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Positivity and Hierarchical Structure of Green's Functions of 2-Point Boundary Value Problems for Bending of a Beam*

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Green's functions to 2-point simple type self-adjoint boundary value problems for bending of a beam under relatively strong tension on an elastic foundation are studied. We have 9 different Green's functions. All are positive-valued and have a suitable hierarchical structure.

Key words: beam deflection, Green's function, positivity, hierarchical structure

1. Introduction

We treat several boundary value problems (BVP's) for a 4th-order linear ordinary differential equation (ODE) on a finite interval,

$$\mathrm{BVP}(m_0, m_1; n_1, n_2) : \begin{cases} \mathcal{L}u \equiv u^{(4)} - pu'' + qu = f(x) & (0 < x < L) \\ u^{(m_i)}(0) = \alpha_i, \ u^{(n_i)}(L) = \beta_i, & (i = 0, 1) \end{cases} \tag{1.1}$$

where $f(x)$ is a given function, α_0, α_1, β_0, β_1 are given constants, and coefficients p, q are positive constants. We can find the above equation, for example, in the field of elastics [4], where $u = u(x)$ represents deflection of a beam under tension $p > 0$, which is supported by uniformly distributed springs with spring constant $q > 0$ on a fixed floor, and $f(x)$ is a density of a load.

$m = (m_0, m_1)$ and $n = (n_0, n_1)$ take 6 different values $(0,1)$, $(0,2)$, $(0,3)$, $(1,2)$, $(1,3)$, $(2,3)$. Among them, we here treat only self-adjoint cases $m, n = (0,1)$, $(0,2)$, $(1,3)$, which also have engineering importance and correspond to clumped, simply-supported and sliding edges, respectively. Therefore, the following 9 kind of BVP's can be considered.

$$\begin{aligned}(m,n) = {} & (0,1,0,1), (0,1,0,2), (0,1,1,3), (0,2,0,1), (0,2,0,2), (0,2,1,3), \\ & (1,3,0,1), (1,3,0,2), (1,3,1,3)\end{aligned}$$

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However, throughout this paper, we often focus our attention only on 6 BVP's among them,

$$(m, n) = (0,1,0,1), (0,1,0,2), (0,1,1,3), (0,2,0,2), (0,2,1,3), (1,3,1,3),$$

taking account of the symmetry.

Even though the above BVP's originate from the age of Euler and Bernoulli, they include many unsolved problems. Since equation (1.1) is linear, its solution is written as

$$u(x) = \sum_{j=0}^{1} \alpha_j A_j(m, n, L; x) + \sum_{j=0}^{1} \beta_j B_j(m, n, L; x) + \int_0^L g(m, n; L, x, y) f(y) \mathrm{d}y, \tag{1.2}$$

where $A_0(m,n,L;x)$, $A_1(m,n,L;x)$, $B_0(m,n,L;x)$, $B_1(m,n,L;x)$ are fundamental solutions satisfying relations,

$$\begin{aligned} &A_i^{(m_j)}(m,n,L;0) = \delta_{i,j}, \ A_i^{(n_j)}(m,n,L;L) = 0, \\ &B_i^{(m_j)}(m,n,L;0) = 0, \ B_i^{(n_j)}(m,n,L;L) = \delta_{i,j} \quad (i,j = 0,1), \end{aligned} \tag{1.3}$$

and $g(m,n,L;x,y)$ is a Green's function. The purpose of the paper is to investigate positivity and mutual relations of fundamental solutions and Green's functions of the 9 different BVP's.

REMARK 1. Related eigenvalue problem

$$\begin{cases} u^{(4)} = \lambda u \quad (0 < x < L) \\ u''(0) = u'''(0) = u(L) = u'(L) = 0 \end{cases} \tag{1.4}$$

was treated by L. Euler. (See Ref. [1])

In this paper, we impose an inequality $(p/2)^2 > q > 0$, $p > 0$, in other words, a tension is relatively much stronger than a spring force. Due to this inequality, we see that the characteristic polynomial,

$$P(\lambda) = \lambda^4 - p\lambda^2 + q = (\lambda^2 - a^2)(\lambda^2 - b^2), \quad p = a^2 + b^2, \quad q = a^2 b^2,$$

has 4 roots $\lambda = \pm a, \ \pm b$. $(a > b > 0)$. Under the above assumptions, 4th-order differential operator $\mathcal{L}$ is decomposed into a product of 2 formal positive operators,

$$\mathcal{L} = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^4 - p\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + q = \left(-\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + a^2\right)\left(-\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + b^2\right), \tag{1.5}$$

which suggests positivity of Green's functions.

This paper is organized as follows. In Section 2, we derive fundamental solutions and Green's functions. In Section 3, it is shown that fundamental solutions are positive- (or negative-)valued. Concrete forms of Green's functions are given in Section 4. Section 5 is devoted to proving a hierarchical structure of Green's functions $g(0,2,0,2,L;x,y)$, $g(0,2,1,3,L;x,y)$, $g(1,3,1,3,L;x,y)$. In Section 6, we prove the positivity of the Green's function $g(0,1,0,1,L;x,y)$, which is the most difficult to prove. Section 7 presents the main theorem in this paper, which shows a hierarchical structure of Green's functions.

Throughout this paper, we use the notations,

$$\mathrm{ch}(x) = \cosh(x), \quad \mathrm{sh}(x) = \sinh(x). \tag{1.6}$$

For later convenience sake, we introduce the function $K_0(x)$ defined by

$$K_0(x) = \frac{1}{a^2 - b^2}(a^{-1}\,\mathrm{sh}(ax) - b^{-1}\,\mathrm{sh}(bx)), \tag{1.7}$$

which is a solution to the following Cauchy problem,

$$\begin{cases} u^{(4)} - (a^2+b^2)u'' + a^2b^2u = 0 & (0 < x < \infty), \\ u(0) = u'(0) = u''(0) = 0, \quad u'''(0) = 1. \end{cases} \tag{1.8}$$

Employing the function $K_0(x)$, we also put $K_j(x) = K_0^{(j)}(x)$ and $K_j = K_j(L)$ $(j = 0,1,2,\ldots)$.

2. Fundamental Solution and Green's Functions

We start with the uniqueness theorem of the solution to $\mathrm{BVP}(m_0,m_1;n_0,n_1)$.

THEOREM 2.1 (Uniqueness). *Let $f(x)$ be a complex-valued continuous function on $[0,L]$ and $\{\alpha_0,\alpha_1,\beta_0,\beta_1\}$ be 4 complex numbers. Then, for an arbitrary set of data $\{f(x);\alpha_0,\alpha_1,\beta_0,\beta_1\}$, a 4 times continuously differentiable solution to $\mathrm{BVP}(m_0,m_1;n_0,n_1)$ is unique.*

Proof. It is enough to show a classical solution $u(x)$, if it exists, is expressed as (1.2). By putting $u_i(x) \equiv u^{(i)}(x)$ $(0 \le i \le 3)$, $f_3(x) \equiv f(x)$, (1.1) is rewritten as follows;

$$\boldsymbol{u}'(x) = A\boldsymbol{u}(x) + \boldsymbol{f}(x) \tag{2.1}$$

$$u_{m_i}(0) = \alpha_i, \quad u_{n_i}(L) = \beta_i \quad (i = 0,1), \tag{2.2}$$

where

$$\boldsymbol{u}(x) = {}^t(u_0,u_1,u_2,u_3)(x), \quad \boldsymbol{f}(x) = {}^t(0,0,0,f_3(x)), \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -q & 0 & p & 0 \end{pmatrix}.$$

A solution to (2.1) is written as follows;

$$\boldsymbol{u}(x) = E(x)\boldsymbol{u}(0) + \int_0^x E(x-y)\boldsymbol{f}(y)\mathrm{d}y. \tag{2.3}$$

The fundamental solution $E(x)$ to the initial value problem is expressed as follows;

$$E(x) = K(x)K(0)^{-1}, \tag{2.4}$$

where

$$K(x) = \Big(\, K_{i+j}(x) \, \Big), \quad K(0) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & p \\ 1 & 0 & p & 0 \end{pmatrix}. \tag{2.5}$$

Substitution of (2.4) into (2.3) gives

$$u_i(x) = (\cdots K_{i+j}(x) \cdots)K(0)^{-1}\boldsymbol{u}(0) + \int_0^x K_i(x-y)f_3(y)\mathrm{d}y, \qquad (0 \le i \le 3) \tag{2.6}$$

From the boundary conditions (2.2), we have

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \cdots & K_{m_0+j}(0) & \cdots \\ \cdots & K_{m_1+j}(0) & \cdots \\ \cdots & K_{n_0+j}(L) & \cdots \\ \cdots & K_{n_1+j}(L) & \cdots \end{pmatrix} K(0)^{-1}\boldsymbol{u}(0) + \int_0^L \begin{pmatrix} 0 \\ 0 \\ K_{n_0}(L-y) \\ K_{n_1}(L-y) \end{pmatrix} f_3(y)\mathrm{d}y. \tag{2.7}$$

