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POLYNOMIAL APPROXIMATIONS ON DISJOINT SEGMENTS

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The problem on polynomial approximation of functions from some class defined on a compact set E of the complex plane is studied. The case where E is the union of a finite number of segments is considered. Bibliography: 12 *titles.*

Let E be a compact set of the complex plane, and let X be a class of functions on E. If E possesses interior points, then the functions from X are analytic in the interior of E. The problem on description of the class X in terms of the rate of polynomial approximation of functions from X is a classical theme of complex analysis. Assertions from this field of approximation theory can be divided into the so-called direct and inverse theorems. Direct theorems assert that any function from X can be approximated by a polynomial of degree at most n with rate $b(n,z)$, where $Z \in E$ or $Z \in \partial E$. Inverse theorems assert that if f can be approximated with rate $c(n,z)$, then $f \in X$. Assertions on the consistency of direct and inverse theorems, i.e., $b(n,z) \asymp c(n,z)$, are essential.

The inverse theorems deal with general compact sets (cf. [1–3]). There are a few papers on direct theorems for a disconnected set E. We mention the paper [4] of Walsh, which was published in the 30s, and only three recent papers [5–7]. This is incommensurable with the hundreds of papers where E is a continuum.

In this paper, we consider the case where E consists of a finite number of segments. Thus, E has no interior points, which is a new (relative to [5]) property. The proof includes operations with continua of types that never appeared in approximation theory.

§1. Construction of P_n. The Geometric Stage

Let $S'_k \subset S_k$ be the segment such that the middle points of S'_k and S_k coincide and the length of S'_k is equal to half of the length of S_k. We take arbitrary points $Z_k \in S'_k$ and construct a continuum $\Gamma(Z_1,\dots,Z_m) \supset E$. Let U_k denote the rectangle such that S_k is the median and the length of the sides that are perpendicular to S_k is equal to 4δ. We choose $\delta > 0$ such that the rectangles U_k are pairwise disjoint. Let $U'_k \subset U_k$ be rectangles such that S_k is the median and the length of the sides that are perpendicular to S_k is equal to 2δ. We choose a point A outside the union of all U_k. We connect A with a vertex T_k of U_k, $k = 1,2,\dots,m$, by a curve of class C^2 such that the obtained curves are pairwise disjoint. We define by γ_0 the union of these curves. Let $\tau_k(Z_k)$ denote the segment of length δ such that $\tau_k(Z_k)$ is perpendicular to S_k, the first endpoint is Z_k, and the second endpoint and T_k lie on the same side of U_k. We connect T_k with the second endpoint of $\tau_k(Z_k)$ by a curve $\gamma_k \subset U_k \setminus U'_k$ of class C^2 such that the curvature radius of γ_k has a lower bound $\sigma_0 > 0$ that is independent of Z_k.

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Let $\Gamma(Z_1,\dots,Z_m)=\bigcup\limits_{k=1}^{m}(S_k\cup\tau_k(Z_k)\cup\gamma_k)\cup\gamma_0$.

§2. Formulation of the Theorem and Preparations

Let E be the join of segments $S_k=[a_k,b_k]$, $k=1,2,\dots,m$, of the complex plane $\mathbb{C}$.

Let w_k, $k=1,2,\dots,m$, be moduli of continuity, let r_k, $k=1,2,\dots,m$, be nonnegative numbers, let $r(r_1,r_2,\dots,r_m)$ be an m-dimensional vector, and let $w(w_1,w_2,\dots,w_m)$ be an m-dimensional modulus of continuity. Let the moduli of continuity $w_1,\dots,w_m$ satisfy the condition

$$\int\limits_0^x\frac{w_j(t)}{t}dt+x\int\limits_x^{A_j}\frac{w_j(t)}{t^2}dt\leqslant cw_j(x)$$

for every $x\in(0,A_j)$, where A_j is the length of S_k. Let $E=\bigcup\limits_{k=1}^{m}S_k$, and let $\Lambda_w^r(E)$ be the set of functions f on E such that $f^{(r_k)}\in\Lambda_{w_k}(S_k)$, where

$$\Lambda_{w_k}(S_k)=\{\varphi\text{ is defined on }S_k,\ |\varphi(z)-\varphi(\zeta)|\leqslant C_\varphi w_k(|z-\zeta|)\quad\forall z,\zeta\in S_k\}.$$

We introduce the function $\rho_h(z)$ (cf. [8]), which defines the approximation rate. We consider Green's function $G(z,\infty)$ for $C\setminus E$ with a pole at infinity. Let $L_h=\{z\in C\setminus E,\ G(z,\infty)=h\}$. For a sufficiently small $h\leqslant h_0$, let L_h^k, $k=1,2,\dots,m$, be the closed part of the level line L_h containing the segments S_k, $k=1,2,\dots,m$. Let $\rho_h(z)=\operatorname{dist}(z,L_h)$.

