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DEFORMED SUPERSYMMETRY, q-OSCILLATOR ALGEBRA, AND RELATED SCATTERING PROBLEMS IN QUANTUM MECHANICS

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We describe extensions of the supersymmetric quantum mechanics (SSQM) (in one dimension) which are characterized by deformed algebras. The supercharges involving higher-order derivatives are introduced, leading to a deformed algebra which incorporates a higher-order polynomial of the Hamiltonian. When supplementing them with dilatations, one finds the class of q-deformed SUSY systems. For a special choice of q-self-similar potentials, the energy spectrum is (partially) generated by the q-oscillator algebra. In contrast to the standard harmonic oscillators, these systems exhibit a continuous spectrum. We investigate the scattering problem in the q-deformed SSQM and introduce the notion of self-similarity in the momentum space for scattering data. An explicit model for the scattering amplitude of a q-oscillator is constructed in terms of a hypergeometric function. This model corresponds to a reflectionless potential with infinitely many bound states. A general method of realization of the q-oscillator algebra on the space of wave functions for a one-dimensional Schrödinger Hamiltonian is developed. It shows the existence of non-Fock irreducible representations associated with the continuous part of the spectrum and directly related to the deformation. Bibliography: 24 titles.

1. Introduction

The deformed symmetries were introduced [1, 2] to characterize the integrability of some lattice models and conformal field theories [3, 4]. Later on, a number of examples of q-deformed symmetries or q-deformed spectrum-generating algebras were found in various quantum systems, in particular, in low-dimensional quantum-mechanical (QM) models.

A simple algebra which is realized on isospectral QM systems is the supersymmetric quantum-mechanical algebra [5, 6] of supercharges and a superhamiltonian. Two supersymmetric partners of a superhamiltonian are related (more technically intertwined) by supercharges which actually generate Darboux transformations [7] between two isospectral systems. Deformations of this algebra are provided by extended supercharges involving higher-order derivatives [8–10]. Such an extension leads to a higher-derivative SUSY (HSUSY) algebra [10] which incorporates a higher order polynomial of the Hamiltonian. On the other hand, when supplementing supercharges with dilatations, one reveals the class of q-deformed potential systems [11] on which the q-deformed oscillator algebra can be realized as a spectrum-generating algebra.

The oscillator or Heisenberg–Weyl algebra was deformed in several ways [12, 13], and its representations have formally been classified [13–15]. Because of the significance of the conventional oscillator in many areas of modern quantum physics, new realizations of the q-oscillator algebra on the wave function space of a particular dynamical model are useful for an understanding of the role of this algebra for physical systems.

It is our goal to describe the interplay between polynomial deformations of the SUSY algebra referred to pairs of isospectral systems and q-deformations introduced into a Schrödinger-type system by dilatations of coordinates. In this way, different realizations of the q-oscillator as a quantum system satisfying the Schrödinger equation are found.

As also for the harmonic oscillator ($q = 1$), the connection between the values of different energy levels and the corresponding wave functions [11] (or reflection and transmission coefficients [10]) becomes a consequence of the q-oscillator algebra.

In order to reproduce a q-oscillator, it is necessary to identify the potentials in the Darboux-connected Hamiltonians up to a constant. This self-similarity property holds for the conventional harmonic oscillator and explains its equidistant energy spectrum. The dilated self-similarity condition [11, 16] naturally selects the potentials that yield the energy spectra and wave functions of a q-deformed oscillator. The relation

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between scattering amplitudes of two q-superpartners leads to the "dual" condition of self-similarity [10] (in the momentum space).

In Sec. 2, we outline the basics of one-dimensional SSQM [5, 6]. In Sec. 3, a SSQM with supercharges of second-order in derivatives is constructed in the most general form, and cases are pointed out where the superpartners cannot be constructed by iterations of two ordinary Darboux transformations. Thereby we classify all irreducible transformations involved in the construction of HSUSY systems with polynomially deformed superalgebras.

In Sec. 4, we study the q-deformations of SUSY and HSUSY induced by the dilatation of the coordinates. The coexistence between a q-deformed SUSY algebra with the ordinary Hamiltonian and the ordinary SUSY algebra with the q-deformed Hamiltonian is established and further generalized to HSUSY.

In Sec. 5, we introduce the closure, or self-similarity, condition into the q-deformed SUSYQM and then construct a q-oscillator model with local potential. The proof of the existence of a regular potential is discussed. Its polynomial generalization is obtained by the q-deformation of a polynomial superalgebra.

In Sec. 6, we consider the q-oscillator algebra in the form [13] and give the classification of its representations in terms of the central element. The decomposition of the Hamiltonian realization from Sec. 5 into irreducible q-oscillator representations is developed. Two types of q-oscillator representations appear and, while the Fock representation refers to the bound states, the non-Fock representations cover the continuous spectrum.

In Sec. 7, the consequences of one-dimensional HSUSY are explored for scattering properties [17] of the partner Hamiltonians [18]. The relation between scattering amplitudes of two q-superpartners [10] is presented in Sec. 8. For scattering amplitudes, the "dual" condition of self-similarity (in the momentum space) is defined. An explicit construction satisfying this "dual" self-similarity is presented in terms of a hypergeometric function. This construction corresponds to the reflectionless potential.

In Sec. 9, we analyze other possible realizations of the q-oscillator on the space of wave functions for a one-dimensional Schrödinger Hamiltonian. In our approach, the local Hamiltonian, in general, is not bilinear in creation and annihilation operators but rather belongs to the universal enveloping q-oscillator algebra, i.e., to the algebra of polynomials (analytic functions) of the generators. Thereby the q-oscillator relations are considered as a kind of q-deformed (nonlinear) dynamical algebra. We state a general method of constructing a local Hamiltonian of Schrödinger type with deformed spectrum generating algebras. The related mapping of energy levels is examined. The different forms of the q-oscillator algebra are described, and the constraints on their realization in terms of a Schrödinger Hamiltonian are obtained.

2. SUSY Quantum Mechanics in one dimension

SSQM is generated [5, 6] by the supercharge operators Q^+ and $Q^- = (Q^+)^\dagger$, which together with the Hamiltonian H of the system satisfy the relations

$$(Q^\pm)^2 = 0, \quad [H, Q^\pm] = 0, \tag{1}$$

$$\{Q^+, Q^-\} = H = Q^2, \tag{2}$$

where $Q = Q^+ + Q^-$ is the Hermitian supercharge operator.

