Journal of Mathematical Sciences, Vol. 99, No. 3, 2000

ON EXACT ORDER OF CONVERGENCE OF RANDOM POLYNOMIALS

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Estimates for the rate of convergence of a random second-order polynomial to the distribution χ^2 in uniform and Lévy metrics are obtained. Also, the low bounds in these metrics are constructed.

1. Introduction and Formulation of Main Results

The increased interest in random polynomials (see, e.g., [1, 2, 5–7, 9, 10]) is due to the fact that in many cases linear models describe real situations not quite explicity. Naturally, the improvement of models by means of symmetric polynomials requires one to investigate their rate of convergence.

Let

$$P_n^{(k)}(x) = P_n^{(k)}(x_1, \dots, x_n), \quad x \in \mathbf{R}^n, \quad n = 1, 2, \dots,$$

be a sequence of homogeneous symmetric kth-degree polynomials. Denote

$$S_{(m)} = x_1^m + x_2^m + \dots + x_n^m.$$

This is a well-known Newton representation of symmetric polynomials that is expressed as follows:

$$P_n^{(k)}(x) = \sum \{a_n(i_1, \dots, i_k) S_{(1)}^{i_1} \dots S_{(k)}^{i_k}, \ i_1 + 2i_2 + \dots + ki_k = k\},$$

where $i_1, \dots, i_m$ are natural numbers.

This means that we can write a homogeneous symmetric polynomial of second degree in the following form:

$$P_n^{(2)} = a_n S_{(1)}^2 + b_n S_{(2)}.$$

In [9], the rate of convergence of a random polynomial

$$Y_n = \frac{1}{n}\left(\left(\sum_{i=1}^n X_i\right)^2 - \sum_{i=1}^n X_i^2\right)$$

to the limit distribution was considered. Here $X_1, X_2, \dots, X_n$ are independent and identically distributed random variables. It was V. M. Zolotarev who first suggested considering the rate of convergence of such polynomials. As in [7], we also consider the second-order polynomial

$$Z_n = a_n\left(\sum_{i=1}^n X_i\right)^2 + b_n \sum_{i=1}^n X_i^2$$

with $a_n = a/n$, $b_n = b/n$, $a, b \in \mathbf{R}$, and $a \cdot b \neq 0$. As compared to [9], we substantially improve here the estimates in a uniform metric. Moreover, we consider here polynomials of a more general kind, and lower estimates for the rate of convergence are obtained also.

For simplicity, assume that $\mathbf{E}X_1 = 0$ and $\mathbf{E}X_1^2 = 1$. It is easy to see that Z_n will converge to the polynomial $Z = aN^2 + b$ as $n \to \infty$. Here N is the standard normally distributed random variable.

Denote $\beta_s = \mathbf{E}|X_1|^s$ and suppose that

$$\beta_{4+2\delta} < \infty, \quad 0 \leq \delta \leq 1. \tag{1}$$

This condition will be needed to apply the estimates in the central limit theorem given in [4] to the squares of random variables under consideration.

Recall the definitions of probability metrics that will be used in this work:

Lévy metric

$$L(X,Y) = L(F_X, F_Y) = \inf\{\varepsilon\colon \ F_X(x) \leq F_Y(x+\varepsilon) + \varepsilon, F_Y(x) \leq F_X(x+\varepsilon) + \varepsilon, \ x \in \mathbf{R}\},$$

where F_Z is a distribution function of the random variable Z;

Proceedings of the Seminar on Stability Problems for Stochastic Models, Vologda, Russia, 1998, Part I.

Lévy–Prokhorov metric

$$\pi(X,Y) = \pi(P_X, P_Y) = \inf\{\varepsilon\colon\ P_X(A) \le P_Y(A^\varepsilon) + \varepsilon, P_Y(A) \le P_X(A^\varepsilon) + \varepsilon,\ A \in \mathcal{B}\},$$

where $\mathcal{B}$ is a system of Borel sets on $\mathbf{R}$ and $A^\varepsilon = \{x\colon\ |x-y| < \varepsilon,\ y \in A\}$ is the ε-neighborhood of the set A;

Ky-Fan metric

$$K(X,Y) = \inf\{\varepsilon > 0\colon\ \mathbf{P}(|X-Y| \ge \varepsilon) \le \varepsilon\};$$

uniform metric

$$\rho(X,Y) = \sup\{|F_X(x) - F_Y(x)|\colon\ x \in \mathbf{R}\}.$$

The main results are the following propositions.