Theorem. *Let $f\in\Lambda_w^r(E)$. Then for every $n=1,2,\dots$ there exists a polynomial $P_n(z)$ of degree at most n such that*

$$|f(z)-P_n(z)|\leqslant c\rho_{1/n}^{r_k}(z)w_k(\rho_{1/n}(z)),$$

where $z\in S_k$, $k=1,\dots,m$, and c is independent of n and z.

We continue the function $f\in\Lambda_w^r(E)$ to $\Gamma(Z_1,\dots,Z_m)$. On the common part of γ_0, we have $f\equiv 0$. The derivatives $f'(z),\dots,f^{(r_k)}(z)$ are defined for $z\in S_k$. On $[Z_k,T_k]$, we define f in the form of the Hermite interpolation polynomial (cf. [9]) f_k with the properties

$$\begin{aligned}&f(Z_k)=f_k(Z_k),\\&f'(Z_k)=f_k'(Z_k),\\&\dots\dots\dots\dots\dots\\&f^{(r_k)}(Z_k)=f_k^{r_k}(Z_k).\end{aligned}$$

The number of these conditions is equal to r_k+1. Let

$$\begin{aligned}&f_k(T_k)=0,\\&f_k'(T_k)=0,\\&\dots\dots\dots\dots\dots\\&f_k^{(r_k)}(T_k)=0,\end{aligned}$$

i.e., the number of these conditions is also equal to r_k+1. Thus, we have $2r_k+2$ conditions.

§3. Construction of an Approximate Polynomial

Let $C\setminus\Gamma(Z_1,\dots,Z_m)=\Omega(Z_1,\dots,Z_m)$. Let $\lambda=\varphi_{Z_1,\dots,Z_m}(z)$ be a function with the normalization

$$\varphi_{Z_1,\dots,Z_m}(\infty)=\infty,\quad \lim_{z\to 0}\frac{\varphi_{Z_1,\dots,Z_m}(z)}{z}>0$$

establishing the conformal mapping of $\Omega(Z_1,\dots,Z_m)$ onto the exterior of the circle $|\lambda|>1$. The inverse mapping is $z=\varphi_{Z_1,\dots,Z_m}(\lambda)$. We choose $R=1+1/n$, where n is commensurable with the degree of the polynomial. Let

$$\xi_{R,\theta}(\zeta)=\xi_{R,\theta}(\zeta_{\{Z_1,\dots,Z_m\}})=\Psi(Re^{-i\theta}\varphi(\zeta)),$$

where $\zeta\in\Omega(Z_1,\dots,Z_m)$. Let $\gamma(Z_1,\dots,Z_m)$ be a contour surrounding $\Gamma(Z_1,\dots,Z_m)$. We consider the circuit of this contour in the positive direction. We take some sufficiently large p that is independent of z.

Let $\zeta\in\gamma(Z_1,\dots,Z_m)$, let $z\in E$, and let

$$K(z,\zeta,\theta)=\frac{1}{\xi_{R,\theta}(\zeta)-z}+\frac{-\zeta+\xi_{R,\theta}(\zeta)}{(\xi_{R,\theta}(\zeta)-z)^2}+\dots+\frac{(-\zeta+\xi_{R,\theta}(\zeta))^{p-1}}{(\xi_{R,\theta}(\zeta)-z)^p},$$
$$F(z,\theta)=\frac{1}{2\pi i}\int\limits_{\gamma(Z_1,\dots,Z_m)} f(\zeta)K(z,\zeta,\theta)\,d\zeta,$$

where f is continued to $\Gamma(Z_1,\dots,Z_m)$. The function $K(z,\zeta,\theta)$ maps $\Omega(Z_1,\dots,Z_m)$ onto the exterior of the circle $|R|>1$. However, for ζ_1 and ζ_2 lying on different sides of a cut the values $K(z,\zeta_1,\theta)$ and $K(z,\zeta_2,\theta)$ need not coincide.