The one-dimensional representation is realized by the 2×2 supercharges

$$Q^- = \begin{pmatrix} 0 & 0 \\ a^- & 0 \end{pmatrix} \quad \text{and} \quad Q^+ = \begin{pmatrix} 0 & a^+ \\ 0 & 0 \end{pmatrix}, \tag{3}$$

where

$$a^\pm = \pm\partial + W(x) \tag{4}$$

and the superhamiltonian is comprised of two ordinary Schrödinger Hamiltonians,

$$\begin{aligned} H &= \begin{pmatrix} h^{(1)} & 0 \\ 0 & h^{(2)} \end{pmatrix} = \begin{pmatrix} a^+a^- & 0 \\ 0 & a^-a^+ \end{pmatrix} = -\partial^2 + \begin{pmatrix} W^2 + W' & 0 \\ 0 & W^2 - W' \end{pmatrix} \\ &= (-\partial^2 + W^2)\mathbf{1} + \sigma_3 W' \\ h^{(i)} &\equiv -\partial^2 + V^{(i)}(x), \end{aligned} \tag{5}$$

where σ_3 is a Pauli matrix.

A direct consequence of Eqs. (1), (2) is that all eigenvalues of H are nonnegative. In terms of components, the algebra defined by Eqs. (1), (2) means that the Hamiltonians $h^{(1)}$ and $h^{(2)}$ in Eqs. (5) are factorized. To express it differently, we can say that with a given factorizable Hamiltonian $h^{(1)}$, one can associate a supersymmetric partner $h^{(2)}$ such that both partners are linked by the intertwining relations

$$h^{(1)}a^+ = a^+h^{(2)} \quad \text{and} \quad a^-h^{(1)} = h^{(2)}a^-. \tag{6}$$

These relations lead to the double degeneracy of all positive energy levels of H belonging to the "bosonic" or "fermionic" sectors specified by the grading operator $\tau = (-)^{\widehat{N_F}} = \sigma_3$, where $\widehat{N_F}$ is the fermion number operator. The grading operator commutes with the Hamiltonian H and anticommutes with the supercharge Q. Thus, the supercharge operator transforms eigenstates with $\tau = +1$ (bosons) into eigenstates with $\tau = -1$ (fermions) and vice versa. The boson and fermion wave functions are eigenfunctions of $h^{(1)}$ and $h^{(2)}$, respectively. They are connected via Eqs. (6) by the operators $a^\pm$:

$$e\sqrt{E}\Psi_E^{(2)} = a^-\Psi_E^{(1)} \quad \text{and} \quad \sqrt{E}\Psi_E^{(1)} = a^+\Psi_E^{(2)}. \tag{7}$$

The existence of zero-energy states depends [6] on the asymptotics of the superpotential $W(x)$ which appears in Eq. (4). For appropriate $W(x)$, they can arise either in the bosonic sector, i.e.,

$$a^-\Psi^{(1)}(x) = 0,$$

or in the fermionic one, i.e.,

$$a^+\Psi^{(2)}(x) = 0.$$

3. Higher-derivative SSQM

Now we study realizations where relations (1) are preserved but deformations of relation (2) are allowed. A nonstandard realization [9, 10] relies on the use of higher-order derivative operators in the definition of the supercharges. Instead of the linear operators of Eq. (4), let us define the second-order differential operators

$$A^+ = (A^-)^\dagger = \partial^2 - 2f(x)\partial + b(x). \tag{8}$$

Then, with $Q = Q^+ + Q^-$, Eq. (2) is transformed to

$$\{Q^+, Q^-\} = Q^2 = K.$$

Thus, the quasihamiltonian K is given by the conventional superalgebra, but it is now a fourth-order differential operator, hence not of the Schrödinger form.

Let us assume that there exists a diagonal Hamiltonian H of Schrödinger type,

$$H = \begin{pmatrix} h^{(1)} & 0 \\ 0 & h^{(2)} \end{pmatrix}, \tag{9}$$

which commutes with the supercharges $Q^\pm$ constructed from $A^\pm$ given in (8). Then, from intertwining relations similar to (6),

$$h^{(1)}A^+ = A^+h^{(2)} \quad \text{and} \quad A^-h^{(1)} = h^{(2)}A^-, \tag{10}$$

it follows that the quasihamiltonian K commutes with H and, furthermore, is given in terms of H by

$$K = H^2 - 2\alpha H + \beta, \tag{11}$$

where α and β are constants since $[K, Q] = 0$ and the spectra of $h^{(1)}$ and $h^{(2)}$ in the one-dimensional problem are nondegenerate. The intertwining relations (10) for H imply that

$$b(x) = f^2(x) - f'(x) - \frac{f''(x)}{2f(x)} + \left(\frac{f'(x)}{2f(x)}\right)^2 + \frac{d}{4f^2(x)}, \tag{12}$$

where

$$d = \beta - \alpha^2. \tag{13}$$

This condition is necessary and sufficient for the existence of a generalized polynomial HSUSY algebra defined by

$$(Q^\pm)^2 = 0, \quad [H, Q^\pm] = 0, \quad \{Q^+, Q^-\} = Q^2 = (H - \alpha)^2 + d. \tag{14}$$

By the use of Eq. (12), the potentials of the superpartner Hamiltonians can be expressed in terms of $f(x)$ and its derivatives,

$$V^{(1),(2)} = \mp 2f'(x) + f^2(x) + \frac{f''(x)}{2f(x)} - \left(\frac{f'(x)}{2f(x)}\right)^2 - \frac{d}{4f^2(x)} + \alpha. \tag{15}$$

The eigenfunctions of $h^{(1)}$ and $h^{(2)}$ are obtained from each other by the action of the second-order differential operators $A^\pm$ according to Eqs. (7).

One can factorize the elementary operators $A^\pm$ in terms of the ordinary superpotentials W_1 and W_2, introduced in Eq. (4),

$$A^+ = a_1^+ a_2^+ = (\partial + W_1(x))\,(\partial + W_2(x)). \tag{16}$$

In certain cases, they are connected by the ladder equation

$$a_1^- a_1^+ = a_2^+ a_2^- + c \qquad \text{or} \qquad -W_1' + W_1^2 = W_2' + W_2^2 + c, \tag{17}$$

where, without loss of generality, we assume $c \geq 0$ (superpotentials, supercharges, and other relevant operators will, thus, depend on c). This equation implies

$$\{Q^+, Q^-\} = \widetilde{H}(\widetilde{H} - c), \tag{18}$$

where, for comparison with Eq. (11),

$$\widetilde{H} = H + \frac{1}{2}(c - 2\alpha) = \begin{pmatrix} a_1^+ a_1^- & 0 \\ 0 & a_2^- a_2^+ + c \end{pmatrix}. \tag{19}$$

Obviously, $c^2 = -4d \geq 0$.

Factorization (16) arises [8, 9] from two successive standard SSQM transformations

$$\begin{pmatrix} h^{(1)} & 0 \\ 0 & h \end{pmatrix} = \begin{pmatrix} a_1^+ a_1^- & 0 \\ 0 & a_1^- a_1^+ \end{pmatrix} \tag{20}$$

and

$$\begin{pmatrix} h - c & 0 \\ 0 & h^{(2)} \end{pmatrix} = \begin{pmatrix} a_2^+ a_2^- & 0 \\ 0 & a_2^- a_2^+ \end{pmatrix} \tag{21}$$

together with the ladder condition (17).