Theorem 1. *If condition* (1) *is fulfilled, then there exists an absolute constant* $c > 0$ *such that*

$$\rho(Z_n, aN^2 + b) \le c\left\{\frac{\beta_3}{\sqrt{n}} + n^{-\frac{\delta+4}{20}}\beta_{4+2\delta}\left(1 + \sqrt{\left|\frac{b}{a}\right| \min\left(\log n, \frac{1}{1-\delta}\right)}\right)\right\}.$$

In particular,

$$\rho(Z_n, aN^2 + b) \le \begin{cases} c\left(\dfrac{\beta_3}{\sqrt{n}} + \sqrt{\left|\dfrac{b}{a}\right|}\,\dfrac{\beta_4}{n^{1/5}}\right), & \delta = 0, \\ c\left(\dfrac{\beta_3}{\sqrt{n}} + \sqrt{\left|\dfrac{b}{a}\right|}\,\dfrac{\beta_6}{n^{1/4}}\right), & \delta = 1. \end{cases}$$

Theorem 2. *If condition* (1) *holds for* $0 < \delta \le 1$, *then there exist absolute constants* $c_1 > 0$, $c_2 > 0$ *such that*

$$L(Z_n, aN^2 + b) \le c_1\left(\frac{\beta_3}{\sqrt{n}} + |b|\sqrt{\beta_4 - 1}\sqrt{\frac{\log n}{n}}\right) + r_n,$$

where $r_n \le c_2 \mathbf{E}|X_1^2 - 1|^{2+\delta}(\beta_4 - 1)^{-1-\delta/2}n^{-\delta/2}$, $0 < \delta \le 1$, $\beta_4 \ne 1$; $r_n = 0$ *if* $\beta_4 = 1$.

Remark. Note that if $\beta_4 = 1$ (or $\beta_4 = \beta_2^2 = 1$), then $L(Z_n, aN^2 + b) = O(n^{-1/2})$. The equality $\beta_4 = 1$ means that $\mathbf{E}(X_1^2 - 1)^2 = 0$ or $X_1^2 = 1$ with probability 1. It is easy to see that in this case the summand $b\sum_{i=1}^n X_i^2/n$ in the polynomial Z_n is equal to b. This shows that the order $\sqrt{\log n/n}$ in Theorem 2 is conditioned by the term $b\sum_{i=1}^n X_i^2/n$.

In Sec. 4, we construct an example showing that the orders of lower bounds in Theorems 1 and 2 are, respectively, $n^{-1/4}$ and $\sqrt{\log n/n}$ (in the case $\delta = 1$).

In [7], the rate of convergence of random polynomials was also pursued. There strict restrictions were imposed on the class of polynomials. The results of that work were obtained in the metrics $\kappa(X,Y) = \int |F_X(u) - F_Y(u)|\,du$ and $\pi(X,Y)$. In the $\pi(X,Y)$ metric, the order of the estimate was $n^{-1/4}$.

By the method of characteristic functions the estimates of the $\rho(XAX^T, \xi A \xi^T)$ were obtained in [1] and [5]. Here $X = (X_1, \dots, X_n)$, $\xi = (\xi_1, \dots, \xi_n)$, ξ_i are standard normal independent random variables, whereas vectors X and ξ are assumed to be independent, and $A = (a_{ij})_{1 \le i,j \le n}$ is a real symmetric matrix with $a_{ii} = 0$. The estimates obtained there are of orders $n^{-1/6}(\log n)^{5/3}$ and $n^{-1/8}$, respectively. Since the metric of the quadratic form Z_n has only one eigenvalue strictly distinct from zero (as $n \to \infty$), the application of these results in our case is impossible because the conditions in [5] are not fulfilled or else it yields a trivial estimate [1] (in [1] and [5], to obtain a meaningful result it is needed that the matrix A should have at least two eigenvalues strictly distinct from zero).

By $c, c_1, c_2, \dots$, we denote positive absolute constants that may differ from line to line or from formula to formula.

2. Proof of Theorem 1. Estimation of the Rate of Convergence in the Uniform Metric

Denote

$$Q(\varepsilon, N^2) = \sup_{u \in \mathbf{R}} \mathbf{P}\{N^2 \in [u, u+\varepsilon]\}, \quad \varepsilon > 0.$$

It is easy to show that

$$Q(\varepsilon, N^2) \le c\sqrt{\varepsilon}. \tag{2}$$

Denoting $S_n = (x_1 + \dots + x_n)/\sqrt{n}$, by the triangle inequality, we obtain

$$\rho(Z_n, aN^2 + b) \le \rho(Z_n, aS_n^2 + b) + \rho(aS_n^2 + b, aN^2 + b). \tag{3}$$

By I_1 and I_2, respectively, denote the first and second terms of the right-hand side of (3).

