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Pension funds insurance individuation

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Abstract. This paper considers an individual's problem and effects on savings for retirement. We show using a deterministic model, a linear utility function and assuming an individual can benefit from a tax break in savings, that under reasonable circumstances, there is only one switch from not saving to starting to save. Further, we draw some implications for the effects of income inequalities expressed by individuals tax brackets on inequality after retirement. Due to the model's simplicity, further research is required which will consider the uncertain effects of income and the future on current savings decisions.

Key words: Insurance, Savings, Control, Pensions

1 Introduction

Pension savers tend to underestimate the financial requirements to support future needs. This may lead to future economic distress as well as to a growing social condition where governments signal their inability to meet future needs. There are further darker clouds pending over the future of pension funds and the need to overhaul past practices. This has a particularly important effect on persons that do not have sufficient wealth to support their future needs and rely mostly on withdrawals made by employers and payments made to social security. This state of affairs has led the search to new approaches in saving rates and saving modes. Explicitly, the traditional pension saving approach consisting in investment in fixed income obligations, generally issued and backed by the government is now amended to provide greater investment flexibility. This flexibility, it is believed, may be much more responsive to market forces on the one hand and better tailored to individual savers needs on the other.

Individuation of pension savings is applied in a number of countries in several forms. In France for example, some pension funds allocate "points to savers" according to their contribution, the value of these points however is recalculated each year based on the aggregate performance of the fund. In the US, TIAA and CREF retirement funds allow savers to select part of their retirement income in terms of several investment vehicles, each exhibiting different risk profiles. Such decisions may be binding over certain periods of time. The current trend in the US is to allow greater flexibility in funds investments by diverting some pre-specified proportion of the fund to the stock market. In Israel, as well as in many other countries, the insufficiency of fixed income retirement plans guaranteed by the government are increasingly compensated by the creation of complementary pension savings that benefit of tax advantages (such as tax defferal schemes and outright tax deductions for parts of the investment). Although these schemes are increasingly popular and varied, they share a common trend of "disintermediation" of pension savings, reducing both the responsibility and the social role of governments in the guarantee of pensions and increased "individuation" of pension schemes – much more tuned to individual possibilities and needs. These trends have a major impact on investment policies of pension funds, on their portfolio and of course express a far more liberal view of social policy.

There are many reasons for governments' disassociation of social and retirement pension funds and their active support to pension individuation. Among the many possible reasons, early retirement, a structural growth in unemployment, increased life expectancy, the inflow of more women in the labor market, a trend towards lower interest rates and so on combine to increase the cost of national pension funds insurance and their dwindling economic opportunities. The credo for smaller government, privatization and disengagement by governments, the growth of global financial markets are also reducing the needs for deficit financing through guaranteed payments to pension funds which have been in the past an important source of current budget deficit financing. This paper provides a systematic assessment of some of these issues by considering a simple deterministic dynamic model. The results obtained provide the conditions for savings. Due to the model's simplicity, further research is required which will consider the uncertain effects of income and the future on current savings decisions.

2 A dynamic model of individual savings and retirement

Assume a saver whose tax bracket is known and given by $\tau > 0$. Let $S(t)$ be the current account state of the saver at time t (this may stand for the number of point accumulated by the saver, the outstanding amount of money saved and accumulated over time etc.). Funds are accumulated by individual savers in three ways: First, through an obligatory payment (usually, paid for by the employer and determined as a function of the saver's salary), $a(w, t)$ where $w(t)$ is the individual salary at time t. Second, through a voluntary contribution $v(t)$, measured in dollars and contributed by the individual saver. This contribution, has usually some tax advantages usually regulated by governments that seek to provide an incentive for retirement saving. For our purpose, we state that the amount contributed at a given time is paid continuously and is proportional to the salary, or

