EVALUATION OF SUPERHARMONIC FUNCTIONS USING LIMITS ALONG LINES

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1. Introduction

If *u* is a superharmonic function on \mathbb{R}^2 , then

$$u(x, y) = \liminf_{t \to y} u(x, t)$$

for all $(x, y) \in \mathbb{R}^2$. This follows from the fact that a line segment in \mathbb{R}^2 is non-thin at each of its constituent points. (See Doob [1, 1.XI] or Helms [7, Chapter 10] for an account of thin sets and the fine topology.) The situation is different in higher dimensions. For example, if u is the Newtonian potential on \mathbb{R}^3 defined by

$$u(x, y, z) = \int_{-1}^{1} \{x^2 + y^2 + (z - t)^2\}^{-1/2} |t| \, dt,$$

then

$$u(0,0,z) = \begin{cases} +\infty & (0 < |z| \le 1) \\ 2 & (z = 0). \end{cases}$$

Corollary 2 below will show that, nevertheless, for nearly every vertical line L, the value of a superharmonic function at any point X of L is determined by its lower limit along L at X.

Throughout this paper, we let $n \ge 3$. A typical point of \mathbb{R}^n will be denoted by X or (X', x), where $X' \in \mathbb{R}^{n-1}$ and $x \in \mathbb{R}$. Given any function $f : \mathbb{R}^n \to [-\infty, +\infty]$ and any point X, we define the *vertical cluster set* of f at X by

 $C_V(f;X) = \{ \ell \in [-\infty, +\infty] : \text{ there is a sequence } (t_m) \text{ in } \mathbb{R} \setminus \{x\} \\ \text{ such that } t_m \to x \text{ and } f(X', t_m) \to \ell \},$

and the *fine cluster set* of f at X by

$$C_F(f;X) = \{\ell \in [-\infty, +\infty] : \text{ for each neighbourhood } N \text{ of } \ell \text{ in } [-\infty, +\infty], \\ \text{ the set } f^{-1}(N) \text{ is non-thin at } X \}.$$

THEOREM. If $f : \mathbb{R}^n \to [-\infty, +\infty]$, then there is a polar subset E' of \mathbb{R}^{n-1} such that $C_V(f;X) \cap C_F(f;X) \neq \emptyset$ whenever $X' \in \mathbb{R}^{n-1} \setminus E'$.

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In [2, Theorem 1], it was shown that there is a subset E' of \mathbb{R}^{n-1} such that $E' \times \{0\}$ is a polar subset of \mathbb{R}^n (whence E' has Hausdorff dimension at most n-2) and $C_V(f;X) \subseteq C_F(f;X)$ whenever $X' \in \mathbb{R}^{n-1} \setminus E'$. In contrast, the exceptional set E' in our Theorem is polar in \mathbb{R}^{n-1} , and hence has Hausdorff dimension at most n-3.

A function f which is continuous at a point with respect to the fine topology will be called *finely continuous* at that point.

COROLLARY 1. If $A \subseteq \mathbb{R}^n$ and $f : \mathbb{R}^n \to [-\infty, +\infty]$ is finely continuous at each point of A, then there is a polar subset E' of \mathbb{R}^{n-1} such that $f(X) \in C_V(f;X)$ whenever $X \in A \setminus (E' \times \mathbb{R})$.

COROLLARY 2. Let u be a superharmonic function on an open subset Ω of \mathbb{R}^n . Then there is a polar subset E' of \mathbb{R}^{n-1} such that

$$u(X) = \liminf_{t \to x} u(X', t) \quad (X \in \Omega \setminus (E' \times \mathbb{R})).$$

COROLLARY 3. If Ω is a finely open subset of \mathbb{R}^n , then there is a polar subset E' of \mathbb{R}^{n-1} such that the set $S_{X'}^{\Omega} = \{t \in \mathbb{R} : (X', t) \in \Omega\}$ has no isolated points whenever $X' \in \mathbb{R}^{n-1} \setminus E'$.