Since the metric ρ is invariant with respect to a shift by a constant, we have $I_2 = \rho(S_n^2, N^2)$. Then

$$\rho(S_n^2, N^2) = \sup_{u\geq 0} |\mathbf{P}\{S_n^2 < u\} - \mathbf{P}\{N^2 < u\}| \leq \sup_{u\geq 0} |\mathbf{P}\{S_n < \sqrt{u}\} - \Phi(\sqrt{u})|$$

$$+ \sup_{u\geq 0} |\mathbf{P}\{S_n < -\sqrt{u}\} - \Phi(-\sqrt{u})| + 2d_n \leq \frac{c\beta_3}{\sqrt{n}} + 2d_n, \tag{4}$$

where $d_n = \sup_{u\in\mathbf{R}} \mathbf{P}\{S_n = u\}$. To estimate d_n we make use of Lemma 6 (see [4, p. 59]). This implies that for each $u > 0$

$$d_n \leq u^{-1} \int\limits_{|t|\leq u} |\mathbf{E}e^{itS_n}|\, dt.$$

Since $|\mathbf{E}e^{itS_n}| \leq e^{-t^2/4}$ for $|t| \leq 3\sqrt{n}/(2\beta_3)$, assuming $u = 3\sqrt{n}/(2\beta_3)$, we obtain

$$d_n \leq c\beta_3/\sqrt{n}. \tag{5}$$

Hence it follows that

$$I_2 \leq c\beta_3/\sqrt{n}. \tag{6}$$

Let us now proceed to the estimation of the term I_1. It is easy to see that

$$\rho(Z_n, aS_n^2 + b) = \rho\left(aS_n^2 + \frac{b}{n}\sum_{i=1}^{n}(X_i^2 - 1), aS^2\right).$$

By Lemma 7 (see [4, p. 30]), for any $\varepsilon > 0$

$$\rho(Z_n, aS_n^2 + b) \leq I_{11} + \sup_{x\in\mathbf{R}} I_{12}(x), \tag{7}$$

where

$$I_{11} = \mathbf{P}\left\{\left|\frac{1}{n}\sum_{i=1}^{n}(X_i^2 - 1)\right| > \varepsilon|b|^{-1}\right\},$$

$$I_{12}(x) = \max(\mathbf{P}(aS_n^2 < x + \varepsilon) - \mathbf{P}(aS_n^2 < x), \mathbf{P}(aS_n^2 < x) - \mathbf{P}(aS_n^2 < x - \varepsilon)).$$

Since $\max(u, v) \leq u + v$ for $u, v \geq 0$, we have

$$\sup_{x\in\mathbf{R}} I_{12}(x) \leq \sup_{x\in\mathbf{R}} |\mathbf{P}\{|a|S_n^2 < x + \varepsilon\} - \mathbf{P}\{|a|S_n^2 < x - \varepsilon\}| \leq \sup_{x\in\mathbf{R}}\left|\mathbf{P}\left(N^2 < \frac{x+\varepsilon}{|a|}\right) - \mathbf{P}\left(N^2 < \frac{x-\varepsilon}{|a|}\right)\right| + \frac{c\beta_3}{\sqrt{n}}$$

$$\leq Q\left(N^2, \frac{2\varepsilon}{|a|}\right) + \frac{c\beta_3}{\sqrt{n}} \leq c\left(\sqrt{\frac{\varepsilon}{|a|}} + \frac{\beta_3}{\sqrt{n}}\right). \tag{8}$$

Estimate (8) follows from (2), (4), and (5). It now remains to estimate the term I_{11}. For this we decompose the random variables X_i^2 in the following way:

$$X_i^2 = U_i' + U_i'', \quad i = 1, 2, \ldots, n,$$

where

$$U_i' = X_i^2 \mathbf{1}\{|X_i| \leq \sqrt[4]{n}\}, \qquad U_i'' = X_i^2 \mathbf{1}\{|X_i| > \sqrt[4]{n}\}.$$

It is easy to see that $\mathbf{E}U_i' + \mathbf{E}U_i'' = 1$. Then for any $\varepsilon > 0$

$$\mathbf{P}\left\{\frac{1}{n}\left|\sum_{i=1}^{n}(X_i^2 - 1)\right| \geq \varepsilon|b|^{-1}\right\} = \mathbf{P}\left\{\frac{1}{n}\left|\sum_{i=1}^{n}(U_i' - \mathbf{E}U_i') + \sum_{i=1}^{n}(U_i'' - \mathbf{E}U_i'')\right| > \frac{\varepsilon}{|b|}\right\}$$

$$\leq \mathbf{P}\left\{\frac{1}{n}\left|\sum_{i=1}^{n}(U_i' - \mathbf{E}U_i')\right| > \frac{\varepsilon}{2|b|}\right\} + \mathbf{P}\left\{\frac{1}{n}\left|\sum_{i=1}^{n}(U_i'' - \mathbf{E}U_i'')\right| > \frac{\varepsilon}{2|b|}\right\}. \tag{9}$$

