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LORENTZ SPACES OF VECTOR-VALUED MEASURES

OSCAR BLASCO AND PABLO GREGORI

Abstract

Given a non-atomic, finite and complete measure space (Ω, Σ, μ) and a Banach space X, the modulus of continuity for a vector measure F is defined as the function $\omega_F(t) = \sup_{\mu(E) \leqslant t} |F|(E)$ and the space $V^{p,q}(X)$ of vector measures such that $t^{-1/p'}\omega_F(t) \in L^q((0, \mu(\Omega)], dt/t)$ is introduced. It is shown that $V^{p,q}(X)$ contains isometrically $L^{p,q}(X)$ and that $L^{p,q}(X) = V^{p,q}(X)$ if and only if X has the Radon–Nikodym property. It is also proved that $V^{p,q}(X)$ coincides with the space of cone absolutely summing operators from $L^{p',q'}$ into X and the duality $V^{p,q}(X^*) = (L^{p',q'}(X))^*$ where $1/p + 1/p' = 1/q + 1/q' = 1$. Finally, $V^{p,q}(X)$ is identified with the interpolation space obtained by the real method $(V^1(X), V^\infty(X))_{1/p',q}$. Spaces where the variation of F is replaced by the semivariation are also considered.

1. *Introduction*

Throughout this paper (Ω, Σ, μ) stands for a non-atomic, finite and complete measure space, X is a (complex or real) Banach space and X^* is its topological dual space. As usual p' denotes the conjugate exponent of p, that is, $1/p + 1/p' = 1$.

Given a complex-valued measurable function f we denote by μ_f the distribution function of f, $\mu_f(\lambda) = \mu(E_\lambda)$ for $\lambda > 0$ where $E_\lambda = \{w \in \Omega : |f(w)| > \lambda\}$, by f^* the non-increasing rearrangement of f, $f^*(t) = \inf\{\lambda : \mu_f(\lambda) \leqslant t\}$ and we write $f^{**}(t) = (1/t)\int_0^t f^*(s)\,ds$.

Then the Lorentz space $L^{p,q}$ consists of those measurable functions f such that $\|f\|^*_{pq} < \infty$, where

$$\|f\|^*_{pq} = \begin{cases} \left\{ \dfrac{q}{p} \displaystyle\int_0^\infty \left[t^{1/p} f^*(t)\right]^q \dfrac{dt}{t} \right\}^{1/q} & 0 < p < \infty,\ 0 < q < \infty, \\ \displaystyle\sup_{t>0} t^{1/p} f^*(t) & 0 < p \leqslant \infty,\ q = \infty. \end{cases}$$

Of course $L^{p,p} = L^p$ and if we put $\|f\|_{pq} = \|f^{**}\|^*_{pq}$ then we get an equivalent norm and $L^{p,q}$ for $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$ are Banach spaces.

Let us recall also that simple functions are dense in $L^{p,q}$ for $q \neq \infty$, and also the duality results, $(L^{p,1})^* = L^{p',\infty}$ for $1 \leqslant p < \infty$, as well as $(L^{p,q})^* = L^{p',q'}$ for $1 < p, q < \infty$.

The reader is referred to [**1**, **12**, **14**, **16**] for these results and for basic information on Lorentz spaces.

All these notions make sense also for vector-valued strongly measurable functions by replacing the modulus by the norm. This leads to the natural definition of Lorentz–Bochner spaces $L^{p,q}(X)$, where the norm is defined by $\|f\|_{L^{p,q}(X)} = \|\,\|f(\cdot)\|_X\|_{pq}$.

In the vector-valued case we still have the density of simple functions, but the corresponding duality $(L^{p,q}(X))^* = L^{p',q'}(X^*)$ only holds for Banach spaces X such

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that X^* has the Radon–Nikodym property. The reader is referred to [**8**] for a proof in the case $p = q$ or to [**10**, Theorem 3.2] for a proof in the more general case of the Köthe–Bochner space $E(X)$ for certain Banach lattices including $L^{p,q}$. An identification of the dual space without assumptions on X can be achieved from some general results on the dual of $E(X)$ where E is a Banach lattice (see [**6**] for a description in terms of weakly measurable functions or [**10**] for a formulation in terms of vector measures).

Since $L^p(X)$ coincides with $L^{p,p}(X)$, let us first mention here that in the particular case of Lebesgue–Bochner spaces $L^p(X)$ the dual can be represented as the space of X^*-valued measures of p'-bounded variation, denoted by $V^{p'}(X^*)$ (see [**9**]).

Our objective is to define a space of vector-valued measures in such a way that it contains $L^{p,q}(X)$ isometrically and that it coincides with $V^p(X)$ for $p = q$.

Following [**10**], one could define $V^{p,q}(X)$ as the space of vector measures such that $\sup_{\pi\in D} \| \sum_{A\in\pi}(F(A)/\mu(A))\chi_A\|_{pq} < \infty$ where the supremum is taken over the set D of all finite partitions π of Ω, but we would like to present a notion independent of the knowledge of Lorentz spaces of functions.

In this paper we present a natural definiton of a modulus of continuity of a vector measure (see Definition 2.1) which will allow to define the space $V^{p,q}(X)$ independently of the notion of $L^{p,q}(X)$ and which extends the previous definition for measures $dG = f d\mu$, and also coincides with the definition presented above (see Corollary 2.14).

In the case $q = \infty$, Marcinkiewicz spaces are denoted $\mathcal{V}^{p,\infty}(X)$ and $V^{p,\infty}(X)$ and defined by the existence of a constant $C > 0$ for which $\|F(A)\| \leqslant C\mu(A)^{1/p'}$ or $|F|(A) \leqslant C\mu(A)^{1/p'}$ for all $A \in \Sigma$.