Since we have

$$\det \begin{pmatrix} \cdots & K_{m_0+j}(0) & \cdots \\ \cdots & K_{m_1+j}(0) & \cdots \\ \cdots & K_{n_0+j}(L) & \cdots \\ \cdots & K_{n_1+j}(L) & \cdots \end{pmatrix} = \begin{cases} K_1^2 - K_0K_2 & (m,n) = (0,1,0,1), \\ K_1K_2 - K_0K_3 & (m,n) = (0,1,0,2), \\ K_2K_3 - K_1K_4 & (m,n) = (0,1,1,3), \\ -(K_1K_2 - K_0K_3) & (m,n) = (0,2,0,1), \\ -(K_2^2 - K_0K_4) & (m,n) = (0,2,0,2), \\ -(K_3^2 - K_1K_5) & (m,n) = (0,2,1,3), \\ -(K_2K_3 - K_1K_4) & (m,n) = (1,3,0,1), \\ -(K_3^2 - K_1K_5) & (m,n) = (1,3,0,2), \\ -(K_4^2 - K_2K_6) & (m,n) = (1,3,1,3), \end{cases}$$

which will be shown to be nonzero in Lemma 3.1, the above matrix possesses an inverse matrix for any choice of self-adjoint boundary condition (m, n). Eliminating

$K(0)^{-1}\boldsymbol{u}(0)$ from (2.6) and (2.7), we obtain the solution,

$$u_0(x) = (\cdots K_j(x) \cdots) \begin{pmatrix} \cdots & K_{m_0+j}(0) & \cdots \\ \cdots & K_{m_1+j}(0) & \cdots \\ \cdots & K_{n_0+j}(L) & \cdots \\ \cdots & K_{n_1+j}(L) & \cdots \end{pmatrix}^{-1}$$

$$\left\{ \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \\ \beta_1 \end{pmatrix} - \int_0^L \begin{pmatrix} 0 \\ 0 \\ K_{n_0}(L-y) \\ K_{n_1}(L-y) \end{pmatrix} f_3(y)\mathrm{d}y \right\} + \int_0^x K_0(x-y) f_3(y)\mathrm{d}y. \tag{2.8}$$

Introducing fundamental solutions defined by

$$\begin{aligned} &(A_0(m,n,L;x), A_1(m,n,L;x), B_0(m,n,L;x), B_1(m,n,L;x)) \\ &\equiv (\cdots K_j(x) \cdots) \begin{pmatrix} \cdots & K_{m_0+j}(0) & \cdots \\ \cdots & K_{m_1+j}(0) & \cdots \\ \cdots & K_{n_0+j}(L) & \cdots \\ \cdots & K_{n_1+j}(L) & \cdots \end{pmatrix}^{-1}, \end{aligned} \tag{2.9}$$

we have

$$u_0(x) = (A_0(m,n,L;x), A_1(m,n,L;x), B_0(m,n,L;x), B_1(m,n,L;x))$$

$$\left\{ \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \\ \beta_1 \end{pmatrix} - \int_0^L \begin{pmatrix} 0 \\ 0 \\ K_{n_0}(L-y) \\ K_{n_1}(L-y) \end{pmatrix} f_3(y)\mathrm{d}y \right\} + \int_0^L Y(x-y)K_0(x-y) f_3(y)\mathrm{d}y, \tag{2.10}$$

where $Y(x) = 1$ $(x \geq 0)$, 0 $(x < 0)$ is the Heaviside step function. We now obtained the solution formula,

$$u_0(x) = \sum_{j=0}^{1} \alpha_j A_j(m,n,L;x) + \sum_{j=0}^{1} \beta_j B_j(m,n,L;x) + \int_0^L g(m,n,L;x,y) f_3(y)\mathrm{d}y, \tag{2.11}$$

$$g(m,n,L;x,y) = Y(x-y)K_0(x-y) - \sum_{j=0}^{1} B_j(m,n,L;x)K_{n_j}(L-y). \tag{2.12}$$

By putting $u_0(x)$, $f_3(x)$ as $u(x)$, $f(x)$ again, it is concluded that a classical solution $u(x)$ to BVP(m,n), if it exists, is expressed as

$$u(x) = \sum_{j=0}^{1} \alpha_j A_j(m,n,L;x) + \sum_{j=0}^{1} \beta_j B_j(m,n,L;x) + \int_0^L g(m,n,L;x,y) f(y)\mathrm{d}y. \tag{2.13}$$

Since the right hand side of the above equation depends only on a set of data $\{f(x); \alpha_0, \alpha_1, \beta_0, \beta_1\}$, it is obvious that a classical solution is unique. ■

On the other hand, a straightforward caluculation shows that $u(x)$ defined by (2.13) gives a classical solution to BVP(m,n).

THEOREM 2.2 (Existence of solution). *Under the same assumption as Theorem* 2.1, $u(x)$ *defined by* (1.2) *is a classical solution to* BVP$(m_0, m_1; n_0, n_1)$ *for an arbitrary set of data* $\{f(x); \alpha_0, \alpha_1, \beta_0, \beta_1\}$.

3. Fundamental Solutions and Their Positivity

The purpose of this section is to prove that the fundamental solutions obtained in the previous section are positive- or negative-valued. To this end, we first expand them by means of $K_i(x)$ $(0 \le i \le 3)$. For later convenience sake, given an arbitrary function $u(x)$, let us rewrite its boundary data of type $(m,n) = (m_0, m_1, n_0, n_1)$ as

$$BD(m,n,L;u(x)) = (u^{(m_0)}(0), u^{(m_1)}(0), u^{(n_0)}(L), u^{(n_1)}(L))$$

If $u(x)$ solves a homogeneous ODE, $u^{(4)} - pu'' + qu = 0$, one can find from the uniqueness theorem that it is expanded as

$$u(x) = \sum_{j=0}^{1} u^{(m_j)}(0) A_j(m,n,L;x) + \sum_{j=0}^{1} u^{(n_j)}(L) B_j(m,n,L;x). \tag{3.1}$$

Since $u(x) = K_i(x)$ solves the same ODE, we have

$$BD(m,n,L;K_i(x)) = (K_{i+m_0}(0), K_{i+m_1}(0), K_{i+n_0}(L), K_{i+n_1}(L)),$$

or, in other words,

$$K_i(x) = \sum_{j=0}^{1} K_{i+m_j}(0) A_j(m,n;L,x) + \sum_{j=0}^{1} K_{i+n_j} B_j(m,n;L,x). \tag{3.2}$$

Note that $K_j = K_j(L)$. It should be noted that the relations,

$$K_{m_i+m_j}(0) = K_{n_i+n_j}(0) = 0 \quad (0 \le i,\ j \le 1),$$

hold under the assumption $m,n = (0,1), (0,2), (1,3)$. Putting $i = m_0, m_1$ in (3.2), we obtain

$$\begin{pmatrix} K_{m_0}(x) \\ K_{m_1}(x) \end{pmatrix} = \begin{pmatrix} K_{m_i+n_j} \end{pmatrix} \begin{pmatrix} B_0(m,n;L,x) \\ B_1(m,n;L,x) \end{pmatrix}. \tag{3.3}$$

Taking the same procedures with respect to the solution $u(x) = K_{n_i}(L-x)$,

$$\begin{pmatrix} K_{n_0}(L-x) \\ K_{n_1}(L-x) \end{pmatrix} = \begin{pmatrix} K_{n_i+m_j} \end{pmatrix} \begin{pmatrix} (-1)^{m_i}\delta_{ij} \end{pmatrix} \begin{pmatrix} A_0(m,n;L,x) \\ A_1(m,n;L,x) \end{pmatrix}. \tag{3.4}$$

Therefore, we have reached the following theorem.

THEOREM 3.1.

$$\begin{pmatrix} A_0(m,n;L,x) \\ A_1(m,n;L,x) \end{pmatrix} = \begin{pmatrix} (-1)^{m_i}\delta_{ij} \end{pmatrix} \begin{pmatrix} K_{n_i+m_j} \end{pmatrix}^{-1} \begin{pmatrix} K_{n_0}(L-x) \\ K_{n_1}(L-x) \end{pmatrix} \tag{3.5}$$

$$\begin{pmatrix} B_0(m,n;L,x) \\ B_1(m,n;L,x) \end{pmatrix} = \begin{pmatrix} K_{m_i+n_j} \end{pmatrix}^{-1} \begin{pmatrix} K_{m_0}(x) \\ K_{m_1}(x) \end{pmatrix} \tag{3.6}$$

It should be noted that the determinants of the above matrices are nonzero, as will be shown in Lemma 3.1. Let us rewrite fundamental solutions under 6 boundary conditions in terms of $K_j(x)$. In the rest of this section, we simply rewrite $A_i(m,n,L;x)$, $B_i(m,n,L;x)$ as $A_i(x)$, $B_i(x)$ as far as it is unmistakable.