According to the Chebyshev inequality,

$$\mathbf{P}\left\{\frac{1}{n}\left|\sum_{i=1}^{n}(U_i'' - \mathbf{E}U_i'')\right| > \frac{\varepsilon}{2|b|}\right\} \leq \left(\frac{2|b|}{\varepsilon}\right)^2 \mathbf{E}\left|\frac{1}{n}\sum_{i=1}^{n}(U_i'' - \mathbf{E}U_i'')\right|^2 \leq \frac{(2|b|)^2}{(\varepsilon\sqrt{n})^2}\mathbf{E}(U_1'' - \mathbf{E}U_i'')^2 \leq \frac{c\beta_{4+2\delta}b^2}{(\varepsilon(\sqrt{n})^{1+\delta/2})^2}. \tag{10}$$

Again, by the Chebyshev inequality and Theorem 19 (see [4, p. 86]), for all $p \geq 2$ we have

$$\mathbf{P}\left\{\frac{1}{n}\left|\sum_{i=1}^{n}(U_i' - \mathbf{E}U_i')\right| > \frac{\varepsilon}{2|b|}\right\} \leq \left(\frac{2|b|}{n\varepsilon}\right)^p \mathbf{E}\left|\sum_{i=1}^{n}(U_i' - \mathbf{E}U_i')\right|^p$$

$$\leq \left(\frac{2|b|}{\varepsilon}\right)^p \left(\frac{p}{n}\right)^p \left\{\sum_{i=1}^{n}\mathbf{E}|U_i' - \mathbf{E}U_i'|^p + \left(\sum_{i=1}^{n}\mathbf{E}(U_i' - \mathbf{E}U_i')^2\right)^{p/2}\right\}$$

$$\leq \left(\frac{2p|b|}{\varepsilon n}\right)^p \left\{(2\sqrt{n})^{p-2}\sum_{i=1}^{n}\mathbf{E}(U_i' - \mathbf{E}U_i')^2 + \left(\sum_{i=1}^{n}\mathbf{E}(U_i' - \mathbf{E}U_i')^2\right)^{p/2}\right\} \leq 2\left(\frac{4|b|p\sqrt{\beta_4}}{\varepsilon\sqrt{n}}\right)^p. \tag{11}$$

Combining estimates (6)–(11), we obtain that for all $\varepsilon > 0$ and $p \geq 2$

$$\rho(Z_n, aN^2 + b) \leq c\left\{\frac{\beta_3}{\sqrt{n}} + \frac{\beta_{4+2\delta}b^2}{(\varepsilon(\sqrt{n})^{1+\delta/2})^2} + \sqrt{\frac{\varepsilon}{|a|}} + \left(\frac{4p|b|\sqrt{\beta_4}}{\varepsilon\sqrt{n}}\right)^p\right\}. \tag{12}$$

Assuming in (12) $p = 6\min\{\log n, 1/(1-\delta)\}$, $\varepsilon = 4p\sqrt{\beta_4}e|b|n^{-(4+\delta)/10}$, we obtain

$$\rho(Z_n, aN^2 + b) \leq c\left\{\frac{\beta_3}{\sqrt{n}} + n^{-(\delta+4)/20}\left(\frac{\beta_{4+2\delta}}{n^{\delta/4}} + \sqrt{\frac{|b|}{|a|}\sqrt{\beta_4}\min\left(\frac{1}{1-\delta}, \log n\right)}\right)\right\}$$

$$+ \exp\left\{-6(1 + \log n^{(1-\delta)/10})\min\left(\log n, \frac{1}{1-\delta}\right)\right\} \leq c\left\{\frac{\beta_3}{\sqrt{n}} + \frac{\beta_{4+2\delta}}{n^{(\delta+4)/20}}\left(1 + \sqrt{\frac{|b|}{|a|}\min\left(\log n, \frac{1}{1-\delta}\right)}\right)\right\}.$$

Theorem 1 is proved.

3. Proof of Theorem 2. Upper Estimates of the Rate of Convergence in the Lévy Metric

According to the triangle inequality,

$$L(Z_n, aN^2 + b) \leq L(Z_n, aS_n^2 + b) + L(aS_n^2 + b, aN^2 + b) = l_1 + l_2, \tag{13}$$

where l_1 and l_2 are, respectively, the first and second summands on the right-hand side of (13), $S_n = (x_1 + \cdots + x_n)/\sqrt{n}$. Since the Lévy metric is invariant to a shift by a constant,

$$l_2 = L(aS_n^2, aN^2).$$

By the inequality $L(X, Y) \leq \rho(X, Y)$,

$$l_2 \leq \rho(aS_n^2, aN^2) = \rho(S_n^2, N^2) \leq c\beta_3/\sqrt{n}. \tag{14}$$

Let us estimate l_1. According to [3, p. 111],

$$l_1 = L(Z_n, aS_n^2 + b) \leq \pi(Z_n, aS_n^2 + b).$$

On the other hand, the Lévy–Prokhorov metric is minimal with respect to the Ky-Fan metric (see [3, p. 59]); therefore

$$\pi(Z_n, aS_n^2 + b) \leq K(Z_n, aS_n^2 + b) = \inf\left\{\varepsilon > 0\colon \mathbf{P}\left(\frac{|b|}{n}\left|\sum_{i=1}^{n}(X_i^2 - 1)\right| \geq \varepsilon\right) < \varepsilon\right\}.$$