To deal with the case $q < \infty$, we define two different moduli of continuity for a vector measure, namely $\widetilde{\omega}_F(t) = \sup_{\mu(E)\leqslant t}\|F(E)\|$ and $\omega_F(t) = \sup_{\mu(E)\leqslant t}|F|(E)$. Then we define the spaces $V^{p,q}(X)$ and $\mathcal{V}^{p,q}(X)$ consisting of vector measures such that $t^{-1/p'}\omega_F(t) \in L^q((0,\mu(\Omega)], dt/t)$ and $x^*F \in V^{p,q}(\mathbb{K})$ for all $x^* \in X^*$ respectively. Also a space where ω_F is replaced by $\widetilde{\omega}_F$ is considered.

The paper is divided into three sections.

In the first section, it is proved that $V^{p,q}(X)$ contains isometrically $L^{p,q}(X)$ and that $L^{p,q}(X) = V^{p,q}(X)$ if and only if X has the Radon–Nikodym property. It is also shown that $V^{p,q}(X^*)$ coincides with the dual of $L^{p',q'}(X)$. The next section deals with identification of the previous spaces of vector-valued measures as spaces of operators. In particular, we show that $\mathcal{V}^{p,q}(X)$ and $V^{p,q}(X)$ can be described as spaces of bounded operators from $L^{p',q'}$ into X and cone absolutely summing operators respectively.

In the last section, we describe the space as an interpolation space obtained by interpolation, using the real method, of two natural spaces of vector measures, namely $V^{p,q}(X) = (V^1(X), V^\infty(X))_{1/p',q}$ where $V^1(X)$ corresponds to the space of μ-continuous measures of bounded variation and $V^\infty(X)$ the subspace of those measures such that $\|F(A)\| \leqslant C\mu(A)$ for all $A \in \Sigma$.

2. *Marcinkiewicz and Lorentz spaces of vector measures*

Let us recall that the variation of a vector measure $F : \Sigma \to X$ at the set E is given by $|F|(E) = \sup_{\pi_E} \sum_{A\in\pi_E} \|F(A)\|$ (π_E stands for a finite partition of E and the supremum is taken over all such partitions) and the semivariation is given by

$\|F\|(A) = \sup_{\|x^*\|=1} |x^*F|(A)$. It is worth mentioning (see [**8**]) that

$$\|F\|(A) \approx \sup_{B\subset A} \|F(B)\|. \tag{1}$$

Let us first introduce the notion of 'modulus of continuity of a vector measure'.

DEFINITION 2.1. Let (Ω, Σ, μ) be a non-atomic finite measure space and write $I = (0, \mu(\Omega)]$. Let X be a Banach space and let F be an X-valued measure. We define, for $t \in I$, the functions

$$\widetilde{\omega}_F(t) = \sup_{\mu(E)\leqslant t} \|F(E)\| \quad \text{and} \quad \omega_F(t) = \sup_{\mu(E)\leqslant t} |F|(E).$$

REMARK 2.2. (a) Taking into account that μ is non-atomic, and using (1), one easily sees that for all $t \in I$,

$$\omega_F(t) = \sup_{\mu(E)=t} |F|(E) \quad \text{and} \quad \widetilde{\omega}_F(t) \approx \sup_{\mu(E)=t} \|F\|(E).$$

(b) $F \ll \mu$ if and only if $\lim_{t\to 0} \widetilde{\omega}_F(t) = 0$ and, for vector measures of bounded variation, it is also equivalent to $\lim_{t\to 0} \omega_F(t) = 0$.

(c) $\omega_F \equiv +\infty$ if and only if there exists $t \in I$ such that $\omega_F(t) = +\infty$ or, equivalently, F is not of bounded variation.

PROPOSITION 2.3. *If $f \in L^1(X)$ and $F(E) = \int_E f\, d\mu$ then $tf^{**}(t) = \omega_F(t)$ for all $t \in I$.*

Proof. The result follows easily from the facts that $|F|(E) = \int_E \|f\| d\mu$ and $\int_0^t f^*(s)\, ds = \sup_{\mu(E)\leqslant t} \int_E \|f\|\, d\mu$. $\square$

PROPOSITION 2.4. *Let F be a vector measure. Then either $\omega_F \equiv +\infty$ on I or ω_F is non-decreasing, continuous and concave.*

Proof. Let us assume that $\omega_F(t) < +\infty$ for some $t \in I$. Hence F has bounded variation and clearly ω_F is non-decreasing.

Let us see first that

$$\omega_F(s+h) - \omega_F(s) \geqslant \omega_F(t+h) - \omega_F(t) \tag{2}$$

for all $0 < s < t < \mu(\Omega)$ and $0 < h < \mu(\Omega) - t$.

Indeed, given $\varepsilon > 0$ and $t, t+h, s \in I$, there exist measurable sets E_t, E_{t+h} and E_s for which $\mu(E_t) = t, \mu(E_{t+h}) = t+h$ and $\mu(E_s) = s$ and

$$\omega_F(t) - |F|(E_t) < \varepsilon, \quad \omega_F(t+h) - |F|(E_{t+h}) < \varepsilon, \quad \omega_F(s) - |F|(E_s) < \varepsilon.$$

Let A_h be a measurable set with $\mu(A_h) = h$ and $A_h \subset E_{t+h} \backslash E_s$. Then

$$\begin{aligned}
\omega_F(s+h) &\geqslant |F|(E_s \cup A_h) = |F|(E_s) + |F|(A_h) \\
&\geqslant \omega_F(s) - \varepsilon + |F|(A_h) + |F|(E_{t+h} \backslash A_h) - \omega_F(t) \\
&\geqslant \omega_F(s) - \varepsilon + |F|(E_{t+h}) - \omega_F(t) \\
&\geqslant \omega_F(s) - \varepsilon + \omega_F(t+h) - \varepsilon - \omega_F(t).
\end{aligned}$$

Thus for any $\varepsilon > 0$ we obtain (2).