Let us choose $c = 2\alpha$, i.e., $\beta = 0$. Then the superpotentials $W_{1,2}$ can be parameterized in terms of $f(x)$ as follows:

$$W_{1,2} = \pm\frac{2f'(x) - c}{4f(x)} - f(x). \tag{22}$$

The ladder equation (17) is meaningful only when the discriminant d is negative; it is the property that allows us to introduce the intermediate Hermitian Hamiltonian in Eqs. (20), (21). If $d > 0$, relation (17) cannot be satisfied; therefore, we refer to this class of second-order derivative SSQM as irreducible.

Such a class represents a new primitive element in constructing supersymmetric ladders and, respectively, polynomial SSQM's. Clearly, the corresponding supercharge cannot possess any zero modes since it is bounded from below by the constant $\sqrt{d}$, as shown by Eqs. (14). In order to avoid singular supercharges and potentials (15) in the case under discussion, we assume that the function $f(x)$ is nodeless. The generalization of the second-order derivative SUSY algebra to differential operators $A^\pm$ of higher order is straightforward and leads to polynomials in H on the right-hand side of Eq. (14). The ladder construction (20), (21) is constructed by primitive elements of the first and second order in derivatives. Thus, we obtain, in general, the following polynomial superalgebra:

$$Q^2 = \prod_{i+2j=n} (H - c_i)\left((H - \alpha_j)^2 + d_j\right) \qquad d_j > 0. \tag{23}$$

4. q-DEFORMED SUSY QUANTUM MECHANICS

In the framework of SSQM, a q-deformed SUSY algebra was constructed in a specific realization [11] exploiting the dilatation operator

$$T_q f(x) = \sqrt{q} f(qx), \quad T_q \partial_x = q^{-1} \partial_x T_q, \tag{24}$$

represented also by the following pseudodifferential operator:

$$T_q = \sqrt{q} \exp(\ln q \ x\partial_x), \quad T_q^\dagger = T_q^{-1}. \tag{25}$$

The q-deformed supercharges $Q^\pm$ are constructed from

$$a_q^+ = (\partial + W(x))\, T_q^\dagger \equiv \hat{a}, \quad a_q^- = T_q(-\partial + W(x)) \equiv \hat{a}^+, \tag{26}$$

as in Eq. (3). The redefinition of Darboux operators $a_q^\pm$ in symbols of "creation" $\hat{a}^+$ and "annihilation" $\hat{a}$ operators is done for the purposes of modeling the q oscillator (see the next section). The "bosonic" and "fermionic" components of the Hamiltonian (see Eq. (5)) are now q-deformed:

$$\begin{aligned} h^{(1)} &\equiv a_q^+ a_q^- = -\partial^2 + W^2(x) + W'(x) \equiv \hat{a}\hat{a}^+, \\ h^{(2)} &\equiv q^2 a_q^- a_q^+ = -\partial^2 + q^2 W^2(qx) - qW_x'(qx) \equiv q^2 \hat{a}^+ \hat{a}. \end{aligned} \tag{27}$$

The coefficient q^2 in $h^{(2)}$ is introduced in order to properly normalize the kinetic term. These Hamiltonians are not intertwined by means of the standard SUSY algebra (see Eq. (6)) but satisfy the q-deformed intertwining relations:

$$q^2 h^{(1)} a_q^+ = a_q^+ h^{(2)}, \quad q^2 a_q^- h^{(1)} = h^{(2)} a_q^-. \tag{28}$$

Thus, a q-deformed SUSY algebra arises with the conventional form of Eqs. (1), (2),

$$\{Q^+, Q^-\}_q = H, \quad [Q^+, H]_q = [H, Q^-]_q = 0, \tag{29}$$

in terms of the q-(anti)commutators (instead of the usual ones)

$$[X, Y]_q \equiv XY - q^2 YX, \quad \{X, Y\}_q \equiv XY + q^2 YX. \tag{30}$$

In the usual sense, the supercharges are not preserved now because they do not commute with the Hamiltonian. As a consequence, the superpartner Hamiltonians are no longer isospectral, but their spectra are related by the q-dilatation, $q^2 E^{(1)} = E^{(2)}$. Notice, however, that there exists a different Hamiltonian (q-Hamiltonian $\hat{H}$), with rescaled $h^{(2)}$, which commutes with supercharges and satisfies the superalgebra

$$\{Q^+, Q^-\} = \begin{pmatrix} h^{(1)} & 0 \\ 0 & q^{-2} h^{(2)} \end{pmatrix} \equiv \hat{H}, \quad [\hat{H}, Q^\pm] = 0. \tag{31}$$

The wave functions are connected by the operators $a_q^\pm$:

$$\psi^{(2)} \propto a_q^- \psi^{(1)}, \quad \psi^{(1)} \propto a_q^+ \psi^{(2)}. \tag{32}$$

Now let us combine the q-deformed SUSY with the polynomial HSUSY (see Sec. 3) quantum mechanics. The factorizable construction of the second-order derivative q-deformed HSUSY algebra can be realized by means of a sequence of two q-deformations (26) with different dilatation parameters q_1 and q_2. The components of the q-supercharge are factorized:

$$A_q^+ = q_1^{-1} a_{q_1}^+ \cdot a_{q_2}^+ = q_1^{-1} (\partial + W_1(x))\, T_{q_1}^\dagger (\partial + \widetilde{W}_2(x))\, T_{q_2}^\dagger = (A_q^-)^\dagger. \tag{33}$$

The supercharge components can be transformed to the form of a product of a second-order derivative operator and a single dilatation operator:

$$A_q^+ = (\partial + W_1(x))(\partial + W_2(x))\, T_q^\dagger, \qquad q = q_1 \cdot q_2; \quad W_2(x) \equiv q_1^{-1} \widetilde{W}_2(q_1^{-1} x). \tag{34}$$

In this form, the generalization of the second order derivative SUSY is straightforward and can be performed following the method of Sec. 3 with the dilatation at the last step both for reducible ($d \le 0$) and for irreducible ($d > 0$) cases.

As in the ordinary q-deformed case, the Hamiltonian H does not commute with the supercharge but satisfies Eqs. (29). The q-deformed SUSY algebra with the Hamiltonian H has the usual form (σ_3 is the Pauli matrix):

$$Q^+Q^- + q^4 Q^- Q^+ = \{Q^+, Q^-\}_{q^2} = (H - \alpha q^{1-\sigma_3})^2 + d \cdot q^{2(1-\sigma_3)}, \qquad [Q^+, H]_q = [H, Q^-]_q = 0. \tag{35}$$

There exists a q-Hamiltonian $\hat{H}$ (31) which commutes with the supercharges. Moreover, in terms of $\hat{H}$, the HSUSY algebra takes the usual form (14).