1). $(m,n) = (0,1,0,1)$

$$D = D(0,1,0,1;L) = K_1^2 - K_0K_2,$$

$$A_0(x) = \frac{1}{D}\left[K_1K_1(L-x) - K_2K_0(L-x)\right],$$

$$A_1(x) = \frac{1}{D}\left[K_0K_1(L-x) - K_1K_0(L-x)\right],$$

$$B_0(x) = A_0(L-x), \ -B_1(x) = A_1(L-x).$$

2). $(m,n) = (0,1,0,2)$

$$D = D(0,1,0,2;L) = K_1K_2 - K_0K_3,$$

$$A_0(x) = \frac{1}{D}\left[K_1K_2(L-x) - K_3K_0(L-x)\right],$$

$$A_1(x) = \frac{1}{D}\left[K_0K_2(L-x) - K_2K_0(L-x)\right],$$

$$B_0(x) = \frac{1}{D}\left[K_2K_1(x) - K_3K_0(x)\right], \ -B_1(x) = \frac{1}{D}\left[K_0K_1(x) - K_1K_0(x)\right].$$

3). $(m,n) = (0,1,1,3)$

$$D = D(0,1,1,3;L) = K_2K_3 - K_1K_4,$$

$$A_0(x) = \frac{1}{D}\left[K_2K_3(L-x) - K_4K_1(L-x)\right],$$

$$A_1(x) = \frac{1}{D}\left[K_1K_3(L-x) - K_3K_1(L-x)\right]$$

$$B_0(x) = \frac{1}{D}\left[K_3K_1(x) - K_4K_0(x)\right], \ -B_1(x) = \frac{1}{D}\left[K_1K_1(x) - K_2K_0(x)\right].$$

4). $(m,n) = (0,2,0,2)$

$$D = D(0,2,0,2;L) = K_2^2 - K_0K_4,$$

$$A_0(x) = \frac{1}{D}\left[K_2 K_2(L-x) - K_4 K_0(L-x)\right],$$
$$-A_1(x) = \frac{1}{D}\left[K_0 K_2(L-x) - K_2 K_0(L-x)\right],$$
$$B_0(x) = A_0(L-x), \ -B_1(x) = -A_1(L-x).$$

5). $(m,n) = (0,2,1,3)$

$$D = D(0,2,1,3;L) = K_3^2 - K_1 K_5,$$
$$A_0(x) = \frac{1}{D}\left[K_3 K_3(L-x) - K_5 K_1(L-x)\right],$$
$$-A_1(x) = \frac{1}{D}\left[K_1 K_3(L-x) - K_3 K_1(L-x)\right],$$
$$B_0(x) = \frac{1}{D}\left[K_3 K_2(x) - K_5 K_0(x)\right], \ -B_1(x) = \frac{1}{D}\left[K_1 K_2(x) - K_3 K_0(x)\right].$$

6). $(m,n) = (1,3,1,3)$

$$D = D(1,3,1,3;L) = K_4^2 - K_2 K_6,$$
$$-A_0(x) = \frac{1}{D}\left[K_4 K_3(L-x) - K_6 K_1(L-x)\right],$$
$$A_1(x) = \frac{1}{D}\left[K_2 K_3(L-x) - K_4 K_1(L-x)\right],$$
$$B_0(x) = -A_0(L-x), \ -B_1(x) = A_1(L-x).$$

We next prove the following theorem, which ensures a definite sign of each fundamental solution.

THEOREM 3.2. *On an interval $0 < x < L$, the following inequalities hold;*

$$A_0(0,1,0,1,L;x) > 0, \ A_1(0,1,0,1,L;x) > 0, \ B_0(0,1,0,1,L;x) > 0, \ B_1(0,1,0,1,L;x) < 0, \tag{3.7}$$
$$A_0(0,1,0,2,L;x) > 0, \ A_1(0,1,0,2,L;x) > 0, \ B_0(0,1,0,2,L;x) > 0, \ B_1(0,1,0,2,L;x) < 0, \tag{3.8}$$
$$A_0(0,1,1,3,L;x) > 0, \ A_1(0,1,1,3,L;x) > 0, \ B_0(0,1,1,3,L;x) > 0, \ B_1(0,1,1,3,L;x) < 0, \tag{3.9}$$
$$A_0(0,2,0,2,L;x) > 0, \ A_1(0,2,0,2,L;x) < 0, \ B_0(0,2,0,2,L;x) > 0, \ B_1(0,2,0,2,L;x) < 0, \tag{3.10}$$
$$A_0(0,2,1,3,L;x) > 0, \ A_1(0,2,1,3,L;x) < 0, \ B_0(0,2,1,3,L;x) > 0, \ B_1(0,2,1,3,L;x) < 0, \tag{3.11}$$
$$A_0(1,3,1,3,L;x) < 0, \ A_1(1,3,1,3,L;x) > 0, \ B_0(1,3,1,3,L;x) > 0, \ B_1(1,3,1,3,L;x) < 0. \tag{3.12}$$

Let us start with the following two lemmas;

LEMMA 3.1. *We have $D(m,n;L) > 0$, in the cases* $(m,n) = (0,1,0,1)$, $(0,1,0,2)$, $(0,1,1,3)$, $(0,2,0,2)$, $(0,1,1,3)$, $(1,3,1,3)$.

LEMMA 3.2. *If $x > 0$, the following inequalities hold*;

$$\text{1).}\quad \left(\frac{K_1(x)}{K_0(x)}\right)' = -\frac{K_1^2(x) - K_0(x)K_2(x)}{K_0^2(x)} < 0, \tag{3.13}$$

$$\text{2).}\quad \left(\frac{K_2(x)}{K_0(x)}\right)' = -\frac{K_1(x)K_2(x) - K_0(x)K_3(x)}{K_0^2(x)} < 0, \tag{3.14}$$

$$\text{3).}\quad \left(\frac{K_3(x)}{K_1(x)}\right)' = -\frac{K_2(x)K_3(x) - K_1(x)K_4(x)}{K_1^2(x)} < 0. \tag{3.15}$$

Proof of Lemma 3.1. It is through simple calculations.

$$D(0,1,0,1;L) = K_1^2 - K_0K_2 = \frac{1}{2ab}\left[\frac{\mathrm{ch}((a+b)L) - 1}{(a+b)^2} - \frac{\mathrm{ch}((a-b)L) - 1}{(a-b)^2}\right] > 0, \tag{3.16}$$

$$D(0,1,0,2;L) = K_1K_2 - K_0K_3 = \frac{1}{2ab}\left[\frac{\mathrm{sh}((a+b)L)}{a+b} - \frac{\mathrm{sh}((a-b)L)}{a-b}\right] > 0, \tag{3.17}$$

$$D(0,1,1,3;L) = K_2K_3 - K_1K_4 = \frac{1}{2}\left[\frac{\mathrm{sh}((a+b)L)}{a+b} + \frac{\mathrm{sh}((a-b)L)}{a-b}\right] > 0, \tag{3.18}$$

$$D(0,2,0,2;L) = K_2^2 - K_0K_4 = \frac{1}{2ab}\left[\mathrm{ch}((a+b)L) - \mathrm{ch}((a-b)L)\right] > 0, \tag{3.19}$$

$$D(0,2,1,3;L) = K_3^2 - K_1K_5 = \frac{1}{2}\left[\mathrm{ch}((a+b)L) + \mathrm{ch}((a-b)L)\right] > 0, \tag{3.20}$$

$$D(1,3,1,3;L) = K_4^2 - K_2K_6 = \frac{ab}{2}\left[\mathrm{ch}((a+b)L) - \mathrm{ch}((a-b)L)\right] > 0. \tag{3.21}$$

■

Proof of Lemma 3.2. Replacing L in (3.16), (3.17) and (3.18) with x, we come to a conclusion given by inequalities (3.13)–(3.15). ■

Employing the above 2 lemmas, we finally prove the Theorem 3.2.

Proof of Theorem 3.2.
1). $(m,n) = (0,1,0,1)$

$$\frac{A_0(x)}{K_0(L-x)} = \frac{1}{D}\left[K_1\frac{K_1(L-x)}{K_0(L-x)} - K_2\right] \underset{x\downarrow 0}{\searrow} \frac{A_0(0)}{K_0} = \frac{1}{K_0} > 0,$$
$$\frac{A_1(x)}{K_0(L-x)} = \frac{1}{D}\left[K_0\frac{K_1(L-x)}{K_0(L-x)} - K_1\right] \underset{x\downarrow 0}{\searrow} \frac{A_1(0)}{K_0} = 0.$$

2). $(m,n)=(0,1,0,2)$

$$\frac{A_0(x)}{K_0(L-x)}=\frac{1}{D}\left[K_1\frac{K_2(L-x)}{K_0(L-x)}-K_3\right]\underset{x\downarrow 0}{\searrow}\frac{A_0(0)}{K_0}=\frac{1}{K_0}>0,$$
$$\frac{A_1(x)}{K_0(L-x)}=\frac{1}{D}\left[K_0\frac{K_2(L-x)}{K_0(L-x)}-K_2\right]\underset{x\downarrow 0}{\searrow}\frac{A_1(0)}{K_0}=0,$$
$$\frac{B_0(x)}{K_0(x)}=\frac{1}{D}\left[K_2\frac{K_1(x)}{K_0(x)}-K_3\right]\underset{x\uparrow L}{\searrow}\frac{B_0(L)}{K_0}=\frac{1}{K_0}>0,$$
$$-\frac{B_1(x)}{K_0(x)}=\frac{1}{D}\left[K_0\frac{K_1(x)}{K_0(x)}-K_1\right]\underset{x\uparrow L}{\searrow}-\frac{B_0(L)}{K_0}=0.$$