Therefore for any $s, t \in I$ we have

$$|\omega_F(s) - \omega_F(t)| \leqslant \omega_F(|s-t|^+) - \omega_F(0^+)$$

where $\omega_F(a^+) = \lim_{h\to 0^+} \omega_F(a+h)$.

Hence ω_F is uniformly continuous in I.

It is easy to see from (2) that, for $s, t \in I$,

$$\omega_F\left(\frac{s+t}{2}\right) \geqslant \frac{\omega_F(s) + \omega_F(t)}{2}.$$

This fact, together with the continuity, gives the concavity of ω_F. □

A measure F is said to belong to $V^p(X)$ (respectively $\mathscr{V}^p(X)$) if

$$\|F\|_p = \sup_{\pi\in D}\left(\sum_{A\in\pi} \frac{\|F(A)\|^p}{\mu(A)^{p-1}}\right)^{1/p} < \infty,$$

$$\left(\text{respectively } \||F|\|_p = \sup_{\pi\in D, \|x^*\|=1}\left(\sum_{A\in\pi} \frac{|x^*F(A)|^p}{\mu(A)^{p-1}}\right)^{1/p} < \infty\right),$$

where the supremum is taken over the set D of all finite partitions π of Ω.

In particular, if $F \in V^p(X)$ then $\|F(A)\| \leqslant C\mu(A)^{1/p'}$ for all measurable sets A.

Definition 2.5. Let (Ω, Σ, μ) be a measure space, X a Banach space and $1 < p \leqslant \infty$. Let us define by $\mathscr{V}^{p,\infty}(\mu, X)$ and $V^{p,\infty}(\mu, X)$ the spaces of X-valued measures for which there exists a constant $C > 0$ such that

$$\|F(A)\| \leqslant C\mu(A)^{1/p'} \text{ for any } A \in \Sigma \quad \left(\text{equivalently } \widetilde{\omega}_F(t) \leqslant Ct^{1/p'}\right)$$

and

$$|F|(A) \leqslant C\mu(A)^{1/p'} \text{ for any } A \in \Sigma \quad \left(\text{equivalently } \omega_F(t) \leqslant Ct^{1/p'}\right)$$

respectively.

We shall consider in these spaces the norms

$$\|F\|_{\mathscr{V}^{p,\infty}(X)} = \sup_{A\in\Sigma} \frac{\|F(A)\|}{\mu(A)^{1/p'}} = \sup_{t\in I} t^{-1/p'}\widetilde{\omega}_F(t),$$

$$\|F\|_{V^{p,\infty}(X)} = \sup_{A\in\Sigma} \frac{|F|(A)}{\mu(A)^{1/p'}} = \sup_{t\in I} t^{-1/p'}\omega_F(t).$$

We shall use the notation $\|\cdot\|_{p,\infty}$ when the context is not ambiguous, keeping the notation $\|\cdot\|_{p\infty}$ for vector-valued functions.

Remark 2.6. (a) For $p = \infty$ we clearly have $V^{\infty,\infty}(X) = \mathscr{V}^{\infty,\infty}(X)$ and the space coincides with $V^\infty(X)$ (see **[9]**).

(b) Observe that, since $1 < p \leqslant \infty$ and $\mu(\Omega) < \infty$, measures in $\mathscr{V}^{p,\infty}(\mu, X)$ are μ-continuous, and measures in $V^{p,\infty}(\mu, X)$ are of bounded variation.

(c) It is clear that, using (1), $F \in \mathscr{V}^{p,\infty}(\mu, X)$ if and only if

$$\|F\|(A) \leqslant C\mu(A)^{1/p'} \quad \text{for any } A \in \Sigma$$

if and only if

$$x^*F \in V^{p,\infty}(\mathbb{K}) \quad \text{for all } x^* \in X^*.$$

There are many ways to define spaces of vector measures which extend the Lorentz spaces. We shall try to find a natural way to define $V^{p,q}(X)$ in such a way that the map $f \mapsto dG = f d\mu$ defines an isometric embedding from $L^{p,q}(X)$ and that the space coincides with $V^p(X)$ and $V^{p,\infty}$ for $p = q$ and $q = \infty$ respectively.

Definition 2.7. Let X be a Banach space, $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$. Let us define $V^{p,q}(\mu, X)$ and $\widetilde{\mathscr{V}}^{p,q}(\mu, X)$ as the spaces of vector measures such that

$$t^{-1/p'}\omega_F(t) \in L^q\left(I, \frac{dt}{t}\right) \quad \text{and} \quad t^{-1/p'}\widetilde{\omega}_F(t) \in L^q\left(I, \frac{dt}{t}\right)$$

respectively. We consider then the norms

$$\|F\|_{V^{p,q}(X)} = \left(\int_I \left(t^{-1/p'}\omega_F(t)\right)^q \frac{dt}{t}\right)^{1/q}$$

and

$$\|F\|_{\widetilde{\mathscr{V}}^{p,q}(X)} = \left(\int_I \left(t^{-1/p'}\widetilde{\omega}_F(t)\right)^q \frac{dt}{t}\right)^{1/q}.$$

We shall use the notation $\|\cdot\|_{p,q}$ when the context is not ambiguous, keeping the notation $\|\cdot\|_{pq}$ for vector-valued functions.

