$

Note that we sum over infinitely many $\mathcal{C}_1, \mathcal{C}_2$. *A priori*, the distribution is only valid for finite partial sums. In terms of configurations we 'put $\mathcal{C}_1$ on $\mathcal{C}_2$' to get $\mathcal{C}_3 = \mathcal{C}_2 \circ \mathcal{C}_1$ (which might be a zero configuration), in fact such a splitting exists and is unique for every non-zero $\mathcal{C}_3$. So the net of finite partial sums over $\mathcal{C}_3$ converges to the infinite expansion (60) of the right-hand side of (62) and Proposition 6.1(3) is proved.

To prove (64), we consider first the special case $g \in \mathcal{C}((S^1)^\Lambda)$:

$$\begin{aligned}
\int_M d\mu\, g \circ S\,\phi &= \lim_{\Lambda_1 \to \mathbb{Z}^d} \int_M d\mu\, g \circ S_{\Lambda_1}\phi \\
&= \lim_{\Lambda_1 \to \mathbb{Z}^d} \int_{(S^1)^{\Lambda_1}} d\mu^{\Lambda_1} g \circ S_{\Lambda_1}\phi_{\Lambda_1} \\
&= \lim_{\Lambda_1 \to \mathbb{Z}^d} \int_{(S^1)^{\Lambda_1}} d\mu^{\Lambda_1} g\, \mathcal{L}_{F^{\Lambda_1}\circ T^{\Lambda_1,\epsilon}}\phi_{\Lambda_1} \\
&= \lim_{\Lambda_1 \to \mathbb{Z}^d} \int_{(S^1)^{\Lambda}} d\mu^{\Lambda} g\, \pi_\Lambda \circ \mathcal{L}_{F^{\Lambda_1}\circ T^{\Lambda_1,\epsilon}} \circ \pi_{\Lambda_1}\phi \\
&= \int_M d\mu\, g\mathcal{L}_{F\circ T^\epsilon}\phi. \qquad (116)
\end{aligned}$$

So (64) is true for $g \in \mathcal{C}((S^1)^\Lambda)$. Taking $g \equiv 1$, we get (65).

Now we show (63), using the special case of (64) for the second equality:

$$\begin{aligned}
\|\mathcal{L}_{F\circ T^\epsilon}\phi\|_{\mathrm{var}} &= \sup_{\Lambda \in \mathcal{F}} \sup_{\substack{g \in \mathcal{C}((S^1)^\Lambda) \\ \|g\|_\infty \le 1}} \int_M d\mu\, g\mathcal{L}_{F\circ T^\epsilon}\phi \\
&= \sup_{\Lambda \in \mathcal{F}} \sup_{\substack{g \in \mathcal{C}((S^1)^\Lambda) \\ \|g\|_\infty \le 1}} \int_M d\mu\, g \circ S\phi \\
&\le \sup_{\Lambda \in \mathcal{F}} \sup_{\substack{g \in \mathcal{C}((S^1)^\Lambda) \\ \|g\|_\infty \le 1}} \|g\|_\infty \|\phi\|_{\mathrm{var}} \\
&= \|\phi\|_{\mathrm{var}}. \qquad (117)
\end{aligned}$$

We can conclude (64) for any $g \in \mathcal{C}(M)$. By assumption $\phi$ and then by (63) $\mathcal{L}_{F\circ T^\epsilon}\phi$ are in $\mathcal{H}^{bv}$, i.e. the integrals in (64) correspond to continuous linear functionals on $\mathcal{C}(M)$. The net $(g_\Lambda)_{\Lambda\in\mathcal{F}}$ converges uniformly to $g$ as $\Lambda \to \mathbb{Z}^d$, as does $(g_\Lambda \circ S)_{\Lambda\in\mathcal{F}}$ to $g \circ S$, so (64) follows by uniform approximation of $g$ by functions $g_\Lambda$ and Proposition 6.1(4) is proved.