The following example shows that the above results are sharp with regard to the size of the exceptional set.

EXAMPLE. Let E' be any polar subset of \mathbb{R}^{n-1} . Then there is a bounded Newtonian potential u on \mathbb{R}^n such that

$$u(X) < \liminf_{t \to x} u(X', t),$$

and hence

$$u(X) \notin C_V(u;X),$$

whenever $X \in E' \times \mathbb{Q}$. Also, there is a finely open set Ω such that $S_{X'}^{\Omega} = \{0\}$ whenever $X' \in E'$.

The proof of the Theorem and Corollaries 1-3 will be given in Section 2, and the details of the Example in Section 3.

2. Proof of the Theorem

We shall require the following.

LEMMA 1. Let $A' \subseteq \mathbb{R}^{n-1}$ and $(X', x) \in \mathbb{R}^{n-1} \times \mathbb{R}$, and let $\varepsilon > 0$. The following are equivalent.

- (a) $A' \times \mathbb{R}$ is thin at (X', x) in \mathbb{R}^n .
- (b) $A' \times (x, x + \varepsilon)$ is thin at (X', x) in \mathbb{R}^n .
- (c) $A' \times (x \varepsilon, x)$ is thin at (X', x) in \mathbb{R}^n .
- (d) A' is thin at X' in \mathbb{R}^{n-1} .

A proof of the equivalence of (a) and (d) above may be found in [5, Lemma 2]

(see [6] for a general abstract result of this type). Clearly, (a) implies both (b) and (c). Since superharmonicity is preserved by reflection, (b) is equivalent to (c). Hence it remains to check that (b) and (c) together imply (a). Since the set

$$\{Y \in A' \times \{x\} : 2^{-j} \le |Y - X| < 2^{1-j}\}$$

is contained in the image of

$$\{Y \in A' \times (x, x + \varepsilon) : 2^{-j} \le |Y - X| < 2^{1-j}\}$$

under the canonical projection from \mathbb{R}^n to $\mathbb{R}^{n-1} \times \{x\}$, and Newtonian capacity decreases under this projection, it follows from Wiener's criterion (see [7, Theorem 10.21]) that $A' \times \{x\}$ is thin at X = (X', x). Hence $A' \times (x - \varepsilon, x + \varepsilon)$ is thin at (X', x), being the union of three such sets, and (a) follows since thinness is a local property.

The following lemma is an easy consequence of the definition of the fine cluster set $C_F(f; X)$.

LEMMA 2. Let
$$A \subseteq \mathbb{R}^n$$
 and $X \in \mathbb{R}^n$. If A is non-thin at X in \mathbb{R}^n , then

$$C_F(f;X) \cap f(A) \neq \emptyset,$$

where $\overline{f(A)}$ denotes the closure of f(A) in $[-\infty, +\infty]$.

For the proof of the Theorem, we shall modify an argument of Hayman (see [8, pp. 472–473]). Let \mathscr{L} denote the collection of open intervals of \mathbb{R} with endpoints in \mathbb{Q} , and let \mathscr{I} denote the collection of finite unions of closed intervals of $[-\infty, +\infty]$ with endpoints in $\mathbb{Q} \cup \{-\infty, +\infty\}$. Also, we define

$$E' = \{ X' \in \mathbb{R}^{n-1} : C_V(f; (X', x)) \cap C_F(f; (X', x)) = \emptyset \text{ for some } x \in \mathbb{R} \}.$$

Let $Y' \in E'$. Then there exists y in \mathbb{R} such that $C_V(f; Y) \cap C_F(f; Y) = \emptyset$, where Y = (Y', y). Since $C_V(f; Y)$ and $C_F(f; Y)$ are compact subsets of $[-\infty, +\infty]$, we can find $I, J \in \mathcal{I}$ and $L \in \mathcal{L}$ such that

$$C_F(f;(Y',x)) \subseteq I \text{ for some value of } x \text{ in } L, \tag{1}$$

$$f(Y', x) \in J$$
 for all but at most one value of x in L, (2)

$$I \cap J = \emptyset. \tag{3}$$

Given any $I, J \in \mathscr{I}$ and $L \in \mathscr{L}$, we now define a subset E'(I, J, L) of \mathbb{R}^{n-1} by writing $Y' \in E'(I, J, L)$ if $Y' \in E'$ and (1) and (2) both hold. The preceding paragraph shows that

$$E' = \bigcup E'(I, J, L), \tag{4}$$

where the union is over all choices of I, J and L which satisfy (3).