Definition 2.8. Let X be a Banach space, $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$. Let us define $\mathscr{V}^{p,q}(\mu, X)$ as the space of vector measures $F : \Sigma \to X$ such that $x^*F \in V^{p,q}(\mathbb{K})$, for all $x^* \in X^*$. We consider then the norm

$$\|F\|_{\mathscr{V}^{p,q}(X)} = \sup_{\|x^*\| \leqslant 1} \|x^*F\|_{V^{p,q}(\mathbb{K})}.$$

Let us now recollect some elementary results about these spaces.

Proposition 2.9. *Let $1 < p \leqslant \infty$ and $1 \leqslant q \leqslant \infty$. Then the following hold:*
(a) *$(\mathscr{V}^{p,q}(X), \|\cdot\|_{\mathscr{V}^{p,q}(X)})$ and $(V^{p,q}(X), \|\cdot\|_{V^{p,q}(X)})$ are Banach spaces.*
(b) *$V^p(X) \subset V^{p,\infty}(X) \subset \mathscr{V}^{p,\infty}(X)$.*
(c) *$V^{p,q}(\mu, X) \subset \widetilde{\mathscr{V}}^{p,q}(X) \subset \mathscr{V}^{p,q}(X)$.*
(d) *$V^{p,q}(X) \subset V^{p,\infty}(X)$ and $\mathscr{V}^{p,q}(X) \subset \mathscr{V}^{p,\infty}(X)$.*

Proof. The proofs of (a), (b) and (c) are easy and left to the reader.
To see (d) note that, since ω_F is non-decreasing, integrating $s^{-q/p'-1}$ from t to $\mu(\Omega)$, we have

$$t^{-1/p'}\omega_F(t) \leqslant C_1 \left(\int_t^{\mu(\Omega)} \left(s^{-1/p'}\omega_F(s)\right)^q \frac{ds}{s}\right)^{1/q} + C_2,$$

for some constants $C_1, C_2 > 0$. A similar estimate holds for $\widetilde{\omega}_F(t)$. □

Example 2.10. Let $1 < p, q < \infty$ and let us take $X = L^{p,q}(\mu, \Omega)$. Consider the $L^{p,q}$-valued measure F given by $F(E) = \chi_E$ for $E \in \Sigma$. Then the following hold:
(i) $\omega_F(t) = +\infty$ for $t \in I$.
(ii) $\widetilde{\omega}_F(t) = t^{1/p}$ for $t \in I$.
(iii) For any $x^* = \phi \in X^* = L^{p',q'}$, the measure $x^*F(E) = \int_E \phi\, d\mu$ for $E \in \Sigma$.

Therefore the following hold:
(i′) $F \notin V^{r,s}(X)$ for any r, s.

(ii′) $F \in \widetilde{\mathscr{V}}^{r,s}(X)$ if and only if $1 \leqslant r \leqslant p'$ and $s = \infty$ or $1 \leqslant r < p'$ and $1 \leqslant s \leqslant \infty$.
(iii′) $F \in \mathscr{V}^{r,s}(X)$ if and only if $r = p'$ and $s \geqslant q'$ or $1 \leqslant r < p'$.

In particular, there exist $F_1 \in \mathscr{V}^{p',q'}(X)\backslash\widetilde{\mathscr{V}}^{p',q'}(X)$ and $F_2 \in \mathscr{V}^{p',\infty}(X)\backslash V^{p',\infty}(X)$.

The importance of the following lemma is evident from the several references to it in the subsequent results.

Lemma 2.11. *Let $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$. The following are equivalent:*
(a) $F \in V^{p,q}(X)$.
(b) *There exists $\varphi \geqslant 0$, $\varphi \in L^{p,q}$ such that $|F|(A) = \int_A \varphi\, d\mu$ for all $A \in \Sigma$.*
Moreover if $|F|(A) = \int_A \varphi\, d\mu$ for all $A \in \Sigma$ for some $\varphi \geqslant 0$ then $\|F\|_{V^{p,q}(X)} = \|\varphi\|_{pq}$.

Proof. Assume that (a) holds and that $q = \infty$. Let us take $F \in V^{p,\infty}(X)$. Then using the Radon–Nikodym theorem we find a non-negative function φ such that $|F|(A) = \int_A \varphi\, d\mu$ for all $A \in \Sigma$. Now observe that

$$\begin{aligned}\|F\|_{V^{p,\infty}(X)} &= \sup_{A\in\Sigma} \frac{|F|(A)}{\mu(A)^{1/p'}} = \sup_{t>0}\left(\sup_{\mu(A)=t} \frac{|F|(A)}{t^{1/p'}}\right)\\ &= \sup_{t>0}\left(t^{-1/p'} \sup_{\mu(A)=t}\int_A \varphi\, d\mu\right) = \sup_{t>0} t^{-1/p'}\int_0^t \varphi^*(s)\, ds\\ &= \sup_{t>0} t^{1/p}\varphi^{**}(t) = \|\varphi\|_{p\infty}.\end{aligned}$$

For $q < \infty$ if $F \in V^{p,q}(X)$ then $F \in V^{p,\infty}(X)$. From the previous case we find a non-negative function φ such that $|F|(A) = \int_A \varphi\, d\mu$ for all $A \in \Sigma$. Now, with a look at Proposition 2.3, it is plain to see that $\|F\|_{p,q} = \|\varphi\|_{pq}$.
The converse also follows from Proposition 2.3. □

With the help of Lemma 2.11 we can prove the following.