Further steps in extension of the higher-order derivative SUSY either lead to Eq. (23) with the q-Hamiltonian and the conventional SUSY or to its q-deformed version [9, 10] with the true Hamiltonian H and the primitive blocks defined in (35):

$$\{Q^+, Q^-\}_{q^n} = \prod_{i+2j=n} (H - c_i \cdot q^{1-\sigma_3}) \left((H - \alpha_j q^{1-\sigma_3})^2 + d_j \cdot q^{2(1-\sigma_3)} \right); \qquad [Q^+, H]_q = [H, Q^-]_q = 0; \quad d_j > 0. \tag{36}$$

5. Local Hamiltonian model of a q-oscillator

This model is constructed from the q-deformed SUSYQM (27), (29) by imposing the closure condition

$$h^{(1)} = h^{(2)} + 1.$$

Then obviously, the operators $\hat{a}$ and $\hat{a}^+$ satisfy the q-commutator relation

$$\hat{a}\hat{a}^+ - q^2 \hat{a}^+ \hat{a} = 1, \tag{37}$$

which is an essential ingredient of the algebraic definition of a q-oscillator [13] (see Sec. 6). The closure condition leads to the q-self-similarity equation for $W(x)$,

$$W'(x) + qW'(qx) + W^2(x) - q^2 W^2(qx) = 1, \tag{38}$$

where the prime stands for the derivative with respect to x. Obviously, for $q = 1$, the ordinary harmonic oscillator is reproduced with $W(x) = x/2$ corresponding to a self-similar potential. Equation (38) can be considered on the entire axis, $x \in (-\infty, +\infty)$, or on the semiaxis, $x \in [0, \infty)$. In order to develop the expansion around $x \simeq 0$ or as $x \to \infty$, it is convenient to express Eq. (38) in the operator form. Let us introduce the dilatation generator for T_q (25),

$$D = x\partial_x, \quad Dx^n = nx^n, \quad T_q = q^{D+\frac{1}{2}}, \tag{39}$$

and put $\bar{W} \equiv xW(x)$. Then, in the operator form, Eq. (38) appears as

$$(D - 1)\bar{W} = \frac{1}{1+q^D}\, x^2 - \frac{1 - q^D}{1 + q^D}\, \bar{W}^2. \tag{40}$$

When expanding

$$\widetilde{W}(x) = \sum_{n=1}^{\infty} c_n x^n$$

around the origin, one derives from (40) the set of recurrent relations for $n > 1$,

$$(n-1)c_n = \frac{1}{1+q^2}\delta_{n,2} - \frac{1-q^n}{1+q^n}\sum_{l=1}^{n-1} c_l c_{n-l}, \tag{41}$$

where c_1 is an arbitrary constant. The choice $c_1^2 = 1/(1-q^2)$ corresponds to the trivial solution where $c_n = 0, \quad n \neq 1$.

From (41) it follows [11, 16] that, for $0 < q \leq 1$, a solution regular on the entire axis arises for $c_1 = 0,\ c_2 = 1/(1+q^2)$. As a consequence, $c_{2j+1} = 0$, $\widetilde{W}(-x) = \widetilde{W}(x)$, and $W(-x) = -W(x)$.

The asymptotic expansion at infinity can be found from (40) in the same manner. We have

$$W(x)|_{|x|>>1} = \pm\frac{1}{\sqrt{1-q^2}} + \frac{b_2(\ln x)}{x^2} + \mathrm{O}(\frac{1}{x^3}), \tag{42}$$

where $b_2(\xi \pm \ln q) = b_2(\xi)$ is an arbitrary periodic function of $\xi \equiv \ln x$ (in particular, $b_2 = \mathrm{const}$). Obviously, it provides a decreasing potential, $V(x) \sim 1/x^2, \quad |x| \gg 1$. Thus, the spectrum of Hamiltonians with regular potential must have a continuous part. If the solution $W(x)$ is chosen to take a positive constant value as $x \to +\infty$ and a negative one as $x \to -\infty$, then it follows from Eqs. (26) that the normalizable ground state of $h^{(1)}$, $\psi_0 \sim \exp(-\int dx W(x))$, exists and represents the zero-mode of the annihilation operator $\hat{a}$: in this case, the q-oscillator model contains a bound state spectrum. Then it makes sense to renormalize the Hamiltonians $h^{(1),(2)}$ so as to set zero energy at the beginning of the continuous spectrum

$$h^{(1)} = H_1 + \frac{1}{1-q^2}, \quad h^{(2)} = H_2 + \frac{q^2}{1-q^2} \tag{43}$$

with the closure condition $H_1 = H_2$.

Now we present the generalization of the above q-oscillator model based on the q-deformation of the polynomial HSUSY algebra (34), (36) developed in the previous section. In order to reproduce q-oscillator relation (37), we can use only the following relations in (36):

$$\hat{a}\hat{a}^+ = H_1^n + \frac{1}{1-q^{2n}}, \quad q^{2n}\hat{a}^+\hat{a} = H_2^n + \frac{q^{2n}}{1-q^{2n}}. \tag{44}$$

If $H_1 = H_2$, then

$$\hat{a}\hat{a}^+ - q^{2n}\hat{a}^+\hat{a} = 1. \tag{45}$$

For $n \geq 2$, the polynomial on the right-hand side of (44) always has pairs of complex conjugate roots. According to the ladder construction (36), the relevant Darboux intertwining operators in $\hat{a}, \hat{a}^+$ are comprised of a number of primitive Darboux transformations of second order in derivatives. There are two nonequivalent sets of q-oscillator models corresponding to even or odd n. For odd n, the above ladder contains one Darboux transformation of first order in derivatives, while for even n it consists only of second-derivative primitive elements.

Let us consider, in particular, the case $n = 2$, where the creation and annihilation operators are given by relations (34). The q-self-similarity equation follows from (15) and (34) and has the form

$$\begin{aligned} V(x) &= f^2(x) - 2f'(x) + \frac{f''(x)}{2f(x)} - \left(\frac{f'(x)}{2f(x)}\right)^2 - \frac{1}{4f^2(x)(1-q^4)} \\ &= q^2 f^2(qx) + 2qf'(qx) + \frac{f''(qx)}{2f(qx)} - \left(\frac{f'(qx)}{2f(qx)}\right)^2 - \frac{q^2}{4f^2(qx)(1-q^4)}. \end{aligned} \tag{46}$$

This equation is much more complicated than (38), and the existence of its regular solution is not obvious (see also the discussion in Sec. 8), though, for the semiaxis problem, the linearization method seems to be convergent and applicable to the proof of existence.