Now suppose (with the aim of obtaining a contradiction) that there is a constituent member $F' = E'(I_0, J_0, L_0)$ of the above union, and a point Z' of F', such that F' is non-thin at Z'. In view of (1), we can choose z in the open interval L_0 such that $C_F(f;Z) \subseteq I_0$, where Z = (Z', z). We choose a positive number ε such that $(z - \varepsilon, z + \varepsilon) \subseteq L_0$. Also, we define

$$F'_1 = \{Y' \in F' : f(Y', x) \in J_0 \text{ whenever } x \in (z, z + \varepsilon)\}$$

and

$$F'_2 = \{Y' \in F' : f(Y', x) \in J_0 \text{ whenever } x \in (z - \varepsilon, z)\}$$

In view of (2), at least one of the sets F'_1 , F'_2 must be non-thin at Z'.

We consider first the case where F'_1 is non-thin at Z'. By Lemma 1, the set $F'_1 \times (z, z + \varepsilon)$ is non-thin at Z in \mathbb{R}^n . Hence $C_F(f;Z) \cap J_0 \neq \emptyset$, by Lemma 2. This yields a contradiction, since $C_F(f;Z) \subseteq I_0$ and $I_0 \cap J_0 = \emptyset$ by (3).

If, instead, F'_2 is non-thin at Z', then we note from Lemma 1 that $F'_2 \times (z - \varepsilon, z)$ is non-thin at Z, and argue similarly.

From the above contradiction we conclude that F' must be thin at each of its points and hence is polar (see [1, 1.XI.6]). Since this is true for each constituent set of the union in (4), and since this is a countable union, the set E' is polar.

This concludes the proof of the Theorem.

If f is finely continuous at a point X, then $C_F(f;X) = f(X)$. Thus Corollary 1 is an immediate consequence of the Theorem.

Corollary 2 is deduced as follows. We extend u to be defined on all of \mathbb{R}^n by assigning arbitrary values on $\mathbb{R}^n \setminus \Omega$. Since u is finely continuous at all points of Ω , it follows from Corollary 1 that there is a polar subset E' of \mathbb{R}^{n-1} such that

 $u(X) \in C_V(u; X)$ whenever $X \in \Omega \setminus (E' \times \mathbb{R})$.

However, by the lower semicontinuity of u,

$$u(X) \leq \ell$$
 for all $\ell \in C_V(u; X)$ $(X \in \Omega)$.

Thus $u(X) \in C_V(u, X)$ if and only if

$$u(X) = \liminf_{t \to x} u(X', t),$$

and the result follows.

Corollary 3 is obtained from Corollary 1 by defining f to be the characteristic function valued 1 on Ω and 0 elsewhere.

3. Details of the Example

The Example was suggested by a construction in [3, Example 4] (or see [4, Example 1.13]). Let E' be any non-empty polar subset of \mathbb{R}^{n-1} . Then there is a superharmonic function v on \mathbb{R}^{n-1} such that $v = +\infty$ on E'. Let A be the open subset of \mathbb{R}^n defined by

$$A = \{ (X', x) : v(X') > |x|^{2-n} \},\$$

and let μ be the Riesz measure associated with the superharmonic function defined on \mathbb{R}^n by

$$(X', x) \longmapsto v(X').$$

In view of the invariance of the above function under vertical translation, μ cannot charge any point of \mathbb{R}^n . Since

$$\liminf_{\substack{X \to (Y',0) \\ X \in A}} \frac{v(X')}{|X - (Y',0)|^{2-n}} \ge \liminf_{\substack{X \to (Y',0) \\ X \in A}} \frac{v(X')}{|x|^{2-n}} \ge 1 > 0 = \mu(\{(Y',0)\}) \quad (Y' \in \mathbb{R}^{n-1}),$$

the set A is thin at each point of $\mathbb{R}^{n-1} \times \{0\}$ (see [1, 1.XI.4 (c)]).