Theorem 2.12. *Let X be a Banach space and $1 < p < \infty, 1 \leqslant q \leqslant \infty$. Then the following hold:*
(a) *$L^{p,q}(X)$ is isometrically embedded into $V^{p,q}(X)$.*
(b) *$L^{p,q}(X) = V^{p,q}(X)$ if and only if X has the Radon–Nikodym property.*

Proof. Let $f \in L^{p,q}(X)$, then the measure $F(E) = \int_E f\, d\mu$ belongs to $V^{p,q}(X)$. This follows from the fact that $|F|(E) = \int_E \|f\|\, d\mu$ and Lemma 2.11 for $\varphi = \|f\|$.
To prove (b), let us assume that $L^{p,q}(X) = V^{p,q}(X)$ and let $T : L^1 \to X$ be a bounded operator. We need to show that T is representable (see [**8**]). Note that $F(E) = T(\chi_E)$ gives a measure in $V^\infty(X)$, and then $F \in V^{p,q}(X)$. Now by the assumption there exists $f \in L^{p,q}(X)$ such that $F(E) = \int_E f\, d\mu$ for all $E \in \Sigma$. This shows that $T(\psi) = \int \psi f\, d\mu$ for all $\psi \in L^1$.
Conversely, let us assume that X has the Radon–Nikodym property and take $F \in V^{p,q}(X)$. Because F is μ-continuous and with bounded variation, $F(E) = \int_E f\, d\mu$ for all $E \in \Sigma$ for some $f \in L^1(X)$. To show that $f \in L^{p,q}(X)$, apply Lemma 2.11 for $\varphi = \|f\|$ again. □

LEMMA 2.13. *Let $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$ or $p = q = \infty$. If $F \in V^{p,q}(X)$ then*

$$\|F\|_{V^{p,q}(X)} \approx \sup\left\{ \sum_i |\alpha_i| \, \|F(A_i)\| : \left\| \sum_i \alpha_i \chi_{A_i} \right\|_{p'q'} \leqslant 1 \right\}$$

$$= \sup\left\{ \sum_i |\alpha_i| \, |F|(A_i) : \left\| \sum_i \alpha_i \chi_{A_i} \right\|_{p'q'} \leqslant 1 \right\}.$$

Proof. If φ is the function in Lemma 2.11, we use [**1**, Theorem 4.7, p. 220] on the dual norm in the space $L^{p,q}$, to show that $\|F\|_{p,q} = \|\varphi\|_{pq}$ is equivalent to $\sup\{\int_\Omega \varphi |f| \, d\mu : \|f\|_{p'q'} 1\}$.

Hence

$$\|F\|_{p,q} \approx \sup\left\{ \int_\Omega \varphi\left(\sum_i |\alpha_i| \chi_{A_i} \right) d\mu : f = \sum_i \alpha_i \chi_{A_i}, \|f\|_{p'q'} \leqslant 1 \right\}$$

$$= \sup\left\{ \sum_i |\alpha_i| \int_{A_i} \varphi \, d\mu : \left\| \sum_i \alpha_i \chi_{A_i} \right\|_{p'q'} \leqslant 1 \right\}$$

$$= \sup\left\{ \sum_i |\alpha_i| |F|(A_i) : \left\| \sum_i \alpha_i \chi_{A_i} \right\|_{p'q'} \leqslant 1 \right\}$$

$$= \sup\left\{ \sum_i |\alpha_i| \|F(A_i)\| : \left\| \sum_i \alpha_i \chi_{A_i} \right\|_{p'q'} \leqslant 1 \right\}.$$

The equality of both suprema in the last step follows easily from the definition of variation $|F|(A_i)$. ☐

COROLLARY 2.14. *Let $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$. Then $F \in V^{p,q}(X)$ if and only if*

$$\sup_{\pi \in D} \left\| \sum_{A \in \pi} \frac{F(A)}{\mu(A)} \chi_A \right\|_{pq} < \infty$$

where the supremum is taken over the set D of all finite partitions π of Ω.

In particular, $V^{p,p}(X) = V^p(X)$ for all $1 < p \leqslant \infty$.

Proof. Let π be a partition in D. Then

$$\left\| \sum_{A \in \pi} \frac{F(A)}{\mu(A)} \chi_A \right\|_{L^{p,q}(X)} = \left\| \sum_{A \in \pi} \frac{\|F(A)\|}{\mu(A)} \chi_A \right\|_{L^{p,q}}$$

$$= \sup\left\{ \sum_{A \in \pi} \frac{\|F(A)\|}{\mu(A)} \int_A \psi \, d\mu : \|\psi\|_{p',q'} = 1 \right\}$$

$$= \sup\left\{ \sum_{A \in \pi} \alpha_A \|F(A)\| : \left\| \sum_{A \in \pi} \alpha_A \chi_A \right\|_{p',q'} = 1 \right\}.$$

In the last equality we have used the fact that $E_\pi(\psi) = \sum_{A \in \pi} \int_A \psi \, d\mu / \mu(A) \chi_A$ defines a bounded operator of norm 1 in $L^{p',q'}$. ☐

Now, using the convenient results for spaces of measurable functions, we obtain the following embeddings.

COROLLARY 2.15. *Let $1 < p < \infty$, $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$, $1 < p_1 \leqslant p_2 < \infty$ and $1 \leqslant q$, $r \leqslant \infty$. Then the following hold:*

(a) $V^{p,q_1}(X) \subset V^{p,q_2}(X)$, $\widetilde{\mathcal{V}}^{p,q_1}(X) \subset \widetilde{\mathcal{V}}^{p,q_2}(X)$ *and* $\mathcal{V}^{p,q_1}(X) \subset \mathcal{V}^{p,q_2}(X)$.