Remark. The constructed continuum has no interior points. Assume that f is an analytic function in a small neighborhood of S_k. For example, let f^* approximate f in a neighborhood of E. Then we can use the Cauchy formula. We modify the contour $\gamma(z_1,\dots,z_m)$. Let τ_k be a contour that surrounds S_k and is close to S_k. Let $z\in S_k$, and let γ_{τ_k} be the contour such that the part of the initial contour surrounding S_k and the part of $\tau_k(Z_k)$ surrounded by τ_k are replaced with τ_k. Then

$$f(z)=\frac{1}{2\pi i}\int\limits_{\tau_k}\frac{f(\zeta)}{\zeta-z}d\zeta=\frac{1}{2\pi i}\int\limits_{\gamma_{\tau_k}}\frac{f(\zeta)}{\zeta-z}d\zeta=\frac{1}{2\pi i}\lim_{\text{width of }\gamma_{\tau_k}\to 0}\int\limits_{\gamma_{\tau_k}}\frac{f(\zeta)}{\zeta-z}d\zeta,$$
$$F(z,\theta)=\frac{1}{2\pi i}\int\limits_{\gamma_{\tau_k}} f(\zeta)K(z,\zeta,\theta)\,d\zeta,\quad z\in S_k.$$

We have

$$F(z,\theta)-f(z)=\frac{1}{2\pi i}\int\limits_{\gamma_\tau\{Z_1,\dots,Z_m\}} f(\zeta)\left(K(z,\zeta,\theta)-\frac{1}{\zeta-z}\right)d\zeta,$$

$$\begin{aligned}K(z,\zeta,\theta)-\frac{1}{\zeta-z}&=\frac{1}{\xi_{R,\theta}(\zeta)-z}+\frac{-\zeta+\xi_{R,\theta}}{(\xi_{R,\theta}-z)^2}+\dots+\frac{(-\zeta+\xi_{R,\theta})^{p-1}}{(\xi_{R,\theta}-z)^p}-\frac{1}{\zeta-z}\\&=-\frac{(\xi_{R,\theta}-\zeta)^p}{(\xi_{R,\theta}-z)^p(\zeta-z)},\end{aligned} \tag{1}$$

$$F(z,\theta)-f(z)=-\frac{1}{2\pi i}\int\limits_{\gamma_\tau\{Z_1,\dots,Z_m\}} f(\zeta)\frac{(\xi_{R,\theta}-\zeta)^p}{(\xi_{R,\theta}-z)^p(\zeta-z)}d\zeta.$$

For $z\in S_k$, we represent the function $f(\zeta)$ from (1) in the form

$$f(\zeta)=f(z)+f'(z)(\zeta-z)+\dots+\frac{1}{r_k!}f^{(r_k)}(z)(\zeta-z)^{(r_k)}+\varphi(\zeta,z). \tag{2}$$

By (1) and (2), we find

$$\begin{aligned}F(z,\theta)-f(z)&=-\frac{1}{2\pi i}\int\limits_{\gamma_\tau}\Big(f(z)+f'(z)(\zeta-z)+\dots+\frac{1}{r_k!}f^{(r_k)}(z)(\zeta-z)^{r_k}\\&\quad+\varphi(\zeta,z)\Big)\frac{(\xi_{R,\theta}-\zeta)^p}{(\xi_{R,\theta}-z)^p(\zeta-z)}d\zeta\\&=-\frac{f(z)}{2\pi i}\int\limits_{\gamma_\tau}\frac{(\xi_{R,\theta}-\zeta)^p}{(\xi_{R,\theta}-z)^p(\zeta-z)}d\zeta-\frac{f'(z)}{2\pi i}\int\limits_{\gamma_\tau}\frac{(\xi_{R,\theta}-\zeta)^p}{(\xi_{R,\theta}-z)^p}d\zeta-\dots\\&\quad-\frac{f^{(r_k)}(z)}{2\pi i r_k!}\int\limits_{\gamma_\tau}\left(\frac{\xi_{R,\theta}-\zeta}{\xi_{R,\theta}-z}\right)^p(\zeta-z)^{r_k-1}d\zeta+\frac{1}{2\pi i}\int\limits_{\gamma_\tau}\frac{\varphi(\zeta,z)}{z-\zeta}\left(\frac{\xi_{R,\theta}-\zeta}{\xi_{R,\theta}-z}\right)^p d\zeta.\end{aligned} \tag{3}$$

Let $\widetilde{\gamma}_l=\tau_l(z_k)+\gamma_l+S_l$, and let $\widetilde{\gamma}_k=\tau'_k(z_k)+\gamma_k+\tau_k$, where $\tau'_k(z_k)$ is the part of $\tau_k(z_k)$ that lies outside τ_k. Then

$$\int\limits_{\gamma_\tau}=\int\limits_{\gamma_k}+\sum_{l\neq k}\int\limits_{\widetilde{\gamma}_l}+\int\limits_{\widetilde{\gamma}_k}. \tag{4}$$