From this it is easy to see that

$$\pi(Z_n, aS_n^2 + b) \le 2L\left(b\frac{\sum_{i=1}^n X_i^2}{n}, b\right). \tag{15}$$

Making use of the invariance property of the Lévy metric and by the triangle inequality, we obtain

$$L\left(b\frac{\sum_{i=1}^n X_i^2}{n}, b\right) = L\left(\frac{b}{\sqrt{n}}\frac{\sum_{i=1}^n \xi_i}{\sqrt{n}}, 0\right) \le L\left(\frac{b}{\sqrt{n}}\frac{\sum_{i=1}^n \xi_i}{\sqrt{n}}, \frac{b\sqrt{\beta_4 - 1}\,N}{\sqrt{n}}\right) + L\left(\frac{b\sqrt{\beta_4 - 1}\,N}{\sqrt{n}}, 0\right), \tag{16}$$

where $\xi_i = X_i^2 - 1$. In [9], for $L_0 = L(\varepsilon N, 0)$ as $\varepsilon \to 0$ the following expression was obtained:

$$L_0 = \sqrt{2}\,|\varepsilon|\sqrt{\log(1/|\varepsilon|) + c + o(1)} = \sqrt{2}\,|\varepsilon|\sqrt{\log(1/|\varepsilon|)} + o(|\varepsilon|\sqrt{\log(1/|\varepsilon|)}\,).$$

Thus

$$L\left(\frac{b\sqrt{\beta_4 - 1}\,N}{\sqrt{n}}, 0\right) = \frac{\sqrt{2(\beta_4 - 1)}\,|b|}{\sqrt{n}}\sqrt{\log(\sqrt{n}/|b|)(1 + o(1))} = |b|\sqrt{\beta_4 - 1}\sqrt{\frac{\log n}{n}}(1 + o(1)).$$

Let us now consider the first summand on the right-hand side of (16). Since $L(X, Y) \le \rho(X, Y)$, we have

$$L\left(\frac{b}{\sqrt{n}}\frac{\sum_{i=1}^n \xi_i}{\sqrt{n}}, \frac{b\sqrt{\beta_4 - 1}\,N}{\sqrt{n}}\right) \le \rho\left(\frac{b}{\sqrt{n}}\frac{\sum_{i=1}^n \xi_i}{\sqrt{n}}, \frac{b\sqrt{\beta_4 - 1}\,N}{\sqrt{n}}\right) = \rho\left(\frac{\sum_{i=1}^n \xi_i}{\sqrt{n}}, N\sqrt{\beta_4 - 1}\right) = r_n,$$

where $r_n \le c\mathbf{E}|X_1^2 - 1|^{2+\delta}(\beta_4 - 1)^{(-2+\delta)/2}(\sqrt{n})^{-\delta}$, $0 < \delta \le 1$, $\beta_4 \ne 1$, and $r_n = 0$ for $\beta_4 = 1$.

Now (13)–(16) yield the proof of Theorem 2.

4. On Lower Bounds for the Estimates in Theorems 1 and 2

Let $a = 1$ and $b = -1$. Then

$$Z_n = \left(\frac{X_1 + \cdots + X_n}{\sqrt{n}}\right)^2 - \sum_{i=1}^n \frac{X_i^2}{n}.$$

Denote

$$\overline{X} = \frac{X_1 + \cdots + X_n}{n}, \qquad S^2 = \frac{1}{n}\sum_{i=1}^n X_i^2 - \overline{X}^2.$$

It is easy to see that

$$Z_n = n\overline{X}^2 - S^2 - \overline{X}^2 = (n-1)\overline{X}^2 - S^2.$$

When $X_i \stackrel{d}{=} N(0,1)$ it is well known that the random variables $\overline{X}$ and S^2 are independent and $nS^2 \stackrel{d}{=} \chi^2(n-1)$. Therefore, we can write $Z_n n/(n-1)$ as the sum of two independent random variables N^2 and $-\chi^2(n-1)/(n-1)$:

$$\frac{n}{n-1}Z_n = N^2 - \frac{\chi^2(n-1)}{n-1}.$$

Let us investigate the distribution function $H(x)$ of the random variable $N^2 - \chi^2(n-1)/(n-1)$ (briefly $N^2 - \chi^2_{n-1}$):

$$H(x) = F_{N^2 - \chi^2_{n-1}}(x) = \int_{-\infty}^{\infty} F_{N^2}(x - z)\,d\mathbf{P}(-\chi^2_{n-1} < z)$$

$$= \int_{\infty}^{-\infty} F_{N^2}(x + z)\,d\mathbf{P}(-\chi^2_{n-1} < -z) = \int_{\max(0,-x)}^{\infty} F_{N^2}(x + z)\,d\mathbf{P}(\chi^2_{n-1} \le z). \tag{17}$$