The deformation of a bosonic oscillator can be defined in terms of the q-commutator, $\hat{a}\hat{a}^+ - q^2\hat{a}^+\hat{a}$, where $\hat{a}$ and $\hat{a}^+ = (\hat{a})^+$ are the annihilation and creation operators and q is a real number which, without loss of generality, will be assumed positive. The commutator can be defined in different ways. We define it [13] as follows:

$$\hat{a}\hat{a}^+ - q^2\hat{a}^+\hat{a} = 1. \tag{47}$$

It is supplemented with the number operator N such that

$$[N, \hat{a}^+] = \hat{a}^+, \quad [N, \hat{a}] = -\hat{a}, \tag{48}$$

where the usual commutators are implied. For the harmonic oscillator, when $q = 1$, the ground state is a zero mode of the annihilation operator, $\hat{a}\psi_0 = 0$, and the number operator N can be normalized to be $N = \hat{a}^+\hat{a}$ so that the zero occupation number is assigned for the ground state.

If $q \neq 1$, the number operator is no longer bilinear in $\hat{a}, \hat{a}^+$. However, there is a central element[1] given by [13],

$$\hat{\zeta} = q^{-2N}\left([N]_q - \hat{a}^+\hat{a}\right), \tag{49}$$

where the q-symbol $[N]_q$ is defined as

$$[N]_q = \frac{1-q^{2N}}{1-q^2}.$$

The operator $\hat{\zeta}$ commutes with all generators of the q-oscillator algebra, which is seen from the relations

$$\hat{a}^+F(N) = F(N-1)\hat{a}^+, \quad \hat{a}F(N) = F(N+1)\hat{a}. \tag{50}$$

Therefore, its eigenvalues ζ enumerate the representations, and, for a given representation with chosen ζ, one can find the connection between the bilinear operators $\hat{a}^+\hat{a}$ and $\hat{a}\hat{a}^+$ and the number operator N,

$$\begin{aligned} \hat{a}^+\hat{a} &= [N]_q - \zeta q^{2N}, \\ \hat{a}\hat{a}^+ &= [N+1]_q - \zeta q^{2N+2}, \end{aligned} \tag{51}$$

where the q-commutator (47) is used. These operators commute with N, and their spectra are generated by the spectrum of N. From Eqs. (51), the commutator can be evaluated,

$$\hat{a}\hat{a}^+ - \hat{a}^+\hat{a} = q^{2N}\left(1 - \frac{\zeta}{\zeta_c}\right), \tag{52}$$

where ζ_c is the critical value of ζ for which the commutator vanishes. We have

$$\zeta_c = -\frac{1}{1-q^2}.$$

Any representation can be described [13, 14] in the basis of eigenfunctions of the number operator with eigenvalues ν_n, $N\psi_n = \nu_n\psi_n$. As in the case of the conventional harmonic oscillator, due to relations (48), all eigenstates can be built from one selected state ψ_0 by means of the ladder operators, $\psi_{n+1} \simeq \hat{a}^+\psi_n$, $\psi_{n-1} \simeq a\psi_n$ and hence $\nu_{n\pm1} = \nu_n \pm 1$. For a chosen ψ_0, the nonequivalent representations are parameterized by the values of ν_0 lying in the unit interval, $0 \leq \nu_0 < 1$, since the shift on an integer number maps one state to another of the same representation. We can redefine the number operator, $N = \tilde{N} + \nu_0$, so that the eigenvalues of $\tilde{N}$ become integer numbers. This redefinition is compatible with

[1] In fact, this element is not unique, since any periodic function $\phi(N) = \phi(N \pm 1)$ also belongs to the central-element subspace, which thereby consists of any algebraic combinations of $\hat{\zeta}$ and $\phi(N)$. However, for further purposes, it is sufficient to select only one central element in the form (49).

the basic commutation relations (47) and (48), and, due to Eq. (49), corresponds to the following change of the central element:

$$\hat{\zeta}' = \hat{\zeta} q^{2\nu_0} - \zeta_c \left(q^{2\nu_0} - 1\right). \tag{53}$$

For $\zeta \neq \zeta_c$, any representation characterized by two parameters ν_0 and ζ is equivalent to the representation ${\nu'}_0 = 0, \zeta'$. In this case, it is sufficient to shift eigenvalues of the number operator to integer numbers and to study the eigenvalues of the central element. For $\zeta = \zeta_c$, the representations are labeled by values of ν_0.

As to the classification of q-oscillator representations in terms of ζ, we have the following three types of nonequivalent representations [13, 14]: the Fock representation $\zeta > \zeta_c$, non-Fock representations for $\zeta < \zeta_c$, and the special representation for $\zeta = \zeta_c$.

The Fock representation is characterized by the existence of a ground state of the number operator—zero mode of the annihilation operator, $a\psi_0 = 0$. Since, from Eqs. (47) and (52) and ${\nu'}_0 = 0$, we have $\zeta' = 0$, the Fock representation is unique and can be constructed for any $0 < q < \infty$.

The non-Fock representations can appear only if $0 < q \leq 1$ and thereby $\zeta_c < 0$. The spectrum of N is unbounded from below. The consistency with Eqs. (51) requires $\zeta < \zeta_c$. Due to relation (53), there is a one-parameter family of irreducible non-Fock representations. Their parameterization can be realized either with the help of the central element by fixing $\nu_0 = 0$ or by means of the parameter $0 \leq \nu_0 < 1$ for a fixed value of ζ. In the sequel, we make use of the second part of the alternative, and, for definiteness, we set $\zeta = 2\zeta_c$.

In the special representation (again for $0 < q \leq 1$), the creation and annihilation operators commute (see Eq. (52)), and it follows from Eq. (47) that the bilinear operators (51) become c-numbers:

$$\hat{a}\hat{a}^+ = \hat{a}^+\hat{a} = \frac{1}{1-q^2} \equiv \rho^2. \tag{54}$$

This representation is generated by a unitary operator U so that $\hat{a} = \rho U$ and $\hat{a}^+ = \rho U^+$. The powers of creation and annihilation operators form a discrete subgroup of the group $U(1)$. In this representation, the number operator cannot be expressed as a function of $\hat{a}$ and $\hat{a}^+$.

Thus, any Hamiltonian realization of the q-oscillator algebra can be decomposed into the above irreducible representations. In what follows, we restrict ourselves to the q-oscillator model with a *local* (in x) Hamiltonian[2] of the Schrödinger type [11].