(b) $V^{p_2,q}(X) \subset V^{p_1,r}(X)$, $\widetilde{\mathcal{V}}^{p_2,q}(X) \subset \widetilde{\mathcal{V}}^{p_1,r}(X)$ *and* $\mathcal{V}^{p_2,q}(X) \subset \mathcal{V}^{p_1,r}(X)$.

Our next step is the description of the dual of the vector-valued Lorentz function spaces in terms of vector measure spaces.

THEOREM 2.16. *Let $1 < p < \infty$ and $1 \leqslant q < \infty$ or $p = q = 1$. Then $V^{p',q'}(X^*) = [L^{p,q}(X)]^*$.*

Proof. Let $F \in V^{p',q'}(X^*)$ and ϕ_F be the functional over $L^{p,q}(X)$ given as usual by $\phi_F(\sum_i x_i\chi_{A_i}) = \sum_i \langle x_i, F(A_i)\rangle$. We have

$$\left|\phi_F\left(\sum_i x_i\chi_{A_i}\right)\right| \leqslant \sum_i \|x_i\|\,\|F(A_i)\|$$

for every simple function. Then Lemma 2.13 gives $\|\phi_F\| \leqslant C\|F\|_{p',q'}$ for some constant $C > 0$.

Now let $\phi \in [L^{p,q}(X)]^*$. For each $A \in \Sigma$, $F_\phi(A)$ is the element of X^* such that $F_\phi(A)(x) = \phi(x\chi_A)$. Observe that F is well-defined and a countably additive measure (by continuity of ϕ). It is plain that $\phi = \phi_{F_\phi}$. Then, we can conclude, since

$$\begin{aligned}
\|F_\phi\|_{p',q'} &\leqslant C' \sup\left\{\sum_i |\alpha_i|\ \|F(A_i)\| : \left\|\sum_i \alpha_i\chi_{A_i}\right\|_{p,q} \leqslant 1\right\}\\
&= C' \sup\left\{\sum_i |\alpha_i|\left|\langle x_i, F(A_i)\rangle\right| : \left\|\sum_i \alpha_i\chi_{A_i}\right\|_{p,q} \leqslant 1,\ \|x_i\| \leqslant 1\ \forall\ i\right\}\\
&= C' \sup\left\{\left|\sum_i \langle x_i, F(A_i)\rangle\right| : f = \sum_i x_i\chi_{A_i} \in L^{p,q}(X),\ \|f\|_{p,q} \leqslant 1\right\}\\
&= C' \sup\left\{\left|\phi_F\left(\sum_i x_i\chi_{A_i}\right)\right| : f = \sum_i x_i\chi_{A_i} \in L^{p,q}(X),\ \|f\|_{pq} \leqslant 1\right\}\\
&= C'\|\phi\|.
\end{aligned}$$

□

COROLLARY 2.17. *Let $1 < p, q < \infty$. Then $[L^{p,q}(X)]^* = L^{p',q'}(X^*)$ if and only if X^* has the Radon–Nikodym property.*

3. *Vector measures and operators*

We comment that a vector measure F provides us with a linear operator T_F acting on characteristic functions as $T_F(\chi_A) = F(A)$, and it can be obviously extended by linearity to the set of step functions.

In this way we clearly identify

$$V^{\infty}(X) = \mathscr{V}^{\infty}(X) = L(L^1, X).$$

In this section we analyse the cases $\mathscr{V}^{p,q}(X)$ and $V^{p,q}(X)$.

To describe the spaces $\mathscr{V}^{p,q}(X)$ in terms of operators, we need the following lemma.

LEMMA 3.1. *Let $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$. If $F \in \mathscr{V}^{p,q}(X)$ then*

$$\|F\|_{\mathscr{V}^{p,q}(X)} \approx \sup\left\{\left\|\sum_i \alpha_i F(A_i)\right\| : \left\|\sum_i \alpha_i \chi_{A_i}\right\|_{p'q'} \leqslant 1\right\}.$$

Proof. Using Lemma 2.13 for the particular case $X = \mathbb{K}$ we have

$$\begin{aligned}
\|F\|_{\mathscr{V}^{p,q}(X)} &= \sup_{\|x^*\|\leqslant 1} \|x^* F\|_{V^{p,q}} \\
&\simeq \sup\left\{\sum_i |\alpha_i||x^* F(A_i)| : \left\|\sum_i \alpha_i \chi_{A_i}\right\|_{p'q'} \leqslant 1,\ \|x^*\| \leqslant 1\right\} \\
&= \sup\left\{\left|\left\langle \sum_i \alpha_i F(A_i), x^*\right\rangle\right| : \left\|\sum_i \alpha_i \chi_{A_i}\right\|_{p'q'} \leqslant 1,\ \|x^*\| \leqslant 1\right\} \\
&= \sup\left\{\left\|\sum_i \alpha_i F(A_i)\right\| : \left\|\sum_i \alpha_i \chi_{A_i}\right\|_{p'q'} \leqslant 1\right\}.
\end{aligned}$$

□

The following results are straightforward corollaries.

THEOREM 3.2. *Let $1 < p < \infty$ and $1 \leqslant q < \infty$. Then*

$$\mathscr{V}^{p',q'}(X) = L(L^{p,q}, X).$$

COROLLARY 3.3. *Let $1 < p < \infty$ and $1 \leqslant q < \infty$. Then*

$$\mathscr{V}^{p',q'}(X^*) = (L^{p,q} \hat{\otimes}_\pi X)^*.$$

From Theorem 3.2, $V^{p',q'}(X)$ is a subspace of the space of operators from $L^{p,q}$ to X. If we want to understand the corresponding class of operators we need to recall the following notion.