We show Proposition 6.1(5) by indirect proof. We have, by definition, $(\mathcal{L}_{F\circ T^\epsilon}\phi)_\Lambda \stackrel{\text{def}}{=} \lim_{\Lambda_1\to\mathbb{Z}^d} \pi_\Lambda \circ \mathcal{L}_{F^{\Lambda_1}\circ T^{\Lambda_1,\epsilon}}\phi_{\Lambda_1}$. If that was negative somewhere there would be a $\Lambda_1 \in \mathcal{F}$ with $\pi_\Lambda \circ \mathcal{L}_{F^{\Lambda_1}\circ T^{\Lambda_1,\epsilon}}\phi_{\Lambda_1}$ having negative values and we could find a non-negative $g \in \mathcal{C}((S^1)^\Lambda)$ such that

$$\int_{(S^1)^\Lambda} d\mu^\Lambda g\, \pi_\Lambda \circ \mathcal{L}_{F^{\Lambda_1}\circ T^{\Lambda_1,\epsilon}}\phi_{\Lambda_1} < 0 \tag{118}$$

However, by Proposition 6.1(4) the integral equals

$$\int_{(S^1)^{\Lambda_1}} d\mu^{\Lambda_1} g \circ S\, \phi_{\Lambda_1} \geq 0. \tag{119}$$

So $\mathcal{L}_{F\circ T^\epsilon}$ is non-negative. □

*Proof of Theorem 7.1.*

$$\begin{aligned}
\nu_{\Lambda_1\cup\Lambda_2} &= \sum_{\mathcal{C}\in E(\Lambda_1\cup\Lambda_2)} \pi_{\Lambda_1\cup\Lambda_2} \circ \mathcal{L}_{\mathcal{C}} h \\
&= \sum_{\substack{\mathcal{C}=\mathcal{C}_1\cup\mathcal{C}_2 \\ b(\mathcal{C})\leq\frac{1}{2}\mathrm{dist}(\Lambda_1,\Lambda_2)}} (\pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}_1} h)(\pi_{\Lambda_2} \circ \mathcal{L}_{\mathcal{C}_2} h) \\
&\quad + \sum_{\substack{\mathcal{C} \\ b(\mathcal{C})>\frac{1}{2}\mathrm{dist}(\Lambda_1,\Lambda_2)}} \pi_{\Lambda_1\cup\Lambda_2} \circ \mathcal{L}_{\mathcal{C}} h.
\end{aligned} \tag{120}$$

In estimating the second summand we note that if we sum in formulae (97) and (105) just over $\mathcal{C}$ for which $b(\mathcal{C}) \geq \frac{1}{2}\mathrm{dist}(\Lambda_1, \Lambda_2)$ ($b(\mathcal{C})$ was defined in (109)), we can take out from $\prod_{k=1}^\infty (\exp(-\tilde{c_2} k^d))^{n_{\beta,k}}$ a factor $\exp(-\xi \frac{1}{2}\mathrm{dist}(\Lambda_1, \Lambda_2))$ (similar to the proof of Proposition 6.1). We do so by choosing a $\kappa \in (0, 1)$ so that

$$\tilde{c_2} + \ln\kappa = c_2 - c_{21} - c_{22} - c_{23} + 3^d \ln\vartheta + \ln\kappa > 0 \tag{121}$$

and by defining $\xi$ by $\exp(-\xi\frac{1}{2}) = \kappa$. Note that such a choice exists as $\tilde{c_2} > 0$ by (100).