3). $(m,n)=(0,1,1,3)$

$$\frac{A_0(x)}{K_1(L-x)}=\frac{1}{D}\left[K_2\frac{K_3(L-x)}{K_1(L-x)}-K_4\right]\underset{x\downarrow 0}{\searrow}\frac{A_0(0)}{K_1}=\frac{1}{K_1}>0,$$
$$\frac{A_1(x)}{K_1(L-x)}=\frac{1}{D}\left[K_1\frac{K_3(L-x)}{K_1(L-x)}-K_3\right]\underset{x\downarrow 0}{\searrow}\frac{A_1(0)}{K_1}=0,$$
$$\frac{B_0(x)}{K_0(x)}=\frac{1}{D}\left[K_3\frac{K_1(x)}{K_0(x)}-K_4\right]\underset{x\uparrow L}{\searrow}\frac{B_0(L)}{K_0}=\frac{K_1K_3-K_0K_4}{K_0D}>0,$$
$$-\frac{B_1(x)}{K_0(x)}=\frac{1}{D}\left[K_1\frac{K_1(x)}{K_0(x)}-K_2\right]\underset{x\uparrow L}{\searrow}-\frac{B_1(L)}{K_0}=\frac{K_1^2-K_0K_2}{K_0D}>0.$$

4). $(m,n)=(0,2,0,2)$

$$\frac{A_0(x)}{K_0(L-x)}=\frac{1}{D}\left[K_2\frac{K_2(L-x)}{K_0(L-x)}-K_4\right]\underset{x\downarrow 0}{\searrow}\frac{A_0(0)}{K_0}=\frac{1}{K_0}>0,$$
$$-\frac{A_1(x)}{K_0(L-x)}=\frac{1}{D}\left[K_0\frac{K_2(L-x)}{K_0(L-x)}-K_2\right]\underset{x\downarrow 0}{\searrow}-\frac{A_1(0)}{K_0}=0.$$

5). $(m,n)=(0,2,1,3)$

$$\frac{A_0(x)}{K_1(L-x)}=\frac{1}{D}\left[K_3\frac{K_3(L-x)}{K_1(L-x)}-K_5\right]\underset{x\downarrow 0}{\searrow}\frac{A_0(0)}{K_1}=\frac{1}{K_1}>0,$$
$$-\frac{A_1(x)}{K_1(L-x)}=\frac{1}{D}\left[K_1\frac{K_3(L-x)}{K_1(L-x)}-K_3\right]\underset{x\downarrow 0}{\searrow}-\frac{A_1(0)}{K_1}=0,$$
$$\frac{B_0(x)}{K_0(x)}=\frac{1}{D}\left[K_3\frac{K_2(x)}{K_0(x)}-K_5\right]\underset{x\uparrow L}{\searrow}\frac{B_0(L)}{K_0}=\frac{K_2K_3-K_0K_5}{K_0D}>0,$$
$$-\frac{B_1(x)}{K_0(x)}=\frac{1}{D}\left[K_1\frac{K_2(x)}{K_0(x)}-K_3\right]\underset{x\uparrow L}{\searrow}-\frac{B_1(L)}{K_0}=\frac{K_1K_2-K_0K_3}{K_0D}>0.$$

6). $(m,n)=(1,3,1,3)$

$$-\frac{A_0(x)}{K_1(L-x)}=\frac{1}{D}\left[K_4\frac{K_3(L-x)}{K_1(L-x)}-K_6\right]\underset{x\downarrow 0}{\searrow}-\frac{A_0(0)}{K_1}=\frac{K_3K_4-K_1K_6}{K_1D}>0,$$

$$\frac{A_1(x)}{K_1(L-x)} = \frac{1}{D}\left[K_2\frac{K_3(L-x)}{K_1(L-x)} - K_4\right] \underset{x\downarrow 0}{\searrow} \frac{A_1(0)}{K_1} = \frac{K_2K_3 - K_1K_4}{K_1D} > 0.$$

∎

4. Green's Functions to BVP$(m_0, m_1; n_0, n_1)$

We start with the theorem with respect to expression of Green's functions given by (2.12).

THEOREM 4.1. *If* $0 < x,\, y < L$, *Green's functions are expressed as in the following five ways*;

$$\begin{aligned} & g(m,n,L;x,y) \\ &= Y(x-y)K_0(x-y) - \sum_{j=0}^{1} B_j(m,n,L;x)K_{n_j}(L-y) && (4.1) \\ &= \sum_{j=0}^{1} A_j(m,n,L;x)(-1)^{m_j+1}K_{m_j}(y) - Y(y-x)K_0(x-y). && (4.2) \\ &= \sum_{j=0}^{1} A_j(m,n,L;x\vee y)(-1)^{m_j+1}K_{m_j}(x\wedge y) && (4.3) \\ &= -\sum_{j=0}^{1} K_{n_j}(L-x\vee y)B_j(m,n,L;x\wedge y) && (4.4) \\ &= -\begin{pmatrix} K_{n_0}(L-x\vee y) & K_{n_1}(L-x\vee y)\end{pmatrix}\begin{pmatrix} K_{m_i+n_j}\end{pmatrix}^{-1}\begin{pmatrix} K_{m_0}(x\wedge y) \\ K_{m_1}(x\wedge y)\end{pmatrix}, && (4.5) \end{aligned}$$

where

$$x\vee y = \max(x,y), \quad x\wedge y = \min(x,y).$$

Moreover, each Green's function is symmetric with respect to x *and* y,

$$g(m,n,L;x,y) = g(m,n,L;y,x). \tag{4.6}$$

Proof. For each fixed y as a function of x, $K_0(x-y)$ has the boundary data,

$$\begin{aligned} & BD(m,n;K_0(x-y)) \\ &= (K_{m_0}(-y), K_{m_1}(-y), K_{n_0}(L-y), K_{n_1}(L-y)) \\ &= ((-1)^{m_0+1}K_{m_0}(y), (-1)^{m_1+1}K_{m_1}(y), K_{n_0}(L-y), K_{n_1}(L-y)), && (4.7) \end{aligned}$$

Applying formula (1.2) to $K_0(x-y)$, we have

$$K_0(x-y) = \sum_{j=0}^{1}\left(A_j(m,n,L;x)(-1)^{m_j+1}K_{m_j}(y) + B_j(m,n,L;x)K_{n_j}(L-y)\right), \tag{4.8}$$

substitution of which into (4.1) gives (4.2). Combining (4.1) and (4.2), we have

$$
\begin{aligned}
g(m,n,L;x,y) =&Y(x-y)\sum_{j=0}^{1} A_j(m,n,L;x)(-1)^{m_j+1}K_{m_j}(y) \\
&-Y(y-x)\sum_{j=0}^{1} B_j(m,n,L;x)K_{n_j}(L-y)
\end{aligned} \tag{4.9}
$$

On the other hand from (3.5), (3.6), we have

$$
-\sum_{j=0}^{1} B_j(m,n,L;x)K_{n_j}(L-y) = \sum_{j=0}^{1} A_j(m,n,L;y)(-1)^{m_j+1}K_{m_j}(x) \tag{4.10}
$$

Combining (4.9) and (4.10), we have

$$
\begin{aligned}
g(m,n,L;x,y) &= Y(x-y)\sum_{j=0}^{1} A_j(m,n,L;x)(-1)^{m_j+1}K_{m_j}(y) \\
&\quad + Y(y-x)\sum_{j=0}^{1} A_j(m,n,L;y)(-1)^{m_j+1}K_{m_j}(x) \\
&= \sum_{j=0}^{1} A_j(m,n,L;x\vee y)(-1)^{m_j+1}K_{m_j}(x\wedge y)
\end{aligned}
$$

which proves (4.3). Equations (4.4) and (4.5) are shown in the same way. The symmetry of Green's functions (4.6) follows from (4.3) because $x\vee y$ and $x\wedge y$ are both symmetric functions with respect to x and y. ■

Let us illustrate concrete forms of Green's functions under 6 boundary conditions.