DEFINITION 3.4 (see [**15**, p. 244]). Let E be a Banach lattice and B a Banach space. A linear operator from E to B is said to be *cone absolutely summing* if there is a constant $C > 0$ such that for every $k \in \mathbb{N}$ and every family $e_1, e_2, \ldots, e_k \in E$ of positive elements we have

$$\sum_{i=1}^{k} \|T(e_k)\|_B \leqslant C \sup_{\|e^*\|_{E^*}\leqslant 1} \sum_{i=1}^{k} |\langle e_i, e^*\rangle|.$$

We denote by $\Pi_+^1(E, B)$ the set of such operators and its norm is given by the infimum of the constants C satisfying the previous inequality.

Remark 3.5. It is rather easy to give an equivalent defintion (see [**15**, p. 244]) by using that if $e_1, e_2, \ldots, e_k$ are positive elements in E then

$$\sup_{\|e^*\|_{E^*} \leqslant 1} \sum_{i=1}^{k} |\langle e_i, e^* \rangle| = \left\| \sum_{i=1}^{k} e_i \right\|. \tag{3}$$

Let us mention here that $L^{p,q}$ are, clearly, Banach lattices and the next theorem allows us to identify the space $\Pi_+^1(L^{p,q}, X)$.

Theorem 3.6. *Let $1 < p < \infty$ and $1 \leqslant q < \infty$ or $p = q = 1$. Then $V^{p',q'}(X) = \Pi_+^1(L^{p,q}, X)$.*

Proof. Let $F \in V^{p',q'}(X)$ and $T_F : L^{p,q} \to X$ be defined as explained at the beginning of this section. Let us see that $T \in \Pi_+^1(L^{p,q}, X)$. If $N \in \mathbb{N}$ and $f_1, f_2, \ldots, f_N$ are non-negative functions in $L^{p,q}$, then Lemma 2.11 gives the existence of $\varphi \in L^{p',q'}$ such that for all $f \in L^{p,q}$

$$\|T_F(f)\| \leqslant \int_\Omega |f| \varphi d\mu.$$

Therefore

$$\sum_{n=1}^{N} \|T_F(f_n)\| \leqslant \sum_{n=1}^{N} \left(\int_\Omega f_n \varphi \, d\mu \right) = \int_\Omega \left(\sum_{n=1}^{N} f_n \right) \varphi \, d\mu \leqslant \left\| \sum_{n=1}^{N} f_n \right\|_{p,q} \|F\|_{p',q'}.$$

Hence, from (3), one obtains $\|T_F\|_{\Pi_+^1} \leqslant \|F\|_{p',q'}$.

Now if $T \in \Pi_+^1(L^{p,q}, X)$ and $F_T : \Sigma \to X$ is defined by $F_T(A) = T(\chi_A)$, it is obvious that $T = T_{F_T}$ and F_T is countably additive. Let $f = \sum \alpha_i \chi_{A_i} \in L^{p,q}$ with $\|f\|_{p,q} \leqslant 1$, then

$$\sum_i |\alpha_i| \, \|F_T(A_i)\| = \sum_i \|T(|\alpha_i| \chi_{A_i})\| \leqslant \|T\|_{\Pi_+^1} \left\| \sum_i |\alpha_i| \chi_{A_i} \right\|_{p,q} \leqslant \|T\|_{\Pi_+^1}.$$

Therefore $\|F_T\|_{p',q'} \leqslant C \|T\|_{\Pi_+^1}$, and the proof is complete. □

4. *Lorentz spaces and interpolation*

We refer the reader to [**1**] or [**2**], where a wide study of interpolation spaces is developed.

Definition 4.1 (the K-functional). Let (X_0, X_1) be a compatible couple of Banach spaces. The *K-functional* can be defined for every $f \in X_0 + X_1$ and $t > 0$ by

$$K(f, t; X_0, X_1) = \inf \left\{ \|f_0\|_{X_0} + t \|f_1\|_{X_1} : f = f_0 + f_1 \right\},$$

where the infimum is taken from all the possible representations $f = f_0 + f_1$ of f with $f_0 \in X_0$ and $f_1 \in X_1$.

Theorem 4.2 [**1**, p. 298]. *Let (Ω, Σ, μ) be a totally σ-finite measure space, then*

$$K(f, t; L^1, L^\infty) = \int_0^t f^*(s) \, ds = t f^{**}(t).$$

DEFINITION 4.3 [1, p. 299]. Let (X_0, X_1) be a compatible couple and suppose that $0 < \theta < 1$, $1 \leqslant q < \infty$ or $0 \leqslant \theta \leqslant 1$, $q = \infty$. The space $(X_0, X_1)_{\theta,q}$ consists of all f in $X_0 + X_1$ for which the functional

$$\|f\|_{\theta,q} = \begin{cases} \left\{ \int_0^\infty [t^{-\theta} K(f,t)]^q \frac{dt}{t} \right\}^{1/q} & 0 < \theta < 1,\ 1 \leqslant q < \infty, \\ \sup_{t>0} t^{-\theta} K(f,t) & 0 \leqslant \theta \leqslant 1,\ q = \infty \end{cases}$$

is finite (here $K(f,t) := K(f,t; X_0, X_1)$).

Let us denote by $V^1(X)$ the space of μ-continuous vector measures of bounded variation and write $\|F\|_{V^1(X)} = |F|(\Omega)$.