The rest of the analysis is as in the proof of Proposition 8.1. We get

$$\left\| \sum_{\substack{\mathcal{C} \\ b(\mathcal{C})>\frac{1}{2}\mathrm{dist}(\Lambda_1,\Lambda_2)}} \pi_{\Lambda_1\cup\Lambda_2} \circ \mathcal{L}_{\mathcal{C}} h \right\|_{\Lambda_1\cup\Lambda_2} \tag{122}$$

$$\begin{aligned}
&\leq \kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)} c_{31} \|h\|_\vartheta \vartheta^{-|\Lambda_1|-|\Lambda_2|} \\
&\leq c_{32} \vartheta^{-|\Lambda_1|-|\Lambda_2|} \kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)}.
\end{aligned} \tag{123}$$

We write for the first summand in (120)

$$\sum_{\substack{\mathcal{C}=\mathcal{C}_1\cup\mathcal{C}_2\\ b(\mathcal{C})\le\frac{1}{2}\mathrm{dist}(\Lambda_1,\Lambda_2)}} (\pi_{\Lambda_1}\circ\mathcal{L}_{\mathcal{C}_1}h)(\pi_{\Lambda_2}\circ\mathcal{L}_{\mathcal{C}_2}h) = \nu_{\Lambda_1}\nu_{\Lambda_2} - \sum_{\substack{\mathcal{C}=\mathcal{C}_1\cup\mathcal{C}_2\\ b(\mathcal{C})>\frac{1}{2}\mathrm{dist}(\Lambda_1,\Lambda_2)}} (\pi_{\Lambda_1}\circ\mathcal{L}_{\mathcal{C}_1}h)(\pi_{\Lambda_2}\circ\mathcal{L}_{\mathcal{C}_2}h) \tag{124}$$

and estimate, in a similar way,

$$\left\| \sum_{\substack{\mathcal{C}=\mathcal{C}_1\cup\mathcal{C}_2\\ b(\mathcal{C})>\frac{1}{2}\mathrm{dist}(\Lambda_1,\Lambda_2)}} (\pi_{\Lambda_1}\circ\mathcal{L}_{\mathcal{C}_1}h)(\pi_{\Lambda_2}\circ\mathcal{L}_{\mathcal{C}_2}h) \right\|_{\Lambda_1\cup\Lambda_2} \le c_{33}\vartheta^{-|\Lambda_1|-|\Lambda_2|}\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)}. \tag{125}$$

Equations (123) and (125) also hold for all $\Lambda_1' \subseteq \Lambda_1$, $\Lambda_2' \subseteq \Lambda_2$ and Theorem 7.1(1) follows:

$$\begin{aligned}\pi_{\Lambda_1}(f\nu) &= \pi_{\Lambda_1}(f\nu_{\Lambda_1\cup\Lambda_2})\\ &= \pi_{\Lambda_1}(f\nu_{\Lambda_1}\nu_{\Lambda_2} - f(\nu_{\Lambda_1}\nu_{\Lambda_2} - \nu_{\Lambda_1\cup\Lambda_2}))\\ &= \nu(f)\nu_{\Lambda_1} - \pi_{\Lambda_1}(f(\nu_{\Lambda_1}\nu_{\Lambda_2} - \nu_{\Lambda_1\cup\Lambda_2}))\end{aligned} \tag{126}$$

and, using $\|\pi_{\Lambda_1}\|_\infty = 1$, we get

$$\|\pi_{\Lambda_1}(f(\nu_{\Lambda_1}\nu_{\Lambda_2} - \nu_{\Lambda_1\cup\Lambda_2}))\|_{\Lambda_1} \le \|f\|_{\Lambda_2}\|\nu_{\Lambda_1}\nu_{\Lambda_2} - \nu_{\Lambda_1\cup\Lambda_2}\|_{\Lambda_1\cup\Lambda_2} \tag{127}$$

and so by Theorem 7.1(1)

$$\|\pi_{\Lambda_1}(f(\nu_{\Lambda_1}\nu_{\Lambda_2} - \nu_{\Lambda_1\cup\Lambda_2}))\|_{\Lambda_1} \le c_{16}\vartheta^{-|\Lambda_1|-|\Lambda_2|}\|f\|_{\Lambda_2}\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)}. \tag{128}$$

This holds for all $\Lambda_1' \subset \Lambda_1$, so Theorem 7.1(2) is proved.