$$
\begin{aligned}
&g(0,1,0,1,L;x,y) \\
&= A_1(0,1,0,1,L;x\vee y)K_1(x\wedge y) - A_0(0,1,0,1,L;x\vee y)K_0(x\wedge y) \\
&= K_1(L-x\vee y)(-B_1(0,1,0,1,L;x\wedge y)) - K_0(L-x\vee y)B_0(0,1,0,1,L;x\wedge y) \\
&= \frac{1}{D(0,1,0,1,L)}\begin{pmatrix}K_0(L-x\vee y) & K_1(L-x\vee y)\end{pmatrix}\begin{pmatrix}K_2 & -K_1\\ -K_1 & K_0\end{pmatrix}\begin{pmatrix}K_0(x\wedge y)\\ K_1(x\wedge y)\end{pmatrix}
\end{aligned} \tag{4.11}
$$

$$
\begin{aligned}
&g(0,1,0,2,L;x,y) \\
&= A_1(0,1,0,2,L;x\vee y)K_1(x\wedge y) - A_0(0,1,0,2,L;x\vee y)K_0(x\wedge y) \\
&= K_2(L-x\vee y)(-B_1(0,1,0,2,L;x\wedge y)) - K_0(L-x\vee y)B_0(0,1,0,2,L;x\wedge y) \\
&= \frac{1}{D(0,1,0,2,L)}\begin{pmatrix}K_0(L-x\vee y) & K_2(L-x\vee y)\end{pmatrix}\begin{pmatrix}K_3 & -K_2\\ -K_1 & K_0\end{pmatrix}\begin{pmatrix}K_0(x\wedge y)\\ K_1(x\wedge y)\end{pmatrix}
\end{aligned} \tag{4.12}
$$

$$
\begin{aligned}
&g(0,1,1,3,L;x,y)\\
&= A_1(0,1,1,3,L;x\vee y)K_1(x\wedge y) - A_0(0,1,1,3,L;x\vee y)K_0(x\wedge y)\\
&= K_3(L-x\vee y)(-B_1(0,1,1,3,L;x\wedge y)) - K_1(L-x\vee y)B_0(0,1,1,3,L;x\wedge y)\\
&= \frac{1}{D(0,1,1,3,L)}\left(K_1(L-x\vee y)\ \ K_3(L-x\vee y)\right)\begin{pmatrix} K_4 & -K_3\\ -K_2 & K_1\end{pmatrix}\begin{pmatrix} K_0(x\wedge y)\\ K_1(x\wedge y)\end{pmatrix}
\end{aligned}
\tag{4.13}
$$

$$
\begin{aligned}
&g(0,2,0,2,L;x,y)\\
&= (-A_1(0,2,0,2,L;x\vee y))K_2(x\wedge y) - A_0(0,2,0,2,L;x\vee y)K_0(x\wedge y)\\
&= K_2(L-x\vee y)(-B_1(0,2,0,2,L;x\wedge y)) - K_0(L-x\vee y)B_0(0,2,0,2,L;x\wedge y)\\
&= \frac{1}{D(0,2,0,2,L)}\left(K_0(L-x\vee y)\ \ K_2(L-x\vee y)\right)\begin{pmatrix} K_4 & -K_2\\ -K_2 & K_0\end{pmatrix}\begin{pmatrix} K_0(x\wedge y)\\ K_2(x\wedge y)\end{pmatrix}
\end{aligned}
\tag{4.14}
$$

$$
\begin{aligned}
&g(0,2,1,3,L;x,y)\\
&= (-A_1(0,2,1,3,L;x\vee y))K_2(x\wedge y) - A_0(0,2,1,3,L;x\vee y)K_0(x\wedge y)\\
&= K_3(L-x\vee y)(-B_1(0,2,1,3,L;x\wedge y)) - K_1(L-x\vee y)B_0(0,2,1,3,L;x\wedge y)\\
&= \frac{1}{D(0,2,1,3,L)}\left(K_1(L-x\vee y)\ \ K_3(L-x\vee y)\right)\begin{pmatrix} K_5 & -K_3\\ -K_3 & K_1\end{pmatrix}\begin{pmatrix} K_0(x\wedge y)\\ K_2(x\wedge y)\end{pmatrix}
\end{aligned}
\tag{4.15}
$$

$$
\begin{aligned}
&g(1,3,1,3,L;x,y)\\
&= A_1(1,3,1,3,L;x\vee y)K_3(x\wedge y) - (-A_0(1,3,1,3,L;x\vee y))K_1(x\wedge y)\\
&= K_3(L-x\vee y)(-B_1(1,3,1,3,L;x\wedge y)) - K_1(L-x\vee y)B_0(1,3,1,3,L;x\wedge y)\\
&= \frac{1}{D(1,3,1,3,L)}\left(K_1(L-x\vee y)\ \ K_3(L-x\vee y)\right)\begin{pmatrix} K_6 & -K_4\\ -K_4 & K_2\end{pmatrix}\begin{pmatrix} K_1(x\wedge y)\\ K_3(x\wedge y)\end{pmatrix}
\end{aligned}
\tag{4.16}
$$

5. Green's Functions without Clumped Edge Condition

In this section, we investigate Green's functions to BVP$(0,2,0,2)$, BVP$(0,2,1,3)$, and BVP$(1,3,1,3)$. For this purpose, we clarify their relation with Green's functions in a whole line and under periodic boundary condition.

First of all, let us consider a BVP in a whole line,

$$
\mathrm{BVP}(-\infty,\infty):\begin{cases} u^{(4)} - pu'' + qu = f(x) & (-\infty < x < \infty),\\ u^{(i)}(x): \mathrm{bdd.} & (0 \le i \le 3).\end{cases}
\tag{5.1}
$$

One can easily confirm the following theorem;

THEOREM 5.1. *Let $f(x)$ be a continuous function on $(-\infty,\infty)$ which satisfies*

$$\int_{-\infty}^{\infty} \mathrm{e}^{-b|x|}|f(x)|\mathrm{d}x < \infty. \tag{5.2}$$

Then BVP$(-\infty,\infty)$ *possesses a unique* 4 *times continously differentiable solution* (*a classical solution*), *which is expressed as*

$$u(x) = \int_{-\infty}^{\infty} g(x-y)f(y)\mathrm{d}y, \tag{5.3}$$

where $g(x)$ is defined by

$$g(x) = G(|x|), \quad G(x) = \frac{1}{2(a^2-b^2)}(b^{-1}\mathrm{e}^{-bx} - a^{-1}\mathrm{e}^{-ax}). \tag{5.4}$$

In the second place, we consider a BVP under a periodic boundary condition,

$$\mathrm{BVP}(P): \begin{cases} u^{(4)} - pu'' + qu = f(x) \quad (-\infty < x < \infty), \\ u(x+2L) = u(x). \end{cases} \tag{5.5}$$

We also assume that $f(x)$ be a function with a period $2L$. Then the next theorem holds.

THEOREM 5.2. *Let $f(x)$ be a continuous function. Then* BVP(P) *has a unique classical solution, which has the form,*

$$u(x) = \int_0^{2L} g_p(2L; x-y)f(y)\mathrm{d}y. \tag{5.6}$$

Green's functions $g_P(2L;x)$ is given by

$$\begin{aligned} g_p(2L;x) &= \sum_{j=-\infty}^{\infty} g(x+2jL) \\ &= \frac{1}{2(a^2-b^2)}\left(\frac{\mathrm{ch}(b(|x|-L))}{b\,\mathrm{sh}(bL)} - \frac{\mathrm{ch}(a(|x|-L))}{a\,\mathrm{sh}(aL)}\right) \quad (|x| \le 2L) \end{aligned} \tag{5.7}$$

and satisfies

$$\begin{aligned} &2(a^2-b^2)\frac{\mathrm{d}}{\mathrm{d}x}g_p(2L;x) \\ &= \mathrm{sgn}(x)\left(\frac{\mathrm{sh}(b(|x|-L))}{\mathrm{sh}(bL)} - \frac{\mathrm{sh}(a(|x|-L))}{\mathrm{sh}(aL)}\right)\begin{cases} <0 & (0<x<L), \\ =0 & (x=L), \\ >0 & (L<x<2L), \end{cases} \end{aligned} \tag{5.8}$$

$$\mathrm{sgn}(x) = -1 \ (x<0), \quad 0 \ (x=0), \quad 1 \ (0<x),$$

and

$$\min_{|x|\leq 2L} g_p(2L;x) = g_p(2L;L) > 0. \tag{5.9}$$

REMARK 2. (5.8) is shown by making use of the inequality,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{sh}(ax)}{\mathrm{sh}(bx)}\right) > 0, \quad (a > b > 0,\ x > 0), \tag{5.10}$$

which also plays an important role in the next section.

The following theorem states that Green's functions to BVP(m,n), where m,n are not equal to $(0,1)$, are expressed by means of those under a periodic boundary condition.