By (4), we can write the last term of (3) in the form

$$\frac{1}{2\pi i}\int\limits_{\gamma_\tau}\frac{\varphi(\zeta,z)}{z-\zeta}\left(\frac{\xi_{R,\theta}-\zeta}{\xi_{R,\theta}-z}\right)^p d\zeta=\frac{1}{2\pi i}\int\limits_{\widetilde{\gamma}_k}\dots+\sum_{l\neq k}\int\limits_{\widetilde{\gamma}_l}\dots+\frac{1}{2\pi i}\int\limits_{\gamma_0}\dots. \tag{5}$$

We estimate the integral over $\widetilde{\gamma}_k$ from (5). By the Jacobi formula

$$\varphi(\zeta,z)=\frac{1}{(r_k-1)!}\int\limits_z^\zeta(\zeta-\sigma)^{r_k-1}(f^{(r_k)}(\sigma)-f^{(r_k)}(z))\,d\sigma.$$

We have

$$\frac{1}{2\pi i}\int\limits_{\tilde{\gamma}_k}\frac{\varphi(\zeta,z)}{z-\zeta}\left(\frac{\xi_{R,\theta}-\zeta}{\xi_{R,\theta}-z}\right)^p d\zeta$$

$$=\frac{1}{2\pi i}\int\limits_{\tilde{\gamma}_k}\frac{1}{(r_k-1)!}\left(\frac{1}{z-\zeta}\int\limits_z^\zeta(\zeta-\sigma)^{r_k-1}(f^{(r_k)}(\sigma)-f^{(r_k)}(z))\,d\sigma\right)\left(\frac{\xi_{R,\theta}-\zeta}{\xi_{R,\theta}-z}\right)^p d\zeta. \quad (6)$$

If $r_k=0$, then $\varphi(\zeta,z)=f(\zeta)-f(z)$. Hence the right-hand side of (6) has the form

$$\frac{1}{2\pi i}\int\limits_{\tilde{\gamma}_k}\frac{f(\zeta)-f(z)}{\zeta-z}\left(\frac{\xi_{R,\theta}-\zeta}{\xi_{R,\theta}-z}\right)^p d\zeta.$$

If $r_k=0$, then this integral converges since the following estimate holds:

$$\int\limits_{\tilde{\gamma}_k}\frac{|f(\zeta)-f(z)|}{|\zeta-z|}|d\zeta|\leqslant c\int\limits_0^{\operatorname{diam}\tilde{\gamma}_k}\frac{w_k(t)}{t}\,dt. \quad (7)$$

For $r_k>0$, the following integral converges:

$$\int\limits_{\tilde{\gamma}_k}\left|\frac{1}{z-\zeta}\int\limits_z^\zeta(\zeta-\sigma)^{r_k-1}(f^{(r_k)}(\sigma)-f^{(r_k)}(z))\,d\sigma\right|d\zeta. \quad (8)$$

Since the integrals (7) and (8) are convergent, we can pass to the limit in (3) and (5) as $\tau_k\to 0$. We have

$$F(z,\theta)-f(z)=-\frac{f(z)}{2\pi i}\int\limits_{\gamma(Z_1,\dots,Z_m)}\frac{(\xi_{R,\theta}-\zeta)^p}{(\xi_{R,\theta}-z)^p(\zeta-z)}\,d\zeta-\dots$$

$$-\frac{f^{(r_k)}(z)}{2\pi i r_k!}\int\limits_{\gamma(Z_1,\dots,Z_m)}\frac{(\xi_{R,\theta}-\zeta)^p(\zeta-z)^{r_k-1}}{(\xi_{R,\theta}-z)^p}\,d\zeta$$

$$+\frac{1}{2\pi i\,(r_k-1)!}\int\limits_{\gamma_k}\left(\frac{1}{z-\zeta}\int\limits_z^\zeta(\zeta-\sigma)^{r_k-1}(f^{(r_k)}(\sigma)-f^{(r_k)}(z))\,d\sigma\right)\left(\frac{\xi_{R,\theta}-\zeta}{\xi_{R,\theta}-z}\right)^p d\zeta$$

$$+\sum_{l\neq k}\frac{1}{2\pi i}\int\limits_{\gamma_l}\varphi(\zeta,z)\frac{(\xi_{R,\theta}-\zeta)^p}{(\xi_{R,\theta}-z)^p(\zeta-z)}\,d\zeta+\frac{1}{2\pi i}\int\limits_{\gamma_0}\varphi(\zeta,z)\frac{(\xi_{R,\theta}-\zeta)^p}{(\xi_{R,\theta}-z)^p(\zeta-z)}\,d\zeta.$$