We set $\phi = f\nu - \nu(f)\nu$. So $\pi_{\Lambda_1}\circ\mathcal{L}^N_{F\circ T^\epsilon}(f\nu) - \nu(f)\nu_{\Lambda_1} = \pi_{\Lambda_1}\circ\mathcal{L}^N_{F\circ T^\epsilon}\phi$. We estimate the $\|\cdot\|_{\Lambda_1,\tilde{\vartheta}}$-norm of the last term as in the proof of Proposition 8.1, but this time using the finer estimates from Theorem 7.1(2)

$$\begin{aligned}\|\phi_{\Lambda(\mathcal{C})}\|_{\Lambda(\mathcal{C})} &\le \vartheta^{-|\Lambda(\mathcal{C})|}c_{11}\vartheta^{-|\Lambda_2|}\|f\|_{\Lambda_2}\kappa^{\mathrm{dist}(\Lambda(\mathcal{C}),\Lambda_2)}\\ &\le c_{11}\vartheta^{-|\Lambda_2|}\|f\|_{\Lambda_2}\vartheta^{-|\Lambda_r(\mathcal{C})|-\sum_{k=1}^\infty(3k)^d n_{\beta,k}}\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)-\sum_{k=1}^\infty kn_{\beta,k}}\end{aligned} \tag{129}$$

where as before $\Lambda(\mathcal{C}) \overset{\mathrm{def}}{=} \tilde{\Lambda}_{\mathcal{C}} \cup \Lambda_r$. So we get analogously to formulae (97) and (98):

$$\begin{aligned}&\tilde{\vartheta}^{|\Lambda_1|}\sum_{\mathcal{C}:\mathrm{length}(\mathcal{C})=N}\|\pi_{\Lambda_1}\circ\mathcal{L}_{\mathcal{C}}\phi_{\Lambda_2}\|_{\Lambda_1}\\ &\quad\le \tilde{\vartheta}^{|\Lambda_1|}\sum_{K=0}^{|\Lambda_1|}\binom{|\Lambda_1|}{K}\sum_{\substack{n_\beta\\ K\le|n_\beta|<\infty}} 4^K\prod_{k=1}^\infty(\exp(c_{21}k^d))^{n_{\beta,k}}(c_1\epsilon)^{|n_\beta|}\\ &\quad\times\prod_{k=1}^\infty(\exp(-c_2k^d))^{n_{\beta,k}}\sum_{L=|n_\beta|}^\infty\binom{L}{|n_\beta|}c_r^{|n_\beta|}\eta^L\prod_{k=1}^\infty(\exp(c_{22}k^d))^{n_{\beta,k}}\\ &\quad\times\prod_{k=1}^\infty(\exp(c_{23}k^d))^{n_{\beta,k}}\eta^{\max\{0,N-L\}}\sum_{l=0}^{|\Lambda_1|-K}\binom{|\Lambda_1|-K}{l}(c_r\eta^N)^l c_h^{|\Lambda_1|-K-l}\\ &\quad\times c_{11}\vartheta^{-|\Lambda_2|}\|f\|_{\Lambda_2}\vartheta^{-l-\sum_{k=1}^\infty(3k)^d n_{\beta,k}}\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)-\sum_{k=1}^\infty kn_{\beta,k}}\end{aligned}$$

$$\leq c_{11}\tilde{\vartheta}^{|\Lambda_1|}\sum_{K=0}^{|\Lambda_1|}\binom{|\Lambda_1|}{K}\sum_{\substack{n_\beta\\ K\leq|n_\beta|<\infty}}4^K(c_1\epsilon c_r)^{|n_\beta|}$$
$$\times\prod_{k=1}^{\infty}(\exp((c_{21}-c_2+c_{22}+c_{23}-3^d\ln\vartheta-\ln\kappa)k^d))^{n_{\beta,k}}$$
$$\times\sum_{L=|n_\beta|}^{\infty}\binom{|\Lambda_1|}{K}\eta^{\max\{L,N\}}(\vartheta^{-1}c_r\eta^N+c_h)^{|\Lambda_1|-K}\vartheta^{-|\Lambda_2|}\|f\|_{\Lambda_2}\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)}. \tag{130}$$