$$0 \leq v(t) \leq v^* \tag{1}$$

where $0 < v^* \leq 1$ is a parameter expressing the maximal amount a saver can devote to a pension fund at a given time. Third, investment yield of the fund generates an income that is reinvested. For our purpose, we shall assume that the investment yield is known and given by a rate of return ρ. As a result, we can represent the saving's process by the following differential equation:

$$\frac{dS(t)}{dt} = a(w,t) + \theta(t)v(t)w(t) + \rho S(t); \quad S(0) = S_0 \tag{2}$$

where $\theta(t)$ is a parameter expressing the transaction cost associated to an individual contribution. Now assume that at time T, a retirement time, the saver stops working-contributing, and collects a pension over his remaining life time, which is a function of the accumulated fund at time T, $S(T)$. In particular, if $1 - F(z)$ is the probability that the saver is still alive at time z (after retirement), then the current value at retirement of the pension is given by:

$$\int_T^\infty \alpha(S(T))[1 - F(z)]e^{-r(z-T)}\,dz \tag{3}$$

which we write for simplicity by $\pi\alpha(S(T))$, $\pi = \int_T^\infty [1 - F(z)]e^{-r(z-T)}\,dz$ where r is a riskless discount rate expressing the time value of money for the saver. The rate thus expresses the valuation of future consumption relative to the present one. As a result, we can formulate an individual saver's objective by the following:

$$\underset{v(t)\in V}{\text{Max}}\; V(T) = e^{-rT}\pi\alpha(S(T)) + \int_0^T Q(t)e^{-rt}\,dt \tag{4}$$

where $Q(t)$ denotes the utility of consumption of the individual saver. Let $c(t)$ be consumption, then if all disposable income is consumed, we have $w(t) = c(t) + (1-\tau)v(t)w(t)$ and therefore,

$$Q(t) = u(c(t)) = u(w(t)(1 - (1-\tau)v(t))) \tag{5}$$

where $u(.)$ is a utility function while $(1-\tau)v(t)w(t)$ is the individual's voluntary contribution net of taxes. The wealthier the individual the larger the taxation rate. A solution of the optimal control problem defined by (1)–(4) provides initial results for discussing the individual's saving problem. We shall summarize these results by the following proposition proved below. Further, since there is no uncertainty, we consider a linear utility function.

Proposition 1: *Assume a linear utility of consumption and for simplicity, assume fixed wage rates, fixed tax brackets. Then:*

(a) *If $r > \rho$, it is not optimal to save.*

(b) *If $\rho > r$, there is only one single saving switching time defined at time T^* and savings occur in $\lfloor T^*, T \rfloor$*

$$\begin{gathered} e^{-rT} - e^{-\rho T} = ke^{-rT^*} - e^{-\rho T^*}; \quad k = \frac{(1-\tau)}{\theta\alpha\pi} > 1 + e^{-rT} - e^{-\rho T}; \\ \frac{dT^*}{dr} > 0, \quad \frac{dT^*}{d\rho} < 0 \quad \text{if } \rho/r < T/T^* \text{ and vice versa;} \quad \frac{dT}{dT^*} > 0 \end{gathered} \tag{6}$$

(c) *If;* $k = \frac{(1-\tau)}{\theta\alpha\pi} < 1 + e^{-rT} - e^{-\rho T}$*; it is optimal to save all of the time.*

Proof: See Appendix 1.

The proposition sets out the conditions for retirement savings. Of course, if tax brackets T^* (deductions) are sufficiently large, life after retirement π and the proportion of savings per period at retirement α are large, then it will pay to save as long as possible and vice versa. Interestingly, the switching time is independent of the amount saved at retirement time because of the risk neutrality (linear) assumption made here. Under uncertainty and risk aversion, this will not be the case. We consider next the optimal retirement time T, summarized by Proposition 2. The proof of this proposition is based of course on the first proposition. Further, we consider only the case $k > 1$ which is the more interesting one.