As $F_{N^2-1}(-1) = 0$, it is obvious that

$$\sup_{x\in\mathbf{R}} |H(x) - F_{N^2-1}(x)| \ge |H(-1) - F_{N^2-1}(-1)| = H(-1) = \int_1^{\infty} F_{N^2}(z - 1)\,d\mathbf{P}(\chi^2_{n-1} \le z). \tag{18}$$

Since

$$dP(\chi^2_{n-1} \le z) = \frac{m^m}{\Gamma(m)} e^{-mz} z^{m-1}\,dz \quad (\text{here } m = (n-1)/2),$$
$$\frac{m^m}{\Gamma(m)} = c_0 e^m \sqrt{m}(1+\varepsilon_m), \qquad c_0 = (e\sqrt{2\pi})^{-1}, \qquad \varepsilon_m = O(1/m), \tag{19}$$

we have

$$H(-1) = c_0(1+\varepsilon_m) \int_1^\infty F_{N^2}(z-1) e^{-m(z-1)} \sqrt{m}\, z^{m-1}\,dz = c_0(1+\varepsilon_m)\sqrt{m} \int_0^\infty F_{N^2}(z) e^{-mz}(z+1)^{m-1}\,dz$$

$$= c\sqrt{m} \int_0^{1/2} F_{N^2}(z) e^{-mz}(z+1)^{m-1}\,dz = c\sqrt{m} \int_0^{1/2} \frac{F_{N^2}(z)}{1+z} (e^{-z}(z+1))^m\,dz.$$

For $0 \le z \le 1/2$,

$$\frac{F_{N^2}(z)}{1+z} = \frac{1}{\sqrt{2\pi}(1+z)} \int_0^z \frac{e^{-u/2}}{\sqrt{u}}\,du \ge \frac{2}{3\sqrt{2\pi} e^{1/4}} \int_0^z \frac{1}{\sqrt{u}}\,du = c_2\sqrt{z}.$$

Thus

$$H(-1) \ge c\sqrt{m} \int_0^{1/2} \sqrt{z}(e^{-z}(z+1))^m\,dz. \tag{20}$$

The inequality $e^{-z}(1+z) \ge e^{-z^2/2}$ $(0 \le z \le 1/2)$ yields

$$H(-1) \ge c\sqrt{m} \int_0^{1/2} \sqrt{z}\, e^{-z^2 m/2}\,dz = \frac{c}{\sqrt[4]{m}} \int_0^{\sqrt{m}/2} \sqrt{z}\, e^{-z^2/2}\,dz = \frac{c}{\sqrt[4]{m}}.$$

Since $m = (n-1)/2$, we have $H(-1) \ge c/\sqrt[4]{n}$. Note that the passage from the random variable $Z_n n/(n-1)$ to Z_n does not change the order $n^{-1/4}$ (see (30)).

Let us deal with the Lévy distance. By the definition,

$$L(N^2-1, N^2-\chi^2_{n-1}) = \inf\{\delta\colon\ H(x-\delta)-\delta \le F_{N^2-1}(x) \le H(x+\delta)+\delta,\ \forall x \in \mathbf{R}\}.$$

Since for $x \le -1$, $F_{N^2-1}(x) = 0$, we have

$$\inf\{\delta\colon\ F_{N^2-\chi^2_{n-1}}(x-\delta)-\delta \le 0 \le F_{N^2-\chi^2_{n-1}}(x+\delta)+\delta,\ x \le -1\} = \delta_0,$$

where δ_0 is the solution of the following equation:

$$H(-1-\delta) = F_{N^2-\chi^2_{n-1}}(-1-\delta) = \delta. \tag{21}$$

For $x > -1$, we have

$$H(x-\delta_0) = \int_{\max(0,-x+\delta_0)}^\infty F_{N^2}(x-\delta_0+z)\,dP(\chi^2_{n-1} \le z) \le F_{N^2}(x+1) \int_{\max(0,-x+\delta_0)}^{1+\delta_0} dP(\chi^2_{n-1} \le z)$$

$$+ \int_{1+\delta_0}^\infty F_{N^2}(x-\delta_0+z)\,dP(\chi^2_{n-1} \le z) = F_{N^2}(x+1) + \int_{1+\delta_0}^\infty (F_{N^2}(x-\delta_0+z) - F_{N^2}(x+1))\,dP(\chi^2_{n-1} \le z).$$