Using (121), we get with the same analysis as (98)–(103):

$$\leq c_{34}\left(\epsilon_2+c_r\tilde{\eta}^N\frac{\tilde{\vartheta}}{\vartheta}+c_h\tilde{\vartheta}\right)^{|\Lambda_1|}\vartheta^{-|\Lambda_2|}\|f\|_{\Lambda_2}\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)}\tilde{\eta}^N. \tag{131}$$

For sufficiently small $\epsilon_2$ and $\tilde{\vartheta}$ the term in brackets is smaller than one. Note that there is no condition on $N$. So we get the same estimates for all $n\geq 0$ and these also hold for $\Lambda\subset\Lambda_1$. So in analogy with (61) we get

$$\|\mathcal{L}^N_{F\circ T^\epsilon}\phi-\mathcal{L}^{N+1}_{F\circ T^\epsilon}\phi\|_{\Lambda_1}\leq c_{35}\vartheta^{-|\Lambda_2|}\|f\|_{\Lambda_2}\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)}\tilde{\eta}^N \tag{132}$$

and as $\mu(\phi)=0$ we conclude Theorem 7.1(3). □

*Proof of Theorem 2.2.* Applying Theorem 7.1(1) we get

$$\begin{aligned}&\left|\int_M \nu\,d\mu\,gf-\left(\int_M\nu\,d\mu\,g\right)\left(\int_M\nu\,d\mu\,f\right)\right|\\&\quad\leq\left|\int_{(S^1)^{\Lambda_1\cup\Lambda_2}}d\mu^{\Lambda_1\cup\Lambda_2}(\nu_{\Lambda_1\cup\Lambda_2}-\nu_{\Lambda_1}\nu_{\Lambda_2})gf\right|\\&\quad\leq\|\nu_{\Lambda_1\cup\Lambda_2}-\nu_{\Lambda_1}\nu_{\Lambda_2}\|_{\Lambda_1\cup\Lambda_2}\|g\|_\infty\|f\|_\infty\\&\quad\leq c_{10}\vartheta^{-|\Lambda_1|-|\Lambda_2|}\|g\|_\infty\|f\|_\infty\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)},\end{aligned} \tag{133}$$

so Theorem 2.2(1) is proved.

We have

$$\begin{aligned}&\left|\int_M\nu\,d\mu\,g\circ\tau\circ S^nf-\left(\int_M\nu\,d\mu\,g\circ\tau\right)\left(\int_M\nu\,d\mu\,f\right)\right|\\&\quad=\left|\int_M d\mu\,g\circ\tau(\pi_{\tau^{-1}(\Lambda_1)}\circ\mathcal{L}^n_{F\circ T^\epsilon}(f\nu)-\nu(f)\nu_{\tau^{-1}(\Lambda_1)})\right|\\&\quad\leq c_{12}c_5^{|\Lambda_1|+|\Lambda_2|}\|f\|_{\Lambda_2}\|g\|_\infty\kappa^{\mathrm{dist}(\tau^{-1}(\Lambda_1),\Lambda_2)}\tilde{\eta}^n.\end{aligned} \tag{134}$$

Here we have used Theorem 7.1(3) and set $c_5\overset{\mathrm{def}}{=}\tilde{\vartheta}^{-1}$. From

$$\mathrm{dist}(\tau^{-1}(\Lambda_1),\Lambda_2)\geq m(\tau)-\max\{\|p-q\|:p\in\Lambda_1,q\in\Lambda_2\} \tag{135}$$

follows

$$\kappa^{\mathrm{dist}(\tau^{-1}(\Lambda_1),\Lambda_2)}\leq c(\Lambda_1,\Lambda_2,\kappa)\kappa^{m(\tau)} \tag{136}$$

where $c(\Lambda_1, \Lambda_2, \kappa)$ is as defined in Theorem 2.2. If $\tau$ and $S$ commute, Theorem 2.2(3) follows from Theorem 2.2(2).