THEOREM 5.3.

$$g(0,2,0,2,L;x,y) = g_p(2L;x-y) - g_p(2L;x+y) \tag{5.11}$$

$$= g_p(4L;x-y) + g_p(4L;x-y-2L) - g_p(4L;x+y) - g_p(4L;x+y-2L) \tag{5.12}$$

$$g(1,3,1,3,L;x,y) = g_p(2L;x-y) + g_p(2L;x+y) \tag{5.13}$$

$$= g_p(4L;x-y) + g_p(4L;x-y-2L) + g_p(4L;x+y) + g_p(4L;x+y-2L) \tag{5.14}$$

$$\begin{aligned} g(0,2,1,3,L;x,y) = {} & g_p(4L;x-y) - g_p(4L;x-y-2L) \\ & - g_p(4L;x+y) + g_p(4L;x+y-2L) \end{aligned} \tag{5.15}$$

Making use of the above theorem, we finally give the main theorem in this section,

THEOREM 5.4. *If* $0 < x,\ y < L$, *we have the following inequalities*;

$$0 < g(0,2,0,2,L;x,y) < g(0,2,1,3,L;x,y) < g(1,3,1,3,L;x,y). \tag{5.16}$$

Proof. Let us first prove $0 < g(0,2,0,2,L;x,y)$. From (5.11), it is enough to show

$$g_p(2L;x-y) - g_p(2L;x+y) > 0 \qquad (0 < x,\ y < L). \tag{5.17}$$

Considering the symmetry, we may suppose $x > y$. If $x+y \leq L$, the positivity is obvious due to the monotone decreasing property of $g_p(2L;x)$ in $0 < x < L$. If $x+y > L$, we have $0 < 2L-x-y < L$ and

$$g_p(2L;x+y) = g_p(2L;2L-x-y) = g_p(2L;x-y+2(L-x)) < g_p(2L;x-y) \tag{5.18}$$

due to the symmetry of $g_p(2L;x)$ with respect to $x = L$.

The second inequality $g(0,2,0,2,L;x,y) < g(0,2,1,3,L;x,y)$, is shown as follows;

$$\begin{aligned}((5.15)-(5.12))/2 &= g_p(4L;x+y-2L) - g_p(4L;x-y-2L)\\ &= g_p(4L;2L-x-y) - g_p(4L;2L-x+y)\\ &= g_p(4L;2L-x-y) - g_p(4L;2L-|x-y|) > 0. \qquad (5.19)\end{aligned}$$

The third inequality $g(0,2,1,3,L;x,y) < g(1,3,1,3,L;x,y)$ follows directly from (5.14), (5.15). ■

6. Positivity of Green's Functions to BVP$(0,1,0,1)$

In this and next section, we put $L = 1$ without loss of generality and rewrite $g(m,n;L,x,y)$ as $g(m,n;x,y)$ for simplicity.

THEOREM 6.1 (Integral representation of $g(0,1,0,1;x,y)$). *In the domain* $0 < x,\ y < 1$,

1). *Green's function $g(0,1,0,1;x,y)$ possesses the following integral representation;*

$$\begin{aligned}&(a+b)^2(a-b)^2\big[\,(a+b)^2\mathrm{ch}(a+b) - (a-b)^2\mathrm{ch}(a-b)\,\big]g(0,1,0,1;x,y)\\ &= Y\int_{-1}^{1}\big[\,(\mathrm{ch}(\alpha(2-|X-tY|)) - \mathrm{ch}(\alpha|X-Y|))(\mathrm{ch}(\beta|X-tY|) - \mathrm{ch}(\beta|X-Y|))\\ &\quad - (\mathrm{ch}(\alpha|X-tY|) - \mathrm{ch}(\alpha|X-Y|))(\mathrm{ch}(\beta(2-|X-tY|)) - \mathrm{ch}(\beta|X-Y|))\,\big]\mathrm{d}t,\end{aligned} \qquad (6.1)$$

$$X = X(x,y) = (x \vee y) \wedge ((1-x) \vee (1-y)),$$

$$Y = Y(x,y) = x \wedge (1-x) \wedge y \wedge (1-y), \qquad (6.2)$$

$$\alpha = (a+b)/2,\ \ \beta = (a-b)/2. \qquad (6.3)$$

2). *Green's function $g(0,1,0,1;x,y)$ is positive-valued.*

The following table illustrates concrete forms and domains of functions X, Y.

$X(x,y)$	$Y(x,y)$	
x	y	$0 < y < x \wedge (1-x)$
$1-y$	$1-x$	$0 < 1-x < y \wedge (1-y)$
$1-x$	$1-y$	$0 < 1-y < x \wedge (1-x)$
y	x	$0 < x < y \wedge (1-y)$

The functions X, Y satisfy inequalities,

$$0 < Y(x,y) < X(x,y) \wedge (1-X(x,y)). \qquad (6.4)$$

Proof. Due to the symmetry of $g(0,1,0,1;x,y)$ with respect to the lines $y=x$ and $y=1-x$ we may suppose $0<y<x\wedge(1-x)$. Then (x,y) is parametrized by means of $0<\eta<\xi<1$ as $\left(1-\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right)$ and Green's function is rewritten as

$$\begin{aligned}
&8(\alpha^2-\beta^2)\alpha\beta((\beta\,\mathrm{sh}(\alpha))^2-(\alpha\,\mathrm{sh}(\beta))^2)g\Big(0,1,0,1;1-\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\Big)\\
&\quad=\big[(\alpha^2-\beta^2)\{\alpha\,\mathrm{ch}(\alpha(1-\xi))\,\mathrm{sh}(\beta(1-\xi))-\beta\,\mathrm{sh}(\alpha(1-\xi))\,\mathrm{ch}(\beta(1-\xi))\}\\
&\quad\quad+\alpha^3\,\mathrm{ch}(\alpha(1-\xi))\,\mathrm{sh}(\beta(1+\xi))+\alpha^2\beta\,\mathrm{sh}(\alpha(1-\xi))\,\mathrm{ch}(\beta(1+\xi))\\
&\quad\quad+\alpha\beta^2\,\mathrm{ch}(\alpha(1+\xi))\,\mathrm{sh}(\beta(1-\xi))+\beta^3\,\mathrm{sh}(\alpha(1+\xi))\,\mathrm{ch}(\beta(1-\xi))\big]\\
&\quad\quad-(\alpha^2-\beta^2)\big[\alpha\,\mathrm{ch}(\alpha(1-\xi))\{\mathrm{sh}(\beta(1+\eta))+\mathrm{sh}(\beta(1-\eta))\}\\
&\quad\quad-\beta\,\mathrm{ch}(\beta(1-\xi))\{\mathrm{sh}(\alpha(1+\eta))+\mathrm{sh}(\alpha(1-\eta))\}\big]\\
&\quad\quad-\alpha\beta\big[\alpha\{\mathrm{sh}(\alpha(1+\eta))\,\mathrm{ch}(\beta(1-\eta))+\mathrm{sh}(\alpha(1-\eta))\,\mathrm{ch}(\beta(1+\eta))\}\\
&\quad\quad+\beta\{\mathrm{ch}(\alpha(1+\eta))\,\mathrm{sh}(\beta(1-\eta))+\mathrm{ch}(\alpha(1-\eta))\,\mathrm{sh}(\beta(1+\eta))\}\big]
\end{aligned}\tag{6.5}$$

through straightforward calculations. Differentiating both sides with respect to η, we obtain

$$\begin{aligned}
&-\frac{\partial}{\partial\eta}\left[8((\beta\,\mathrm{sh}(\alpha))^2-(\alpha\,\mathrm{sh}(\beta))^2)g\Big(0,1,0,1;1-\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\Big)\right]\\
&\quad=(\mathrm{ch}(\alpha(1+\eta))-\mathrm{ch}(\alpha(1-\xi)))(\mathrm{ch}(\beta(1-\eta))-\mathrm{ch}(\beta(1-\xi)))\\
&\quad\quad-(\mathrm{ch}(\alpha(1-\eta))-\mathrm{ch}(\alpha(1-\xi)))(\mathrm{ch}(\beta(1+\eta))-\mathrm{ch}(\beta(1-\xi)))\\
&\hspace{20em}(0<\eta<\xi<1).
\end{aligned}\tag{6.6}$$

Substitution of $\eta=\xi$ in (6.5) gives 0. Combining this fact with (6.6) and integrating both sides of the above equation with η, we obtain the integral formula (6.1).

Finally we prove that the integrand given by (6.6) is positive-valued, which guarantees 2) in the above Theorem. By putting, $(x,y,z)=(1+\eta,1-\eta,1-\xi)$, which satisfy inequality $0<z<y<x$, right-hand side of (6.6) is rewritten as

$$\begin{aligned}
&=(\mathrm{ch}(\alpha x)-\mathrm{ch}(\alpha z))(\mathrm{ch}(\beta y)-\mathrm{ch}(\beta z))-(\mathrm{ch}(\alpha y)-\mathrm{ch}(\alpha z))(\mathrm{ch}(\beta x)-\mathrm{ch}(\beta z))\\
&=\alpha\beta\left\{\int_z^x\mathrm{sh}(\alpha\varphi)\mathrm{d}\varphi\int_z^y\mathrm{sh}(\beta\psi)\mathrm{d}\psi-\int_z^y\mathrm{sh}(\alpha\varphi)\mathrm{d}\varphi\int_z^x\mathrm{sh}(\beta\psi)\mathrm{d}\psi\right\}\\
&=\alpha\beta\left\{\int_y^x\mathrm{sh}(\alpha\varphi)\mathrm{d}\varphi\int_z^y\mathrm{sh}(\beta\psi)\mathrm{d}\psi-\int_z^y\mathrm{sh}(\alpha\varphi)\mathrm{d}\varphi\int_y^x\mathrm{sh}(\beta\psi)\mathrm{d}\psi\right\}\\
&=\alpha\beta\left\{\int_y^x\mathrm{sh}(\alpha\varphi)\mathrm{d}\varphi\int_z^y\mathrm{sh}(\beta\psi)\mathrm{d}\psi-\int_z^y\mathrm{sh}(\alpha\psi)\mathrm{d}\psi\int_y^x\mathrm{sh}(\beta\varphi)\mathrm{d}\varphi\right\}\\
&=\alpha\beta\int_y^x\mathrm{d}\varphi\int_z^y\mathrm{d}\psi\,\mathrm{sh}(\beta\varphi)\mathrm{sh}(\beta\psi)\left(\frac{\mathrm{sh}(\alpha\varphi)}{\mathrm{sh}(\beta\varphi)}-\frac{\mathrm{sh}(\alpha\psi)}{\mathrm{sh}(\beta\psi)}\right)>0.
\end{aligned}$$

Last inequality is shown by means of the inequality (5.10). ■

7. Positivity and Hierarchical Structure of Green's Functions

In this section, we propose the main theorem in this paper.