Since

$$F_{N^2}(x-\delta_0+z) \le F_{N^2}(x+1) + F_{N^2}(-1-\delta_0+z), \tag{22}$$

relation (22) yields

$$H(x-\delta_0) \le F_{N^2-1}(x) + \delta_0, \quad x > -1, \tag{23}$$

as $\delta_0 = \int_{1+\delta_0}^{\infty} F_{N^2}(-1-\delta_0+z)\, d\mathbf{P}(\chi^2_{n-1} \le z)$. Note that $F_{N^2}(x+1) = F_{N^2-1}(x)$. For $-1 < x \le -\delta_0$, we have

$$H(x+\delta_0) = \int_{-x-\delta_0}^{\infty} F_{N^2}(x+z+\delta_0)\, d\mathbf{P}(\chi^2_{n-1} \le z) \ge \int_{-x-\delta_0}^{1-\delta_0} F_{N^2}(x+z+\delta_0)\, d\mathbf{P}(\chi^2_{n-1} \le z) + F_{N^2}(x+1) \int_{1-\delta_0}^{\infty} d\mathbf{P}(\chi^2_{n-1} \le z)$$

$$= F_{N^2}(x+1) - \int_0^{-x-\delta_0} F_{N^2}(x+1)\, d\mathbf{P}(\chi^2_{n-1}) - \int_{-x-\delta_0}^{1-\delta_0} (F_{N^2}(x+1) - F_{N^2}(x+z+\delta_0))\, d\mathbf{P}(\chi^2_{n-1} \le z).$$

The inequalities (for $-1 < x \le -\delta_0$)

$$F_{N^2}(x+1) \le \begin{cases} F_{N^2}(x+z+\delta_0) + F_{N^2}(1-z-\delta_0), & -x-\delta_0 \le z \le 1-\delta_0, \\ F_{N^2}(1-z-\delta_0), & 0 \le z \le -x-\delta_0 \end{cases}$$

yield that for $-1 < x \le -\delta_0$ we have

$$H(x+\delta_0) \ge F_{N^2}(x+1) - \int_0^{-x-\delta_0} F_{N^2}(1-z-\delta_0)\, d\mathbf{P}(\chi^2_{n-1} \le z) - \int_{-x-\delta_0}^{1-\delta_0} F_{N^2}(1-z-\delta_0)\, d\mathbf{P}(\chi^2_{n-1} \le z)$$

$$= F_{N^2-1}(x) - \int_0^{1-\delta_0} F_{N^2}(1-z-\delta_0)\, d\mathbf{P}(\chi^2_{n-1} \le z). \tag{24}$$

For $-\delta_0 < x < \infty$, we have

$$H(x+\delta_0) = \int_0^{\infty} F_{N^2}(x+z+\delta_0)\, d\mathbf{P}(\chi^2_{n-1} \le z) = \int_0^{1-\delta_0} F_{N^2}(x+z+\delta_0)\, d\mathbf{P}(\chi^2_{n-1} \le z) + \int_{1-\delta_0}^{\infty} F_{N^2}(x+z+\delta_0)\, d\mathbf{P}(\chi^2_{n-1} \le z)$$

$$\ge F_{N^2}(x+1) + \int_0^{1-\delta_0} (F_{N^2}(x+z+\delta_0) - F_{N^2}(x+1))\, d\mathbf{P}(\chi^2_{n-1} \le z).$$

As

$$F_{N^2}(x+1) \le F_{N^2}(x+z+\delta_0) + F_{N^2}(1-z-\delta_0), \quad 0 \le z \le 1-\delta_0,$$

thus

$$H(x+\delta_0) \ge F_{N^2-1}(x) - \int_0^{1-\delta_0} F_{N^2}(1-z-\delta_0)\, d\mathbf{P}(\chi^2_{n-1} \le z).$$

Therefore, for all $x > -1$

$$H(x+\delta_0) \ge F_{N^2-1}(x) - \int_0^{1-\delta_0} F_{N^2}(1-z-\delta_0)\, d\mathbf{P}(X^2_{n-1} \le z).$$

We will show that

$$I = \int_0^{1-\delta_0} F_{N^2}(1-z-\delta_0)\, d\mathbf{P}(X^2_{n-1} \le z) < \delta_0.$$

It is easy to see that

$$I = \int_0^{1-3\delta_0} F_{N^2}(1 - z - \delta_0)\, d\mathbf{P}(X_{n-1}^2) + \int_{1-3\delta_0}^{1-\delta_0} F_{N^2}(1 - z - \delta_0)\, d\mathbf{P}(X_{n-1}^2 \le z)$$

$$\le F_{N^2}(1 - \delta_0) \int_0^{1-3\delta_0} d\mathbf{P}(X_{n-1}^2 \le z) + F_{N^2}(2\delta_0) \int_{1-3\delta_0}^{1-\delta_0} d\mathbf{P}(X_{n-1}^2 \le z)$$

$$\le F_{N^2}(1 - \delta_0) \int_0^{1-3\delta_0} d\mathbf{P}(X_{n-1}^2 \le z) + c\delta_0^{3/2} \qquad (25)$$

because $F_{N^2}(2\delta_0) \le c\sqrt{\delta_0}$. Now let us evaluate the integral