We prove Theorem 2.2(4) by approximating $g$ and $f$ by functions for which we can apply estimate Theorem 2.2(2). For any $\gamma > 0$ we can choose $\Lambda_1 \in \mathcal{F}$ so large that $\|g - g_{\Lambda_1}\|_\infty \leq \gamma$. Furthermore, there exists an $\tilde{f}_{\Lambda_2} \in \mathcal{H}(A_\delta^{\Lambda_2})$ with $\|f - \tilde{f}_{\Lambda_2}\|_\infty \leq \gamma$ (sup-norm on $(S^1)^{\mathbb{Z}^d}$). So

$$\begin{aligned}
&\left| \int_M \nu\, d\mu\, g \circ \tau \circ S^n f - \left( \int_M \nu\, d\mu\, g \circ \tau \right) \left( \int_M \nu\, d\mu\, f \right) \right| \\
&\quad \leq \left| \int_M \nu\, d\mu\, (g - g_{\Lambda_1}) \circ \tau \circ S^n f \right| + \left| \int_M \nu\, d\mu\, g_{\Lambda_1} \circ \tau \circ S^n\, (\tilde{f}_{\Lambda_2} - f) \right| \\
&\qquad + \left| \int_M \nu\, d\mu\, g_{\Lambda_1} \circ \tau \circ S^n \tilde{f}_{\Lambda_2} - \left( \int_M \nu\, d\mu\, g_{\Lambda_1} \circ \tau \right) \left( \int_M \nu\, d\mu\, \tilde{f}_{\Lambda_2} \right) \right| \\
&\qquad + \left| \left( \int_M \nu\, d\mu\, g_{\Lambda_1} \circ \tau \right) \left( \int_M \nu\, d\mu\, (f - \tilde{f}_{\Lambda_2}) \right) \right| \\
&\qquad + \left| \left( \int_M \nu\, d\mu\, (g - g_{\Lambda_1}) \circ \tau \right) \left( \int_M \nu\, d\mu\, f \right) \right| \\
&\quad \leq \|g - g_{\Lambda_1}\|_\infty \|f\|_\infty + \|g_{\Lambda_1}\|_\infty \|f - \tilde{f}_{\Lambda_2}\|_\infty \\
&\qquad + c(\Lambda_1, \Lambda_2, \kappa) c_5^{|\Lambda_1| + |\Lambda_2|} \|g_{\Lambda_1}\|_\infty \|\tilde{f}_{\Lambda_2}\|_{\Lambda_2} \tilde{\eta}^{n(\sigma)} \kappa^{m(\sigma)} \\
&\qquad + \|g_{\Lambda_1}\|_\infty \|f - f_{\Lambda_2}\|_\infty + \|g - g_{\Lambda_1}\|_\infty \|f_{\Lambda_2}\|_\infty \\
&\quad \leq (2\|f\|_\infty + 2\|g\|_\infty + 3\gamma)\gamma \\
&\qquad + c(\Lambda_1, \Lambda_2, \kappa) c_5^{|\Lambda_1| + |\Lambda_2|} (\|g\|_\infty + \gamma) \|\tilde{f}_{\Lambda_2}\|_{\Lambda_2} \tilde{\eta}^{n(\sigma)} \kappa^{m(\sigma)}
\end{aligned} \tag{137}$$

and this gets arbitrarily small as we can first choose $\gamma$, and then (depending on $\gamma$) $\Lambda_1$, $\Lambda_2$ and $\tilde{f}_{\Lambda_2}$ and finally $\max\{m(\sigma), n(\sigma)\}$.

Theorem 2.2(5) follows from Theorem 2.2(4) and the commutation of the $\tau_{e_i}$ with $S$. □

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