For a fixed large p, we fix ν such that $\nu>p$ and ν is independent of n. Let

$$J_n(\theta)=C_{n,\nu}\left(\frac{\sin n\theta}{\sin\theta}\right)^{2\nu},\quad \int\limits_{-\pi}^{\pi}C_{n,\nu}\left(\frac{\sin n\theta}{\sin\theta}\right)^{2\nu}d\theta=1.$$

We introduce the polynomial

$$g_n(z)=\int\limits_{-\pi}^{\pi}J_n(\theta)F(z,\theta)\,d\theta.$$

§4. Estimate of $g_n(z)-f(z)$

Let $z\in S_k$. We have

$$
\begin{aligned}
g_n(z)-f(z) &= \int_{-\pi}^{\pi} J_n(\theta)(F(z,\theta)-f(z))\,d\theta \\
&= \int_{-\pi}^{\pi} J_n(\theta)\Bigg[\sum_{N=0}^{r_k}\left(-\frac{f^{(N)}(z)}{2\pi i N!}\right)\int\limits_{\gamma(Z_1,\dots,Z_m)} \frac{(\xi_{R,\theta}-\zeta)^p(\zeta-z)^{N-1}}{(\xi_{R,\theta}-z)^p}\,d\zeta \\
&\quad+\frac{1}{2\pi i\,(r_k-1)!}\int_{\gamma_k}\left(\frac{1}{z-\zeta}\int_z^{\zeta}(\zeta-\sigma)^{r_k-1}(f^{(r_k)}(\sigma)-f^{(r_k)}(z))\,d\sigma\right)\left(\frac{\xi_{R,\theta}-\zeta}{\xi_{R,\theta}-z}\right)^p d\zeta \qquad (9)\\
&\quad-\sum_{l\neq k}\frac{1}{2\pi i}\int_{\gamma_l}\varphi(\zeta,z)\frac{(\xi_{R,\theta}-\zeta)^p}{(\xi_{R,\theta}-z)^p(\zeta-z)}\,d\zeta \\
&\quad-\frac{1}{2\pi i}\int_{\gamma_0}\varphi(\zeta,z)\frac{(\xi_{R,\theta}-\zeta)^p}{(\xi_{R,\theta}-z)^p(\zeta-z)}\,d\zeta\Bigg]\,d\theta.
\end{aligned}
$$

We write the first term of (9):

$$
-\frac{f^{(N)}(z)}{2\pi i N!}\int_{-\pi}^{\pi} J_n(\theta)\Bigg[\int\limits_{\gamma(Z_1,\dots,Z_m)} \frac{(\xi_{R,\theta}-\zeta)^p(\zeta-z)^{N-1}}{(\xi_{R,\theta}-z)^p}\,d\zeta\Bigg]\,d\theta,
$$

where $N>0$ and $\xi_{R,\theta}=\Psi(Re^{-i\theta}\varphi(\zeta))$. We recall that $\gamma(z_1,\dots,z_m)$ is a contour surrounding $\Gamma(Z_1,\dots,Z_m)$. We choose a sufficiently large r_0 such that the circle of radius $r_0/2$ surrounds the continuum, i.e., $|\zeta|<r_0/2$. Let $c_{r_0}=\{|\zeta|=r_0\}$. By the Cauchy theorem, we have

$$
\int\limits_{\gamma(Z_1,\dots,Z_m)} \dots\,d\zeta=\int_{c_{r_0}}\dots\,d\zeta,
$$

i.e.,

$$
\begin{aligned}
\int_{-\pi}^{\pi} J_n(\theta)\Bigg[\int\limits_{\gamma(Z_1,\dots,Z_m)} \frac{(\xi_{R,\theta}-\zeta)^p(\zeta-z)^{N-1}}{(\xi_{R,\theta}-z)^p}\,d\zeta\Bigg]\,d\theta &= \int_{-\pi}^{\pi} J_n(\theta)\left(\int_{c_{r_0}} \frac{(\xi_{R,\theta}-\zeta)^p(\zeta-z)^{N-1}}{(\xi_{R,\theta}-z)^p}\,d\zeta\right)d\theta \\
&= \int_{c_{r_0}}(\zeta-z)^{N-1}\left(\int_{-\pi}^{\pi} J_n(\theta)\left(\frac{\xi_{R,\theta}-\zeta}{\xi_{R,\theta}-z}\right)^p d\theta\right)d\zeta.
\end{aligned}
$$