Proposition 2: *The optimal retirement time is found by the following:*

(a) *If* $r < \rho$ *and* $k \leq 1$ *then* $T^* = 0$ *as stated in the proposition and the optimal retirement time is given by reaching the wealth state at time T given by:*

$$S(T) = \frac{[((1-\tau)v^* - \alpha\pi\theta v^* - 1]w + \alpha\pi a(w)}{(\rho - r)\alpha\pi} \tag{7}$$

(b) *If* $r \leq \rho$ *and* $k > 1$ *then* $T^* < T$ *and the optimal retirement time is given by the solution of (equation (8)):*

a. $$C(T) = \frac{[w + \alpha\pi a(w) + (\rho - r)\alpha\pi S(T)]}{w[(1-\tau) + \alpha\pi\theta]v^*}$$

$$C(T)k(r/\rho) - C(T)e^{-(\rho-r)T^*} = e^{\rho T}(1 - e^{-(\rho-r)T})$$

b. $$e^{-rT} - e^{-\rho T} = ke^{-rT^*} - e^{-\rho T^*}$$

Proof:

Let the case $r < \rho$, and $k > 1$

$$V(T) = S(0)e^{-r(0)} + \int_0^{T^*} Ae^{-rt}\,dt + \int_{T^*}^{T} Be^{-rt}\,dt$$

$$A = [w - w[(1-\tau) - \alpha\pi\theta]v^* + \alpha\pi a(w) + (\rho - r)\alpha\pi S];$$

$$B = [w + \alpha\pi a(w) + (\rho - r)\alpha\pi S]$$

and therefore, the first order necessary condition for the optimal retirement time is found by $\frac{dV(T)}{dT} = 0$ leading to a solution of:

$$\frac{dV(T)}{dT} = Ae^{-rT^*}\frac{dT^*}{dT} + Be^{-rT}\frac{dT}{dT} - Be^{-rT^*}\frac{dT^*}{dT} = 0 \quad \text{or}$$

$$\frac{B(T)e^{-r(T-T^*)}}{B(T^*) - A(T^*)} = \frac{dT^*}{dT} \quad \text{with} \quad \begin{aligned} B(T^*) - A(T^*) &= w[(1-\tau) + \alpha\pi\theta]v^* \\ B(T) &= [w + \alpha\pi a(w) + (\rho - r)\alpha\pi S(T)] \end{aligned}$$

From proposition 1, we have however:

$$\frac{dT^*}{dT} = \frac{e^{(\rho-r)T} - 1 > 0}{k(r/\rho)e^{-rT^*} - e^{-\rho T^*}} > 0 \rightarrow if \quad k(r/\rho)e^{(\rho-r)T^*} > 1$$

and therefore,

$$\frac{B(T)e^{-r(T-T^*)}}{B(T^*) - A(T^*)} = \frac{e^{(\rho-r)T} - 1}{k(r/\rho)e^{-rT^*} - e^{-\rho T^*}} \quad \text{or}$$

$$C(T) = \frac{[w + \alpha\pi a(w) + (\rho - r)\alpha\pi S(T)]}{w[(1-\tau) + \alpha\pi\theta]v^*} = \frac{e^{\rho T} - e^{rT}}{k(r/\rho) - e^{-(\rho-r)T^*}}$$

Further manipulations lead to:

$$C(T)k(r/\rho) - C(T)e^{-(\rho-r)T^*} = e^{\rho T}(1 - e^{-(\rho-r)T})$$

as stated in the proposition. Now set $k < 1$ and

$$V(T) = S(0)e^{-r(0)} + \int_0^T (B - A)e^{-rt}\, dt$$

$$A = [((1-\tau) - \alpha\pi\theta)v^* w]; \quad B = [w + \alpha\pi a(w) + (\rho - r)\alpha\pi S]$$

and therefore, $\frac{dV(T)}{dT} = 0 \rightarrow A = B$ with

$$S(T) = \frac{[((1-\tau)v^* - \alpha\pi\theta v^* - 1]w + \alpha\pi a(w)}{(\rho - r)\alpha\pi} \qquad \text{Q.E.D}$$