Let us decompose the q-oscillator representation given by the model (26)–(38) into irreducible representations. Obviously, the bound state spectrum having the true ground state forms the Fock representation, and the continuous spectrum consists of the set of non-Fock representations. From the q-oscillator relations (51), we find the number operator as a function of the Hamiltonian for both cases,

$$N = \frac{\ln\left[(1-q^2)^2 H^2\right]}{4 \ln q}, \tag{55}$$

where the nonlinear operator relation can be interpreted in terms of the spectral decomposition for the Hamiltonian H. For the Fock representation, this connection was found in [11], and here we extend it to the entire set of non-Fock representations for positive energies. Accordingly, the central element can be defined as follows:

$$\hat{\zeta} = \zeta_c(1 + \mathrm{sign} H), \quad H\psi = E\psi, \tag{56}$$

and, in the Fock representation, we have $\zeta = 0$, $E < 0$, and $E_{n+1} = q^2 E_n$, while, in the non-Fock representation, we obtain $\zeta = 2\zeta_c$ and $E > 0$. Both discrete and continuous sequences of energy levels have $E = 0$ as the accumulation point.

The special representation could be realized on zero-energy states (at the threshold between discrete and continuous spectra) where $\zeta = \zeta_c$. However, for this particular model, it can be proved ([20]) that the physical states[3] for zero energy do not exist because the two zero-energy solutions have increasing

[2]There are also many possibilities to construct nonlocal Hamiltonians (see, for e.g., [12, 19]), which we do not discuss here.
[3]By physical states, we mean the wave functions that remain bounded at infinity.

asymptotic behavior $\sim \sqrt{x}$. Therefore, the special representation does not appear in the decomposition of the space of physical wave functions.

Thus, we have shown that the q-oscillator system with the local Hamiltonian (43) is composed of two types of irreducible representations, in particular, the continuous part of the spectrum is covered by non-Fock representations parameterized by the second invariant in the interval $0 \leq \nu_0 < 1$ corresponding to the energy interval $|\zeta_c| \geq E > q^2|\zeta_c|$,

$$\mathcal{H} = \mathcal{H}_F \bigoplus \int_{0 \leq \nu_0 < 1} d\mu(\nu_0) \mathcal{H}_{NF}^{\nu_0}, \tag{57}$$

where $\mathcal{H}$ denotes the appropriate Hilbert space of wave functions (of both bound states and scattering states). The q-oscillator generators act on scattering wave functions as pseudodifferential operators in accordance with (26).

For the polynomial q-SUSY realization, the equations for N and ζ are the same as for $n = 1$ provided that one makes the substitution $q \longrightarrow q^n$, $H \longrightarrow H^n$. There are two nonequivalent sets of q-oscillator models corresponding to even or odd n. The Hamiltonians belonging to an odd algebra, in general, have the representation content of the $n = 1$ model. In the even case, it is clear that the Fock representation is not involved in the decomposition of the related q-oscillator wave function space since $\|\hat{a}^+\hat{a}\| > 1/(1-q^{2n})$. From a regular nodeless solution $f(x)$ of Eq. (46), we obtain the q-oscillator model with nonnegative Hamiltonian. Negative energy levels are allowed only for the Hamiltonians unbounded from below because of the properties of non-Fock representations. Then such Hamiltonians must be associated with singular potentials.

7. SSQM AND THE SCATTERING PROBLEM

Consider the one-dimensional scattering problem on the line, i.e., $x \in (-\infty, +\infty)$, for the Hamiltonian

$$h^{(1)} = -\partial^2 + V^{(1)}(x) = a^+a^- \tag{58}$$

with a potential that tends (fairly rapidly) to its constant asymptotic value,

$$\begin{gathered} V^{(1)}(x) \longrightarrow C, \\ x \rightarrow \pm\infty, \end{gathered} \tag{59}$$

where the discrete energy spectrum of $h^{(1)}$ is bounded by C $(0 \leq E_n \leq C)$,

$$h^{(1)}\Psi_n^{(1)}(x) = E_n\Psi_n^{(1)}(x), \quad n = 0, 1, .., \tag{60}$$

while the continuous spectrum $E(k) \geq C$ is given by

$$h^{(1)}\Psi_k^{(1)}(x) = E(k)\Psi_k^{(1)}(x); \quad E(k) = k^2 + C. \tag{61}$$

The scattering wave functions satisfy the asymptotic conditions (see, e.g., [17])

$$\Psi_{k,-\infty}^{(1)} = e^{+ikx} + R^{(1)}(k)e^{-ikx}, \tag{62}$$

$$\Psi_{k,+\infty}^{(1)} = T^{(1)}(k)e^{+ikx}, \tag{63}$$

where $R^{(1)}(k)$ and $T^{(1)}(k)$ are the reflection and transmission coefficients, respectively.

The ladder operators $a^{\pm}$ (4) are asymptotically expressed in the form

$$a_{\pm\infty}^- = -\partial + W_{\pm}, \quad a_{\pm\infty}^+ = \partial + W_{\pm}, \tag{64}$$

with

$$W_{\pm} = \lim_{x\to\pm\infty} W(x), \quad W_{\pm}^2 = C. \tag{65}$$

Then the asymptotic scattering wave function of the partner Hamiltonian $h^{(2)}$ is proportional to

$$a^-_{-\infty}\Psi^{(1)}_{k,-\infty} = (-ik + W_-)\ [e^{ikx} - R^{(1)}(k)\frac{k - iW_-}{k + iW_-}e^{-ikx}], \tag{66}$$

while

$$a^-_{+\infty}\Psi^{(1)}_{k,+\infty} = (-ik + W_-)\ [T^{(1)}(k)\frac{k + iW_+}{k + iW_-}e^{ikx}]. \tag{67}$$

Hence the transmission and reflection coefficients associated to $h^{(2)}$ have the form [8, 18]

$$T^{(2)}(k) = T^{(1)}(k)\frac{k + iW_+}{k + iW_-}, \tag{68}$$

$$R^{(2)}(k) = -R^{(1)}(k)\frac{k - iW_-}{k + iW_-}. \tag{69}$$

It is well known [17] that the transmission coefficient contains physical poles in the upper half of the complex k-plane, and their positions correspond to the energies of bound states $E_j = -\kappa_j^2 + C$, $k_j = i\kappa_j$. Thus, the difference in physical pole structure of $T^{(1)}$ and $T^{(2)}$ depends on the signs of $W_{\pm}$. The unitary property always holds for $T^{(2)}$ and $R^{(2)}$ whenever it holds for $T^{(1)}$ and $R^{(1)}$.

The scattering problem for HSSQM, as defined in Eqs. (14), (15), can be obtained from Eqs. (68), (69) by iteration:

$$\begin{aligned} T^{(2)}(k) &= T^{(1)}(k)\frac{(k + iW_{2+})(k + iW_{1+})}{(k + iW_{2-})(k + iW_{1-})}, \\ R^{(2)}(k) &= R^{(1)}(k)\frac{(k - iW_{2-})(k - iW_{1-})}{(k + iW_{2-})(k + iW_{1-})}. \end{aligned} \tag{70}$$

Assuming that $f_{\pm} = \lim_{x\to\pm\infty} f(x)$ are constant (so that the potentials $V^{(1),(2)}$ are finite as $x \to \pm\infty$) and using Eq. (22), we obtain the following asymptotic values of the superpotentials:

$$W_{1\pm} = \frac{c}{4f_{\pm}} - f_{\pm}, \quad W_{2\pm} = -\frac{c}{4f_{\pm}} - f_{\pm}. \tag{71}$$

To obtain the same asymptotic values of the potentials V_i at $+\infty$ and at $-\infty$, we must have $f_+^2 = f_-^2$ or $16f_+^2 f_-^2 = c^2$.