THEOREM 7.1 (Main Theorem). *If* $0 < x,\ y < 1$, *Green's functions constitute a hierarchical structure shown in Figure* 1, *in which "*$g(m,n;x,y) \to g(m',n';x,y)$*" represents that* $g(m',n';x,y)$ *is greater than* $g(m,n;x,y)$ *at every point* $(x,y) \in (0,1)\times(0,1)$.

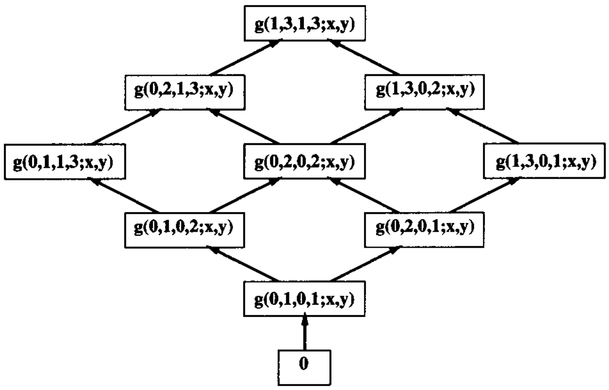

Fig. 1. Hierarchical structures of Green's functions to 9 BVP's.

Considering the symmetry $g(m,n,L;x,y) = g(n,m,L;L-x,L-y)$, we have only to prove the left half of the Figure 1, that is,

$$0 < g(0,1,0,1;x,y) < g(0,1,0,2;x,y) < \left\{ \begin{matrix} g(0,1,1,3;x,y) \\ g(0,2,0,2;x,y) \end{matrix} \right\}$$
$$< g(0,2,1,3;x,y) < g(1,3,1,3;x,y). \tag{7.1}$$

Among them, we have already shown inequalities $0 < g(0,2,0,2;x,y) < g(0,2,1,3;x,y) < g(1,3,1,3;x,y)$ and $0 < g(0,1,0,1;x,y)$ in Theorem 5.4 and 6.1, respectively. Inequalities which remain unproved are as follows;

$$g(0,1,0,1;x,y) < g(0,1,0,2;x,y),\ g(0,1,0,2;x,y) < g(0,2,0,2;x,y),$$
$$g(0,1,1,3;x,y) < g(0,2,1,3;x,y),\ g(0,1,0,2;x,y) < g(0,1,1,3;x,y).$$

THEOREM 7.2.

$$g(0,1,0,2;x,y) - g(0,1,0,1;x,y) = (-B_1(0,1,0,1;x))(-B_1(0,1,0,2;y)) > 0 \tag{7.2}$$

Proof. We may assume $0 < y < x < 1$. Subtracting (4.11) from (4.12), we have

$$
\begin{aligned}
g(0,1,0,2;x,y) - g(0,1,0,1;x,y) = (A_1(0,1,0,2;x)) - A_1(0,1,0,1;x))K_1(y) \\
- (A_0(0,1,0,2;x) - A_0(0,1,0,1;x))K_0(y).
\end{aligned}
\tag{7.3}
$$

Let us prove the following lemma;

LEMMA 7.1.

$$
A_0(0,1,0,2;x) - A_0(0,1,0,1;x) = -\frac{K_1}{D(0,1,0,2)}B_1(0,1,0,1;x) \tag{7.4}
$$

$$
A_1(0,1,0,2;x) - A_1(0,1,0,1;x) = -\frac{K_0}{D(0,1,0,2)}B_1(0,1,0,1;x) \tag{7.5}
$$

Proof. Since the boundary data of fundamental solutions are calculated as

$$
\begin{aligned}
&BD(0,1,0,1;A_0(0,1,0,2;x)) \\
&\quad = \{A_0(0,1,0,2;0), A_0'(0,1,0,2;0), A_0(0,1,0,2;1), A_0'(0,1,0,2;1)\} \\
&\quad = \{1,0,0,A_0'(0,1,0,2;1)\}, \\
&BD(0,1,0,1;A_0(0,1,0,1;x)) \\
&\quad = \{A_0(0,1,0,1;0), A_0'(0,1,0,1;0), A_0(0,1,0,1;1), A_0'(0,1,0,1;1)\} \\
&\quad = \{1,0,0,0\},
\end{aligned}
$$

we have

$$
BD(0,1,0,1;A_0(0,1,0,2;x) - A_0(0,1,0,1;x)) = \{0,0,0,A_0'(0,1,0,2;1)\}.
$$

Noticing that

$$
A_0'(0,1,0,2;1) = \frac{1}{D(0,1,0,2)}(K_1(-K_3(0)) - K_3(-K_1(0))) = -\frac{K_1}{D(0,1,0,2)}
$$

we obtain (7.4) from the uniqueness theorem.

Taking the same procedures, we have

$$
\begin{aligned}
&BD(0,1,0,1;A_1(0,1,0,2;x) - A_1(0,1,0,1;x)) \\
&\quad = \{0,1,0,A_1'(0,1,0,2;1)\} - \{0,1,0,0\} \\
&\quad = \{0,0,0,A_1'(0,1,0,2;1)\} = \left\{0,0,0,-\frac{K_0}{D(0,1,0,2)}\right\},
\end{aligned}
$$

from which we obatin (7.5). ■

By utilizing the Lemma 7.1, (7.3) gives

$$
\begin{aligned}
&g(0,1,0,2;x,y) - g(0,1,0,1;x,y) \\
&\quad = (-B_1(0,1,0,1;x))\frac{1}{D(0,1,0,2)}(K_0K_1(y) - K_1K_0(y))
\end{aligned}
$$

$$= (-B_1(0,1,0,1;x))(-B_1(0,1,0,2;y)) > 0. \tag{7.6}$$

■

THEOREM 7.3.

$$g(0,2,0,2;x,y) - g(0,1,0,2;x,y) = (-A_1(0,2,0,2;x))A_1(0,1,0,2;y) > 0 \tag{7.7}$$

Proof. We assume $0 < y < x < 1$. Subtraction of (4.12) from (4.14) gives

$$\begin{aligned} &g(0,2,0,2;x,y) - g(0,1,0,2;x,y) \\ &\quad = K_2(1-x)\{(-B_1(0,2,0,2;y)) - (-B_1(0,1,0,2;y)\} \\ &\qquad - K_0(1-x)(B_0(0,2,0,2;y) - B_0(0,1,0,2;y)). \end{aligned} \tag{7.8}$$

We first prepare the following lemma;

LEMMA 7.2.

$$B_0(0,2,0,2;x) - B_0(0,1,0,2;x) = \frac{K_2}{D(0,2,0,2)} A_1(0,1,0,2;x), \tag{7.9}$$

$$(-B_1(0,2,0,2;x)) - (-B_1(0,1,0,2;x)) = \frac{K_0}{D(0,2,0,2)} A_1(0,1,0,2;x). \tag{7.10}$$

Proof. Calculation of boundary data gives

$$\begin{aligned} &BD(0,1,0,2; B_0(0,2,0,2;x) - B_0(0,1,0,2;x)) \\ &\quad = BD(0,1,0,2; B_0(0,2,0,2;x)) - BD(0,1,0,2; B_0(0,1,0,2;x)) \\ &\quad = \{0, B_0'(0,2,0,2;0), 1, 0\} - \{0,0,1,0\} \\ &\quad = \{0, B_0'(0,2,0,2;0), 0, 0\} = \left\{0, \frac{K_2}{D(0,2,0,2)}, 0, 0\right\}. \end{aligned}$$

Together with the uniqueness theorem, we obtain (7.9). Similarly, simple calculation shows

$$\begin{aligned} &BD(0,1,0,2; (-B_1(0,2,0,2;x)) - (-B_1(0,1,0,2;x))) \\ &\quad = \{0, -B_1'(0,2,0,2;0), 0, 0\} = \left\{0, \frac{K_0}{D(0,2,0,2)}, 0, 0\right\}, \end{aligned}$$

from which we obtain (7.10). ■

From Lemma 7.2, we have

$$\begin{aligned} &g(0,2,0,2;x,y) - g(0,1,0,2;x,y) \\ &\quad = \frac{1}{D(0,2,0,2)}(K_0K_2(1-x) - K_2K_0(1-x))A_1(0,1,0,2;y) \\ &\quad = (-A_1(0,2,0,2;x))A_1(0,1,0,2;y) > 0, \end{aligned} \tag{7.11}$$

which completes the proof of Theorem 7.3. ■

THEOREM 7.4.