Of course, $V^\infty(X) \subset V^1(X)$ and $V^{p,q}(X) \subset V^1(X)$ for all $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$.

THEOREM 4.4. *Let $F \in V^1(X)$ and $t > 0$. Then*

$$\omega_F(t) \leqslant K(F, t; V^1(X), V^\infty(X)) \leqslant 2\omega_F(t).$$

Proof. Assume that $F = G + H$ with $G \in V^1(X)$ and $H \in V^\infty(X)$. It follows from $|F| \leqslant |G| + |H|$ that $\omega_F(t) \leqslant \omega_G(t) + \omega_H(t)$.

Note that

$$\omega_G(t) = \sup_{\mu(E)\leqslant t} |G|(E) \leqslant |G|(\Omega) = \|G\|_{V^1(X)}$$

and

$$\omega_H(t) = \sup_{\mu(E)\leqslant t} |H|(E) \leqslant t \sup_{\mu(E)\leqslant t} \frac{|H|(E)}{\mu(E)} = t\|H\|_{V^\infty(X)}.$$

Then we get the first estimate by taking the infimum over all decompositions.

Assume now that $\omega_F(t)$ is finite. Since $|F|$ is a μ-continuous measure, we can find a function $\varphi \geqslant 0$ such that $|F|(A) = \int_A \varphi \, d\mu$ for every measurable set A.

We take $E = \{w \in \Omega : \varphi(w) > \varphi^*(t)\}$. Since μ_φ and φ^* are mutually right-continuous inverse functions, then $\mu(E) \leqslant t$.

Let us define $G(A) = F(E \cap A)$ for $A \in \Sigma$. In this case $|G|(A) = \int_A \varphi_G \, d\mu$ for all A, where $\varphi_G = \varphi\chi_E$. It is easy to check that

$$\varphi_G^*(s) = \begin{cases} \varphi^*(s) & 0 < s < \mu(E), \\ 0 & s \geqslant \mu(E) \end{cases}$$

and from this we can deduce that $\|G\|_{V^1(X)} = \|\varphi_G\|_{L^1} = \int_0^{\mu(E)} \varphi^*(s) \, ds$.

Defining $H(A) = F(A) - G(A) = F((\Omega \backslash E) \cap A)$ for all $A \in \Sigma$, we have $|H|(A) = \int_A \varphi_H \, d\mu$ for every A, where $\varphi_H = \varphi\chi_{\Omega\backslash E}$. It is clear that

$$\mu_{\varphi_H}(\lambda) = \begin{cases} \mu_\varphi(\lambda) - \mu_\varphi(\varphi^*(t)) & 0 < \lambda \leqslant \varphi^*(t), \\ 0 & \lambda > \varphi^*(t). \end{cases}$$

Therefore

$$\varphi_H^*(s) = \varphi^*(s + \mu_\varphi(\varphi^*(t))).$$

Hence

$$\sup_A \frac{|H|(A)}{\mu(A)} \leqslant \sup_{s>0} \frac{\int_0^s \varphi_H^*(\theta)\,d\theta}{s} \leqslant \lim_{s\to 0} \varphi_H^*(s) = \varphi^*(\mu_\varphi(\varphi^*(t))) \leqslant \varphi^*(t)$$

and consequently $t\|H\|_{V^\infty(X)} \leqslant t\varphi^*(t)$.

Now, using the decomposition of F and the properties of φ^*, we obtain

$$\|G\|_{V^1(X)} + t\|H\|_{V^\infty(X)} \leqslant \int_0^{\mu(E)} \varphi^*(s)\,ds + t\varphi^*(t) \leqslant 2\int_0^t \varphi^*(s)\,ds = 2\omega_F(t).$$

□

THEOREM 4.5. *If $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$, then*

$$V^{p,q}(X) = (V^1(X), V^\infty(X))_{\theta,q},$$

where $1/p = 1 - \theta$.

Using the reiteration theorem (see [**1**, p. 311]) and Corollary 2.14 we obtain the following.

THEOREM 4.6. *If $1 < p_1, p_2 < \infty$ and $1 \leqslant q \leqslant \infty$, then*

$$V^{p,q}(X) = (V^{p_1}(X), V^{p_2}(X))_{\theta,q},$$

where $1/p = (1-\theta)/p_1 + \theta/p_2$.

Let us mention something about the interpolation when we also change the spaces in which the measures take values. For the difference between $(V^{p_1}(X_1), V^{p_2}(X_2))_{\theta,q}$ and $V^{p,q}((X_1, X_2)_{\theta,q})$ we recall that $V^p(X) = L^p(X)$ if X has the Radon–Nikodym property and simply refer the reader to the paper by M. Cwikel [**7**] where it is shown that for spaces of vector-valued functions the equality may happen only for $q = p$ where $1/p = 1 - \theta/p_1 + \theta/p_2$.

Even in the case $q = p$ the expected interpolation result does not hold for vector-valued measures. The reader is referred to [**5**] for the difference between $(V^{p_1}(X_1), V^{p_2}(X_2))_{\theta,p}$ and $V^{p,q}((X_1, X_2)_{\theta,p})$. Although there the authors deal with interpolation between spaces of vector-valued harmonic functions, instead of vector-valued measures, they can be identified according to the results in [**4**] or [**3**].

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Oscar Blasco and Pablo Gregori
Departamento de Análisis Matemático
Universidad de Valencia
Doctor Moliner, 50
E-46100 Burjassot (Valencia)
Spain

oscar.blasco@uv.es
pablo.gregori@uv.es