For $d > 0$ and nonsingular potentials, the function $f(x)$ is nodeless and, in order to have equal asymptotics (59), it is necessary that $f_+ = f_- = f_\infty$. The transmission and reflection coefficients are connected as follows:

$$T^{(2)} = T^{(1)}, \qquad R^{(2)} = R^{(1)}\ \frac{(k + if_\infty)^2 - \frac{d}{4f_\infty^2}}{(k - if_\infty)^2 - \frac{d}{4f_\infty^2}}. \tag{72}$$

As in (59), the coincidence of the transmission coefficients is caused by equal asymptotic values for each potential at $\pm\infty$.

8. q-OSCILLATOR SCATTERING AND DUAL SELF-SIMILARITY

As to the scattering problem for the above models, the asymptotic relations (66)–(69) should be modified with regard for Eq. (26). As a result, one finds the connection between the transmission and reflection coefficients $h^{(1)} = a_q^+ a_q^-$ and $h^{(2)} = q^2 a_q^- a_q^+$ given by (27), which now involves the dilatation parameter q:

$$\begin{aligned} T^{(2)}(k) &= T^{(1)}(\frac{k}{q})\frac{(k + iW_+q)}{(k + iW_-q)}, \\ R^{(2)}(k) &= -R^{(1)}(\frac{k}{q})\frac{(k - iW_-q)}{(k + iW_-q)}. \end{aligned} \tag{73}$$

Thus, both in the discrete and continuous parts of the spectrum, we remove the spectrum degeneracy typical for the ordinary SSQM and come to the scaling relations.

The q-deformation of scattering data allows us to formulate a dual self-similarity condition, which, in general, does not imply (standard) self-similarity of potentials. As in Sec. 5, we again formulate this condition as a closure condition, but for the scattering amplitudes in the momentum space:

$$T^{(2)}(k) = T^{(1)}(k), \quad R^{(2)}(k) = R^{(1)}(k), \tag{74}$$

which is combined with Eq. (73). Their solutions for $q < 1$ and $W_+ = -W_- > 0$ have the form

$$\begin{aligned} T^{(1)}(k) &= \prod_{n=1}^{\infty} \frac{(k + iW_+q^n)}{(k - iW_+q^n)} t(\ln k) \equiv \frac{(k - iW_+)}{(k + iW_+)} {}_1\Phi_0(-1; iW_+/k) t(\ln k), \\ R^{(1)}(k) &= \prod_{n=1}^{\infty} \frac{(k + iW_+q^n)}{(k - iW_+q^n)} r(\ln k) \equiv \frac{(k - iW_+)}{(k + iW_+)} {}_1\Phi_0(-1; iW_+/k) r(\ln k), \end{aligned} \tag{75}$$

where $|t|^2 + |r|^2 = 1$ and $t(z)(r(z))$ are periodic (antiperiodic) functions,

$$t(z + \ln q) = t(z), \quad r(z + \ln q) = -r(z),$$

and the self-similar transmission and reflection coefficients are parameterized by the hypergeometric function ${}_1\Phi_0(a; z)$ [24].

If one imposes the condition of vanishing of the reflection coefficient for large k, which amounts to the validity of the Born expansion [17], one is led to assume that $r(\ln k) = 0$ and to put $t(\ln k) = 1$, respectively. It is clear that the underlying reflectionless potential has infinitely many bound states with an accumulation point around zero. Since the Born expansion is valid, this means that such a potential is slowly decreasing and oscillating.

By arguments that led to (70) and (72), one can derive q-deformed HSUSY relations between the transmission and reflection coefficients. For the even series, it follows from (44) and (72) that the transmission coefficient is trivial.

For the case of the q-oscillator, one can repeat the preceding arguments concerning the validity of the Born expansion for a well-decreasing potential and conclude that the reflection coefficient should vanish. Therefore, for $n = 2j$, such a q-oscillator should not cause any scattering, and, in this case, it is unlikely to have regular, decreasing q-oscillator potential on the entire axis. Still there is an open possibility to constructing the q-oscillator for even n with regular potential on the semi-axis. As to the odd series, $n = 2j + 1$, the construction of a nontrivial q-oscillator potential does not seem to encounter any obstacles on the entire axis.

Now we proceed to the general description of deformed dynamical algebras which are realized on wave functions of the Schrödinger operator and, in general, induce a *nonlinear* mapping of energy levels.

9. Generalized realizations of q-oscillator models

To describe the q-oscillator algebra in a generalized form, we introduce new creation and annihilation operators A^+ and A by means of a transformation preserving the commutation relations with the number operator (48):

$$\hat{a} = F(N)A, \quad \hat{a}^+ = A^+F^*(N).$$

We may assume that the function $F(x)$ is real since its phase factor does not play any role in the relations to be derived below. From (50) and (47), the q-commutator with a new deformation parameter is reproduced if

$$F^2(N - 1) = C_q F^2(N), \tag{76}$$

where C_q is a positive c-number; this choice preserves the bosonic character of the algebra. The solution of the previous equation has the form

$$F^2(N) = C_q^{-N} \phi^{-1}(N), \quad \phi(N - 1) = \phi(N).$$

The function ϕ is periodic, positive, and, in fact, takes a definite value for a particular representation, and, therefore, is invariant. From Sec. 6, it follows that $\phi = \phi(\nu_0)$. The basic q-commutator takes the form

$$AA^+ - q^2 C_q A^+ A = C_q^N \phi(N). \tag{77}$$

The central element is modified as follows:

$$\hat{\zeta} = q^{-2N} \left([N]_q - C_q^{1-N} \phi^{-1}(N) A^+ A \right). \tag{78}$$

The bilinear operators are also modified,

$$A^+ A = C_q^{N-1} \phi(N)\, a^+ a, \quad AA^+ = C_q^N \phi(N)\, aa^+, \tag{79}$$

as compared with Eqs. (51). The additional q-commutator is constructed with a new deformation parameter:

$$AA^+ - C_q A^+ A = \left(1 - \frac{\zeta}{\zeta_c}\right) C_q^N q^{2N} \phi(N). \tag{80}$$

The classification of representations is similar to that in Sec. 6, since we have not introduced any new algebra but have chosen different elements of the same universal enveloping algebra as basic generators. The special choice $C_q = 1/q$ leads to the algebra from [12], whereas $C_q = 1/q^2$ leads to the algebra introduced in [21].