$$g(0,2,1,3;x,y) - g(0,1,1,3;x,y) = (-A_1(0,2,1,3;x))A_1(0,1,1,3;y) > 0 \quad (7.12)$$

Proof. We suppose $0 < y < x < 1$. Subtracting (4.13) from (4.15), we have

$$\begin{aligned} &g(0,2,1,3;x,y) - g(0,1,1,3;x,y) \\ &\quad = K_3(1-x)\{(-B_1(0,2,1,3;y)) - (-B_1(0,1,1,3;y))\} \\ &\qquad - K_1(1-x)(B_0(0,2,1,3;y) - B_0(0,1,1,3;y)). \end{aligned} \quad (7.13)$$

We first prove the following lemma;

LEMMA 7.3.

$$B_0(0,2,1,3;x) - B_0(0,1,1,3;x) = \frac{K_3}{D(0,2,1,3)}A_1(0,1,1,3;x) \quad (7.14)$$

$$(-B_1(0,2,1,3;x)) - (-B_1(0,1,1,3;x)) = \frac{K_1}{D(0,2,1,3)}A_1(0,1,1,3;x) \quad (7.15)$$

Proof. Through simple calculations, we have

$$\begin{aligned} &BD(0,1,1,3;B_0(0,2,1,3;x) - B_1(0,1,1,3;x)) \\ &\quad = \{0, B_0'(0,2,1,3;0), 0, 0\} = \left\{0, \frac{K_3}{D(0,2,1,3)}, 0, 0\right\}. \end{aligned}$$

Together with the uniqueness theorem, we obtain (7.14). In the same way, making use of the boundary data,

$$BD(0,1,1,3;(-B_1(0,2,1,3;x)) - (-B_1(0,1,1,3;x))) = \left\{0, \frac{K_1}{D(0,2,1,3)}, 0, 0\right\}$$

we obtain (7.15). ■

Owing to Lemma (7.3), we have

$$\begin{aligned} &g(0,2,1,3;x,y) - g(0,1,1,3;x,y) \\ &\quad = \frac{1}{D(0,2,1,3)}(K_1K_3(1-x) - K_3K_1(1-x))A_1(0,1,1,3;y) \\ &\quad = (-A_1(0,2,1,3;x))A_1(0,1,1,3;y) > 0 \end{aligned} \quad (7.16)$$

which completes the proof of Theorem 7.4. ■

The final theorem is the most difficult to prove.

THEOREM 7.5.

$$
\begin{aligned}
&g(0,1,1,3;x,y)-g(0,1,0,2;x,y)\\
&\quad = B_0(0,1,1,3;x)(-B_1(0,1,0,2;y))\\
&\qquad -(-B_1(0,1,1,3;x))[p(-B_1(0,1,0,2;y))-B_0(0,1,0,2;y)]>0
\end{aligned}
\tag{7.17}
$$

Proof. Let $0<y<x<1$. Subtraction of (4.12) from (4.13) gives

$$
\begin{aligned}
g(0,1,1,3;x,y)-g(0,1,0,2;x,y) &= [\,A_1(0,1,1,3;x)-A_1(0,1,0,2;x)\,]K_1(y)\\
&\quad -[\,A_0(0,1,1,3;x)-A_0(0,1,0,2;x)\,]K_0(y)
\end{aligned}
\tag{7.18}
$$

We first prepare the following lemma.

LEMMA 7.4.

$$
\begin{aligned}
&A_0(0,1,1,3;x)-A_0(0,1,0,2;x)\\
&\quad = \frac{K_1B_0(0,1,1,3;x)-(pK_1-K_3)(-B_1(0,1,1,3;x))}{D(0,1,0,2)}
\end{aligned}
\tag{7.19}
$$

$$
\begin{aligned}
&A_1(0,1,1,3;x)-A_1(0,1,0,2;x)\\
&\quad = \frac{K_0B_0(0,1,1,3;x)-(pK_0-K_2)(-B_1(0,1,1,3;x))}{D(0,1,0,2)}
\end{aligned}
\tag{7.20}
$$

Proof. Straightforward calculations give

$$
\begin{aligned}
&BD(0,1,1,3;A_0(0,1,1,3;x)-A_0(0,1,0,2;x))\\
&\quad = \{0,0,-A_0'(0,1,0,2;1),A_0'''(0,1,0,2;1)\}\\
&\quad = \frac{1}{D(0,1,0,2)}\{0,0,K_1,pK_1-K_3\}\,,
\end{aligned}
$$

which proves (7.19). Similarly, we have

$$
\begin{aligned}
&BD(0,1,1,3;A_1(0,1,1,3;x)-A_1(0,1,0,2;x))\\
&\quad = \{0,0,-A_1'(0,1,0,2;1),-A_1'''(0,1,0,2;1)\}\\
&\quad = \frac{1}{D(0,1,0,2)}\{0,0,K_0,pK_0-K_2\}\,,
\end{aligned}
$$

which proves (7.20). ■

Employing Lemma 7.4, we have

$$
\begin{aligned}
&g(0,1,1,3;x,y)-g(0,1,0,2;x,y)\\
&= \frac{1}{D(0,1,0,2)}[\{K_0B_0(0,1,1,3;x)-(pK_0-K_2)(-B_1(0,1,1,3;x))\}K_1(y)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \{K_1 B_0(0,1,1,3;x) - (pK_1 - K_3)(-B_1(0,1,1,3;x))\} K_0(y)\big] \\
&= B_0(0,1,1,3;x)\frac{1}{D(0,1,0,2)}(K_0K_1(y) - K_1K_0(y)) \\
&\quad - (-B_1(0,1,1,3;x))\frac{1}{D(0,1,0,2)} \\
&\qquad \cdot \{p(K_0K_1(y) - K_1K_0(y)) - (K_2K_1(y) - K_3K_0(y))\} \\
&= B_0(0,1,1,3;x)(-B_1(0,1,0,2;y)) \\
&\quad - (-B_1(0,1,1,3;x))(p(-B_1(0,1,0,2;y)) - B_0(0,1,0,2;y))
\end{aligned} \tag{7.21}
$$

Finally, we prove that the right hand side of (7.17) is positive. Dividing its both sides by a positive function $(-B_1(0,1,1,3;x))(-B_1(0,1,0,2;y))$, we have

$$
\frac{g(0,1,1,3;x,y) - g(0,1,0,2;x,y)}{(-B_1(0,1,1,3;x))(-B_1(0,1,0,2;y))} = \frac{B_0(0,1,1,3;x)}{-B_1(0,1,1,3;x)} + \frac{B_0(0,1,0,2;y)}{-B_1(0,1,0,2;y)} - p \tag{7.22}
$$

Since a differential of the first term of the right hand side of (7.22) gives

$$
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{B_0(0,1,1,3;x)}{-B_1(0,1,1,3;x)}\right) = \frac{(K_2K_3 - K_1K_4)(K_1^2(x) - K_0(x)K_2(x))}{(K_1K_1(x) - K_2K_0(x))^2} > 0, \tag{7.23}
$$

we have

$$
\frac{B_0(0,1,1,3;x)}{-B_1(0,1,1,3;x)} = \frac{K_3K_1(x) - K_4K_0(x)}{K_1K_1(x) - K_2K_0(x)} \underset{x\downarrow 0}{\searrow} \frac{K_3}{K_1}. \tag{7.24}
$$

The limit value K_3/K_1 is obtained applying twice L'Hospital's theorem. This is the most delicate point of our proof. In the same way, a differential of the second term gives

$$
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{B_0(0,1,0,2;x)}{-B_1(0,1,0,2;x)}\right) = \frac{(K_1K_2 - K_0K_3)(K_1^2(x) - K_0(x)K_2(x))}{(K_0K_1(x) - K_1K_0(x))^2} > 0, \tag{7.25}
$$

from which one can find

$$
\frac{B_0(0,1,0,2;x)}{-B_1(0,1,0,2;x)} = \frac{K_2K_1(x) - K_3K_0(x)}{K_0K_1(x) - K_1K_0(x)} \underset{x\downarrow 0}{\searrow} \frac{K_2}{K_0}. \tag{7.26}
$$

Here L'Hospital's theorem is used twice again. Hence, we have from (7.22)

$$
\begin{aligned}
\frac{g(0,1,1,3;x,y) - g(0,1,0,2;x,y)}{(-B_1(0,1,1,3;x))(-B_1(0,1,0,2;y))} &\geq \frac{K_3}{K_1} + \frac{K_2}{K_0} - p \\
&= \frac{1}{2(a^2 - b^2)K_0K_1}\left(\frac{\mathrm{sh}(2a)}{a} - \frac{\mathrm{sh}(2b)}{b}\right) > 0,
\end{aligned} \tag{7.27}
$$

which completes the proof of Theorem 7.5. ■

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