We choose a large r_0 that depends on $\Gamma(Z_1,\dots,Z_m)$. For every θ, we have $|\xi_{R,\theta}-z|\geqslant b_1$ and $b_0\leqslant|\zeta-z|\leqslant 2r_0$. Hence

$$
\left|\int_{c_{r_0}}(\zeta-z)^{N-1}\int_{-\pi}^{\pi} J_n(\theta)\left(\frac{\xi_{R,\theta}-\zeta}{\xi_{R,\theta}-z}\right)^p d\theta\,d\zeta\right|\leqslant \max(b_0^{N-1},(2r_0)^{N-1})\cdot 2\pi r_0\frac{1}{b_1^p}\max\int_{-\pi}^{\pi} J_n(\theta)|\xi_{R,\theta}-\zeta|^p\,d\theta.
$$

As is known (cf. [10, 11]), for the last factor we have

$$\int_{-\pi}^{\pi} J_n(\theta)|\xi_{R,\theta}-\zeta|^p\,d\theta \leqslant b_2(\nu,p)r_0^p\frac{1}{n^p}.$$

For $z\in S_k$ and sufficiently large ν, the following inequality holds:

$$\int_{-\pi}^{\pi} J_n(\theta)\left|\frac{\xi_{R,\theta}-\zeta}{\xi_{R,\theta}-z}\right|^l\frac{d\theta}{|\xi_{R,\theta}-z|^\nu}\leqslant c\frac{|\xi_R-\zeta|^l}{|\xi_R-z|^{l+\nu}},\quad l,\nu\leqslant p. \tag{10}$$

If p is sufficiently large (depending on the geometric situation and the numbers $r_1,\dots,r_m$), then for any j we find (cf. [1, 11, 12])

$$\int_{\gamma(Z_1,\dots,Z_m)}\frac{|\zeta-z|^\nu|\zeta-\xi_R|^p}{|\xi_R-z|^p}|d\zeta|\leqslant c\rho_{1/n}^{\nu+1}(Z_1,\dots,Z_m;z),\quad 0\leqslant\nu\leqslant r_j, \tag{11}$$

where c is independent of $Z_1,\dots,Z_m$ and $\rho_h(Z_1,\dots,Z_m)$ is the maximal of the three distances to L_h.

We similarly estimate the third term and the fourth term of (9). It is more difficult to estimate the second term:

$$s_2=\int_{-\pi}^{\pi}J_n(\theta)\,d\theta\frac{1}{2\pi i\,(r_k-1)!}\int_{\gamma_k}\left(\frac{1}{z-\zeta}\int_z^\zeta(\zeta-\sigma)^{r_k-1}(f^{(r_k)}(\sigma)-f^{(r_k)}(z))\,d\sigma\right)d\zeta\left(\frac{\xi_{R,\theta}-\zeta}{\xi_{R,\theta}-z}\right)^p d\zeta.$$

By the definition of the class of functions under consideration, we have $|f^{(r_k)}(\sigma)-f^{(r_k)}(z)|\leqslant w_k(|\sigma-z|)$, $\sigma\in S_k$. Consequently,

$$\left|\frac{1}{z-\zeta}\int_z^\zeta(\zeta-\sigma)^{r_k-1}(f^{(r_k)}(\sigma)-f^{(r_k)}(z))\,d\sigma\right|\leqslant\frac{1}{|z-\zeta|}\int_z^\zeta|\zeta-\sigma|^{r_k-1}w_k(|\sigma-z|)\,|d\sigma|$$

$$\leqslant\frac{1}{|z-\zeta|}|\zeta-\sigma|^{r_k-1}w_k(|z-\zeta|)\int_z^\zeta 1\,|d\sigma|=\frac{1}{|z-\zeta|}|z-\zeta|^{r_k}w_k(|z-\zeta|)=|\zeta-z|^{r_k-1}w_k(|z-\zeta|).$$

Thus,

$$s_2\leqslant c\int_{-\pi}^{\pi}J_n(\theta)\,d\theta\int_{\gamma_k}|\zeta-z|^{r_k-1}w(|\zeta-z|)\left|\frac{\xi_{R,\theta}-\zeta}{\xi_{R,\theta}-z}\right|^p d\zeta.$$