Now let w be a bounded continuous potential on \mathbb{R}^n which determines thinness (see [1, 1.XI.10]); that is, w has the property that for any set $F \subset \mathbb{R}^n$,

 $\widehat{R}_{w}^{F}(X) < w(X)$ if and only if F is thin at X,

where \widehat{R}_w^F is the regularized reduction of w on F relative to superharmonic functions on \mathbb{R}^n . Let $u_0 = \widehat{R}_w^A$. Then u_0 is a bounded potential on \mathbb{R}^n , and $u_0 < w$ on $\mathbb{R}^{n-1} \times \{0\}$. However, since $v = +\infty$ on E', it is clear that $E' \times (\mathbb{R} \setminus \{0\}) \subseteq A$, so $u_0 = w$ on $E' \times (\mathbb{R} \setminus \{0\})$. It follows that

$$u_0(X', x) = w(X', x) \to w(X', 0) > u_0(X', 0) \quad (x \to 0, \ X' \in E').$$
(5)

Finally, let (q_m) be an enumeration of \mathbb{Q} , and let

$$u(X', x) = \sum_{m} 2^{-m} u_0(X', x - q_m) \quad ((X', x) \in \mathbb{R}^n).$$

It follows from the dominated convergence theorem that for any $k \in \mathbb{N}$ and $X' \in E'$,

$$\sum_{m \neq k} 2^{-m} u_0(X', x - q_m) \to \sum_{m \neq k} 2^{-m} u_0(X', q_k - q_m)$$

as $x \to q_k$. Hence

$$\liminf_{x \to q_k} u(X', x) = 2^{-k} \liminf_{x \to 0} u_0(X', x) + \sum_{m \neq k} 2^{-m} u_0(X', q_k - q_m)$$

> $2^{-k} u_0(X', 0) + \sum_{m \neq k} 2^{-m} u_0(X', q_k - q_m) = u(X', q_k),$ (6)

by (5). Since (6) holds for every k and every $X' \in E'$, the first assertion of the Example is established. Also, if we define Ω to be the fine interior of $\mathbb{R}^n \setminus A$, then it is clear that $S_{X'}^{\Omega} = \{0\}$ whenever $X' \in E'$.

References

- 1. J. L. DOOB, Classical potential theory and its probabilistic counterpart (Springer, New York, 1984).
- M. R. Essén and S. J. GARDINER, 'Limits along parallel lines and the classical fine topology', J. London Math. Soc. (2) 59 (1999) 881–894.
- **3.** S. J. GARDINER, 'Superharmonic extension and harmonic approximation', *Ann. Inst. Fourier (Grenoble)* 44 (1994) 65–91.
- 4. S. J. GARDINER, *Harmonic approximation*, London Math. Soc. Lecture Note Ser. 221 (Cambridge University Press, 1995).
- S. J. GARDINER, 'The Lusin–Privalov theorem for subharmonic functions', Proc. Amer. Math. Soc. 124 (1996) 3721–3727.
- W. HANSEN, 'Abbildungen harmonischer Räume mit Anwendung auf die Laplace und Wärmeleitungsgleichung', Ann. Inst. Fourier (Grenoble) 21 (1971) 203–216.
- 7. L. L. HELMS, Introduction to potential theory (Wiley, New York, 1969).
- P. J. RIPPON, 'The boundary cluster sets of subharmonic functions', J. London Math. Soc. (2) 17 (1978) 469–479.

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