Now we proceed to nonlinear realizations of the spectrum (generating deformed algebras) in quantum mechanics. Such realizations are of interest for the algebraic description of physical systems whose spectra are only approximately related to the harmonic or q-harmonic oscillators. We start with the generalized intertwining relations

$$H_1 A^+ = A^+ g(H_2), \quad g(H_2) A = A H_1, \tag{81}$$

where $H_{1,2}$ are Hamiltonians of two quantum systems with related energy spectra and wave functions, $H_i \psi_i = E_i \psi_i$, $E_1 = g(E_2)$, and $\psi_1 = A^+ \psi_2$. In the case of a polynomial SSQM, we have $g(x) = q^2 x$. By virtue of (81), $[A^+ A, H_1] = [AA^+, g(H_2)] = 0$. If we assume, in addition, that the function g is invertible, we obtain $[AA^+, H_2] = 0$. Hence the bilinear operators commute with the Hamiltonians and represent, in general, symmetry operators [22]. In one-dimensional QM, they are functions of Hamiltonians:

$$A^+ A = \sigma_1(H_1), \quad AA^+ = \sigma_2(H_2), \tag{82}$$

where σ_i are arbitrary invertible functions. Then, for analytic functions $f(z)$, we have

$$f(\sigma_1(H_1)) A^+ = \sum_{m=0}^{\infty} c_m \sigma_1^m(H_1) A^+ = A^+ \sum_{m=0}^{\infty} c_m \sigma_2^m(H_2) = A^+ f(\sigma_2(H_2)). \tag{83}$$

Now let us define the inverse functions

$$\sigma_i(\pi_i(z)) = \pi_i(\sigma_i(z)) = z, \tag{84}$$

and choose $f(z) = \pi_i(z)$. Then the mapping g and its inverse g^- are determined by these functions:

$$g(z) = \pi_1(\sigma_2(z)), \quad g^-(z) = \pi_2(\sigma_1(z)). \tag{85}$$

With the help of the inverse mapping, we obtain the second set of intertwining relations:

$$A^+ H_2 = g^-(H_1) A^+. \tag{86}$$

To find links between energy levels of the same dynamical system, we impose the self-similarity condition $H_1 = H_2$. In this way, we discover a deformed dynamical algebra of a Hamiltonian. The ladder procedure connects different levels and eigenstates belonging to the following sequence:

$$\begin{array}{rl} A^+: & \cdots \longrightarrow g^-(g^-(E)) \longrightarrow g^-(E) \longrightarrow E \longrightarrow g(E) \longrightarrow g(g(E)) \longrightarrow \cdots \\ A: & \cdots \longleftarrow g^-(g^-(E)) \longleftarrow g^-(E) \longleftarrow E \longleftarrow g(E) \longleftarrow g(g(E)) \longleftarrow \cdots \end{array} \tag{87}$$

For physical systems of oscillator type with Hamiltonians bound from below, these operators can generate the entire spectrum of a model, whereas, on the continuous part of the spectrum, they connect only subsets of levels.

Imposing relations (77) and (80) on the functions $\sigma_{1,2}$ given by (82), we realize the q-oscillator algebra provided that the following constraints for a representation $[\zeta, \nu_0]$ are fulfilled:

$$\ln(\sigma_2(z) - C_q\sigma_1(z)) = \left(1 + \frac{2\ln q}{\ln C_q}\right) \ln\left[\sigma_2(z) - q^2 C_q \sigma_1(z)\right] - \frac{2\ln q}{\ln C_q} \ln\phi(\nu_0) + \ln\left(1 - \frac{\zeta}{\zeta_c}\right) \tag{88}$$

for $C_q \neq 1$, and

$$\sigma_2(z) - q^2\sigma_1(z) = 1 \tag{89}$$

for $C_q = 1$ and $\phi = 1$. In the latter case, the energy mapping has the form

$$E \longrightarrow \pi_1\left(q^2\sigma_1(E) + 1\right). \tag{90}$$

In particular, for $q = 1$, this mapping represents the generalization of the harmonic oscillator spectrum.

The content of irreducible representations of the q-oscillator algebra in a particular model depends on the position of fixed points of the mapping (87). If a fixed-point energy value is higher than the ground state energy, then the Fock representation exists and is realized in between them. If there are no fixed points for finite energies, then the Fock representation spans the entire Hilbert space of wave functions. On the contrary, if the fixed point coincides with the ground state energy, the non-Fock representations appear only in the decomposition into irreducible representations.

Thus, the general strategy suggested by our approach is to find an algebra generating the spectrum by means of the properties of the bound-state spectrum and scattering coefficients, i.e., to determine the functions $\sigma_{1,2}$ satisfying conditions (88). In practice, this can be done only approximately, and the required perturbation theory will be studied elsewhere. In order to find the related potential, it is necessary to extend the inverse scattering method to potentials with $1/x^2$ asymptotics at infinity [16].

We believe that the analysis of fixed points of functional mappings (87) can lead to a better understanding of the physical meaning (see [23] and the references therein) of a q-deformed algebra. The number of fixed points might represent a sort of topological invariant under "smooth" perturbations of a potential. The problem of realization of the special representation on wave functions of a local Hamiltonian (a fixed point of the energy mapping) remains open.

We also note that there exist other generalizations [23] of the q-oscillator algebra of the form

$$\hat{a}\hat{a}^+ - \Phi_1^2(N)\hat{a}^+\hat{a} = \Phi_2^2(N), \tag{91}$$

where Φ_i are sufficiently regular real functions. In fact, one can redefine the basic elements of the universal enveloping algebra,

$$\hat{a} = M(N)A, \quad \hat{a}^+ = A^+M(N), \quad M^* = M, \tag{92}$$

so as to replace the function $\Phi_1(N)$ by a constant c. We restrict ourselves to transformations which do not change the commutation relations with the number operator. The required function $M(N)$ satisfies the following equation:

$$\Phi_1(N)M(N+1) = cM(N). \tag{93}$$

In the special representation $\{\zeta, \nu_0\}$, its solution has the form

$$M(n+\nu_0) = M(\nu_0)c^{-n}\prod_{l=0}^{n-1}\Phi_1(l+\nu_0), \tag{94}$$

where $M(\nu_0)$ is an arbitrary function for $0 \le \nu_0 < 1$. In qualitative agreement with [23], we obtain the q-deformed algebra (77) but with an arbitrary function $\Phi_2(N)$ on the right-hand side. We note, however, that we have the additional freedom (92) to modify $\Phi_2(N)$, which does not change the enveloping algebra and was not considered in [23]. Thus, nonequivalent q-deformed algebras are only those which cannot be related by these "gauge" transformations.

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