In view of the inequalities (10) and (11), we have

$$|S_2|\leqslant c\int_{\gamma_k}|\zeta-z|^{r_k-1}w_k(|\zeta-z|)\,|d\zeta|\int_{-\pi}^{\pi}J_n(\theta)\left|\frac{\xi_{R,\theta}-\zeta}{\xi_{R,\theta}-z}\right|^p d\theta\quad[\xi_R=\xi_{R,0}]$$

$$\leqslant c\int_{\gamma_k}|\zeta-z|^{r_k-1}w_k(|\zeta-z|)\left|\frac{\xi_{R,0}-\zeta}{\xi_{R,0}-z}\right|^p|d\zeta|$$

$$\leqslant c\int_{\gamma_k}|\zeta-z|^{r_k-1}\frac{|\zeta-z|+\rho}{\rho}w_k(\rho)\left|\frac{\xi_{R,0}-\zeta}{\xi_{R,0}-z}\right|^p|d\zeta|$$

$$\leqslant c\frac{1}{\rho}w(\rho)\rho^{(r_k+1)}+cw(\rho)\rho^{r_k}=c\rho^{r_k}w(\rho),$$

where $\rho \equiv \rho_{1/n}(Z_1,\dots,Z_m,z)$. In (9), $g_n(z)$ depends not only on z, but also on $Z_1,\dots,Z_m$. Thus, we obtain the estimate

$$|g_n(Z_1,\dots,Z_m;z)-f(z)| \leqslant c\rho^{r_k}w(\rho), \quad z \in S_k.$$

§5. The Final Stage

On $[a'_1,b'_1]\times\dots\times[a'_m,b'_m]$, we consider a function of the form

$$\Lambda(Z_1,\dots,Z_m;z) = \Lambda_1(Z_1)\dots\Lambda_m(Z_m)\times(z-Z_1)^{r_1+1}\dots(z-Z_m)^{r_m+1}$$

such that

$$\int\limits_{[a'_1,b'_1]}\dots\int\limits_{[a'_m,b'_m]} \Lambda(Z_1,\dots,Z_m;z)|dZ_1|\dots|dZ_m| = \prod_{k=1}^{m}\int\limits_{[a'_k,b'_k]} \Lambda_k(Z_k)(z-Z_k)^{r_k+1}|dZ_k| \equiv 1.$$

We can assume that the functions $\Lambda_k(Z_k)$ are bounded.

Lemma 1. *In the definition of Λ, we can assume that each $\Lambda_k(Z_k)$ is a polynomial.*

Proof. There exists a polynomial $t_r(x)$ such that $\int_0^1 t_r(x)(z-x)^r\,dx \equiv 1$ for every r. The last condition holds if and only if

$$\int_0^1 x^r t_r(x)\,dx = (-1)^r, \quad \int_0^1 x^\nu t_r(x)\,dx = 0, \quad \nu = 0,1,\dots,r-1.$$

We make a linear change of the variable. We obtain a polynomial $\Lambda_k(Z_k)$ such that

$$\int\limits_{[a'_k,b'_k]} \Lambda_k(Z_k)(z-Z_k)^{r_k+1}|dZ_k| \equiv 1.$$

The lemma is proved.

Since $\Lambda_k(Z_k)$ is bounded, we have

$$|\Lambda(Z_1,\dots,Z_m;z)| \leqslant c\prod_{k-1}^{m}|z-Z_k|^{r_k+1}. \tag{12}$$

The inequality (12) implies that

$$\int\limits_{[a'_1,b'_1]}\dots\int\limits_{[a'_m,b'_m]} f(z)\Lambda(Z_1,\dots,Z_m;z)|dZ_1|\dots|dZ_m| \equiv f(z). \tag{13}$$

Let

$$\widetilde{g}_n(z) = \int\limits_{[a'_1,b'_1]}\dots\int\limits_{[a'_m,b'_m]} g_n(Z_1,\dots,Z_m;z)\Lambda(Z_1,\dots,Z_m;z)|dZ_1|\dots|dZ_m|.$$

The polynomial $\widetilde{g}_n$ has degree at most $n+r_*$, where $r_* = r_1 + \ldots + r_m + m$. From (12) and (13) we derive the relations

$$|f(z)-\widetilde{g}_n(z)| = \left| f(z) - \int\limits_{[a'_1,b'_1]} \cdots \int\limits_{[a'_m,b'_m]} g_n(Z_1,\ldots,Z_m;z)\Lambda(Z_1,\ldots,Z_m;z)|dZ_1|\ldots|dZ_m| \right|$$

$$= \left| \int\limits_{[a'_1,b'_1]} \cdots \int\limits_{[a'_m,b'_m]} f(z)\Lambda(Z_1,\ldots,Z_m;z)|dZ_1|\ldots|dZ_m| - \right.$$

$$\left. - \int\limits_{[a'_1,b'_1]} \cdots \int\limits_{[a'_m,b'_m]} g_n(Z_1,\ldots,Z_m;z)\Lambda(Z_1,\ldots,Z_m;z)|dZ_1|\ldots|dZ_m| \right|$$

$$\leqslant \int\limits_{[a'_1,b'_1]} \cdots \int\limits_{[a'_m,b'_m]} |f(z)-g_n(Z_1,\ldots,Z_m;z)|\Lambda(Z_1,\ldots,Z_m;z)|dZ_1|\ldots|dZ_m|.$$