The results obtained in the two propositions above can be studied further. We consider first the effects of the tax Brackets. Note that from the optimality condition we have that saving for retirement will increase when tax brackets are increased. In other words, the richer a person, the more he will save for retirement. In this sense, individuation of retirement will increase inequalities. By the same token we can study the effects of an extended life of savers by the comparative static of parameter α. Other effects can be studied as well, as this will be the case in subsequent research. For example, so far we have assumed a linear utility function of consumption. When we have a nonlinear utility of consumption, we can expect greater savings when the saver is risk averse. However this will depend on the discount rate used. Further, uncertainty regarding premium at retirement will also affect the savings rate such that for a risk averse person there may be an incentive to start saving earlier.

3 Conclusion

This paper has considered a pension saving insurance problem using an intertemporal framework. We have assumed a deterministic framework in order to obtain tractable and yet revealing results regarding the propensity to save for retirement. Introducing uncertainties in wage rates, funds rates of return, retirement time and life, provide avenues for further study although these uncertainties are likely to have few substantial effects on the problem's conclusion. The essential conclusions we have obtained include the condition for a single switch. This means that generally, a saver will decide at some time prior to retirement, to start saving. When such a decision is taken, this decision will be maintained consistently till the retirement time. Because of the linear objective used, saving rates were found to be of the bang-bang type. Of course, for a nonlinear utility of consumption, this result is likely to be amended. This would be the case, if the utility of consumption were to be some convex function in consumption, yielding thereby interior solutions.

The effects of taxes were also considered important, reflecting savers wage income. The richer a saver, the higher the tax rate and therefore, there are advantages for saving to high wage earners. In other words, *saving for retirement will tend to maintain over time income inequality since a high wage earners will save and later on collect greater benefits*. Of course, in some countries, such as in France, tax is deferred to the retirement time. At this time however, chances are that the individual's tax rate will be smaller and thereby providing another indirect benefit to savers. If the tax rate at retirement equal τ', our results are then still valid, if we replace the parameter α by $\alpha(1-\tau')$ which measures the income received at retirement net of takes.

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Appendix 1: Proof of Proposition 1

The problem (1)–(4) has an objective we can write as follows (where the time index is dropped for convenience and $r(0)=0$):

$$V(T) = S(0)e^{-r(0)} + \int_0^T \left[Q(t) + \alpha\pi \frac{dS}{dt} - r\alpha\pi S \right] e^{-rt}\, dt$$

$$= S(0)e^{-r(0)} + \int_0^T [w(1-(1-\tau)v) + \alpha\pi[a(w) + \theta v w + \rho S] - r\alpha\pi S] e^{-rt}\, dt$$

$$= S(0)e^{-r(0)} + \int_0^T [w(1-(1-\tau)v + \alpha\pi\theta v) + \alpha\pi a(w) - (r-\rho)\alpha\pi S] e^{-rt}\, dt$$

The Hamiltonian is given by:

$$H(S, v, z, t) = [\alpha\pi a(w) + w(1 - (1 - \tau)v + \alpha\pi\theta v) - (r - \rho)\alpha\pi S]e^{-rt} + z[a(w) + \theta v w + \rho S]$$

where $z(t)$ is a Lagrange multiplier given by:

$$\frac{dz(t)}{dt} = -\frac{\partial H}{\partial S} = (r - \rho)\alpha\pi e^{-rt} - \rho z; \quad z(T) = 0$$

The solution of this differential equation, an ordinary linear differential equation, is:

$z(t) = z(0) + \alpha\pi e^{-rt} \lfloor 1 - e^{-(\rho - r)t} \rfloor$ and therefore,

$$z(0) = -\alpha\pi e^{-rT} \lfloor 1 - e^{-(\rho - r)T} \rfloor$$

Using the final boundary,

$$z(t) = \alpha\pi e^{-rt} \lfloor e^{-(\rho - r)t} - e^{-\rho T + rt} - 1 + e^{-r(T-t)} \rfloor$$

The optimal control is thus:

$$v = \begin{cases} v^* & \text{if } \phi(t) \geq 0 \\ 0 & \text{if } \phi(t) < 0 \end{cases}$$

where

$$\phi(t) = [-(1 - \tau) + \alpha\pi\theta]e^{-rt} + z[\theta]$$

and therefore,

$$\frac{\phi(t)e^{rt}}{\theta\alpha\pi} = -k + e^{rt}[e^{-\rho t} - e^{-\rho T} + e^{-rT}]; \quad k = \frac{(1 - \tau)}{\theta\alpha\pi} \quad \text{and}$$

$$v = \begin{cases} v^* & \text{if } [e^{-\rho t} - e^{-\rho T} + e^{-rT}] \geq k e^{-rt} \\ 0 & \text{if } [e^{-\rho t} - e^{-\rho T} + e^{-rT}] < k e^{-rt} \end{cases}$$

The condition for not saving initially is:

$$\lfloor 1 - e^{-\rho T} + e^{-rT} \rfloor < k$$

and for T large this is equivalent to: $1 < k$ and therefore, the smaller the tax bracket the more we are likely not to save and vice versa, the larger the tax bracket the more likely we are to save. In other words, the richer a person the greater the propensity to save. Of course, if $k < 1$, we start by saving if:

$$[1 - k] \geq e^{-rT}(e^{-(\rho - r)T} - 1)$$

which is always the case. Say that initially we do not save, then a switch occurs if the switching function is increasing. That is if:

$$\frac{d}{dt}(e^{-rT} - e^{-\rho T} - ke^{-rt} + e^{-\rho t}) > 0 \quad \text{or}$$

$$rke^{-rt} - \rho e^{-\rho t} > 0 \rightarrow (rk/\rho) > e^{-(\rho - r)t}.$$

Thus if $\rho > r$ we have a single switch that occurs at the following time: $e^{-rT} - e^{-\rho T} = ke^{-rT^*} - e^{-\rho T^*}$. If $\rho < r$ then if initially we do not save, we never switch thereafter to saving. In other words the necessary condition for saving is $\rho > r$ or the rate of return is greater than the discount rate. In this case, by implicit differentiation we have:

$$\frac{dT^*}{dr} = \frac{-re^{-rT} + kre^{-rT^*}}{kre^{-rT^*} - \rho e^{-\rho T^*}} \rightarrow \frac{1 - (1/k)e^{-r(T - T^*)}}{1 - (\rho/kr)e^{-(\rho - r)T^*}} > 0$$

$$\frac{dT^*}{d\rho} = \frac{\rho e^{-rT} - \rho e^{-\rho T^*}}{kre^{-rT^*} - \rho e^{-\rho T^*}} \rightarrow \frac{e^{-rT + \rho T^*} - 1}{(kr/\rho)e^{-(r - \rho)T^*} - 1}$$

$$e^{-rT + \rho T^*} > 1 \rightarrow \rho/r > T/T^*; \quad (kr/\rho)e^{(\rho - r)T^*} > 1$$

$$\frac{dT^*}{dT} = \frac{e^{(\rho - r)T} - 1 > 0}{k(r/\rho)e^{-\rho T}e^{-rT^*} - e^{-\rho T^*}e^{-\rho T}} > 0 \rightarrow \mathit{if} \quad k(r/\rho)e^{(\rho - r)T^*} > 1$$

and the earlier we start saving, the earlier we retire and vice versa, the later we start saving the later we retire as indicated by the proposition. Further, the larger the discount rate the greater the propensity not to save while the larger the rate of return, the larger the propensity to save.