To complete the proof, we need some properties of the distances $\rho_h(z)$ and $\rho^{\cdot}_h(Z_1,\ldots,Z_m;z)$. These properties can be easily obtained in the geometric situation under consideration (cf. [1, 10]).

Lemma 2. *For $z \in S_k$, we have*

$$\rho_{1/n}(z) \asymp \frac{1}{n}\sqrt{|z-a_k||z-b_k|+\frac{1}{n^2}}. \tag{14}$$

If

$$\left|z-\frac{a_k+b_k}{2}\right| \leqslant \frac{1}{4}|b_k-a_k|,$$

then

$$\rho^{\cdot}_{1/n}(Z_1,\ldots,Z_m;z) \asymp \frac{1}{n}\frac{1}{\sqrt{|z-Z_k|^2+1/n}};$$

if $z \in S_k$ and

$$\left|z-\frac{a_k+b_k}{2}\right| > \frac{1}{4}|b_k-a_k|,$$

then

$$\rho^{\cdot}_{1/n}(Z_1,\ldots,Z_m;z) \asymp \frac{1}{n}\sqrt{|z-a_k|\,|z-b_k|+\frac{1}{n^2}}. \tag{15}$$

We present some important corollaries. On $[a'_k,b'_k]$, we have $\rho_{1/n}(z) \asymp \frac{1}{n}$ in view of (14). Thus, by (14) and (15), we have

$$\rho^{\cdot}_{1/n}(Z_1,\ldots,Z_m;z) \leqslant c\rho_{1/n}(z)\frac{1}{|z-Z_k|}, \quad z \in S_k. \tag{16}$$

From (11) and (16) and the inequality $w_k(AS) \leqslant Aw_k(S)$, $A \geqslant 1$, $z \in S_k$, we find

$$\begin{aligned} |g_n(Z_1,\ldots,Z_m;z)-f(z)| &\leqslant c(\rho_{\frac{1}{n}}(z))^{r_k}\frac{1}{|z-Z_k|^{r_k}}w_k\left(\rho_{\frac{1}{n}}(z)\frac{1}{|z-Z_k|}\right) \\ &\leqslant c_1(\rho_{\frac{1}{n}}(z))^{r_k}\frac{1}{|z-Z_k|^{r_k}}\frac{1}{|z-Z_k|}w_k(\rho_{\frac{1}{n}}(z)) = c_1(\rho_{\frac{1}{n}}(z))^{r_k}w_k(\rho_{\frac{1}{n}}(z))\frac{1}{|z-Z_k|^{r_k+1}}. \end{aligned} \tag{17}$$

If $j \neq k$ and $z \in S_k$, then $c_2 \leqslant |z - Z_j| \leqslant c_3$.

We complete the proof of the theorem. We introduce the function

$$\prod(Z_1, \dots, Z_m; z) = \prod_{k=1}^{m} \frac{1}{|z - Z_k|^{r_k+1}}.$$

By (17), for $z \in \bigcup_{k=1}^{m} S_k$ we obtain the inequality

$$|g_n(Z_1, \dots, Z_m; z) - f(z)| \leqslant c\rho_{1/n}^{r_k}(z) w_k(\rho_{1/n}(z)) \prod(Z_1, \dots, Z_m; z). \tag{18}$$

Taking into account (12) and (18), for $z \in S_k$ we find

$$|f(z) - \widetilde{g}_n(z)| \leqslant c\rho_{1/n}^{r_k}(z) w_k(\rho_{1/n}(z)) \int\limits_{[a'_1, b'_1]} \dots \int\limits_{[a'_m, b'_m]} \prod(Z_1, \dots, Z_m; z) \prod_{k=1}^{m} |z - Z_k|^{r_k+1} |dZ_1| \dots |dZ_m|$$

$$= \dots = c\rho_{1/n}^{r_k}(z) w_k(\rho_{1/n}(z)).$$

The theorem is proved.

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