$$I' = \int_0^{1-3\delta_0} d\mathbf{P}(X_{n-1}^2 \le z).$$

By (19),

$$I' = c_0(1 + \varepsilon_m)e^m\sqrt{m} \int_0^{1-3\delta_0} e^{-mz} z^{m-1}\, dz = \frac{e^m c_0}{\sqrt{m}}(1 + \varepsilon_m) \int_0^{1-3\delta_0} \frac{d(e^{-z}z)^m}{1 - z}$$

$$\le \frac{e^m c_0(1 + \varepsilon_m)}{3\delta_0\sqrt{m}} \int_0^{1-3\delta_0} d(e^{-z}z)^m = c_0 \frac{(1 + \varepsilon_m)(e^{3\delta_0}(1 - 3\delta_0))^m}{3\delta_0\sqrt{m}}.$$

Since $e^{3u}(1 - 3u) \le e^{-u}(1 + u)$, $0 \le u \le 1/3$, we have

$$I' \le \frac{27}{51}\left(1 - \frac{1}{e}\right) c_0 \frac{c_0(1 + \varepsilon_m)(e^{-\delta_0}(1 + \delta_0))^m}{\delta_0\sqrt{m}}.$$

Together with (25) this yields

$$I \le c\delta_0^{3/2} + \frac{27}{51} F_{N^2}(1 + \delta_0)\left(1 - \frac{1}{e}\right) c_0(1 + \varepsilon_m) \frac{(e^{-\delta_0}(1 + \delta_0))^m}{\delta_0\sqrt{m}}.$$

Later we shall show that (see (29))

$$c_0(1 + \varepsilon_m)F_{N^2}(1 + \delta_0) \frac{(1 - 1/e)(e^{-\delta_0}(1 + \delta))^m}{\delta_0\sqrt{m}} < \delta_0.$$

Thus

$$I \le c\delta_0^{3/2} + \frac{27}{51}\delta_0 < \delta_0.$$

This means that for $x > -1$

$$H(x + \delta_0) \ge F_{N^2 - 1}(x) - \delta_0.$$

Therefore,

$$L(N^2 - 1, N^2 - \chi_{n-1}^2) = \delta_0,$$

where δ_0 is solution of Eq. (21). Let us solve this equation. Using (19), we write (21) in the form

$$\delta = c_0 e^m \sqrt{m}(1 + \varepsilon_m) \int_{1+\delta}^{\infty} F_{N^2}(-1 - \delta + z) e^{-mz} z^{m-1}\, dz, \quad c_0 = \frac{1}{e\sqrt{2\pi}}.$$

Denote

$$J = \int_{1+\delta}^{\infty} F_{N^2}(-1-\delta+z)e^{-mz}z^{m-1}\,dz. \tag{26}$$

If is not difficult to obtain that

$$\int_{1+\delta}^{\infty} F_{N^2}(-1-\delta+z)e^{-mz}z^{m-1}\,dz < \int_{1+\delta}^{\infty} e^{-mz}z^{m-1}\,dz = \int_{1+\delta}^{\infty} \frac{(1-z)e^{-mz}z^{m-1}\,dz}{1-z} = \frac{1}{m}\int_{1+\delta}^{\infty} \frac{d(S(z))^m}{1-z},$$

where $S(z) = e^{-z}z$. Integrating by parts, we obtain

$$\frac{1}{m}\int_{1+\delta}^{\infty} \frac{d(S(z))^m}{1-z} = \frac{1}{m}\frac{(e^{-(1+\delta)}(1+\delta))^m}{\delta} - \frac{1}{m}\int_{1+\delta}^{\infty} \frac{S^m(z)}{(1-z)^2}\,dz < \frac{1}{m}\frac{(e^{-(1+\delta)}(1+\delta))^m}{\delta}.$$

Therefore,

$$J < \frac{1}{m}\frac{(e^{-(1+\delta)}(1+\delta))^m}{\delta}. \tag{27}$$

Let us estimate J below. By the Taylor formula,

$$F_{N^2}(-1-\delta+z) = F_{N^2}(z) + c_1\frac{e^{-z/2}}{\sqrt{z}}\sum_{k=1}^{\infty}\frac{(-1)^k(1+\delta)^k}{k!}h_k(z), \tag{28}$$

where $c_1 h_k(z)e^{-z/2}/\sqrt{z} = F_{N^2}^{(k)}(z)$, $c_1 = 1/\sqrt{2\pi}$. Since $|h_k(z)| \downarrow 0$, $z \to \infty$, and $h_k(z) < 0$ for $k = 2p$, and $h_k(z) > 0$ for $k = 2p-1$ ($p \in N$), all terms in the second summand on the right-hand side of (28) are negative. As $1/(1-z) \geq 1/(-\delta)$, putting (28) into (26) we obtain

$$\begin{aligned}
J &= \int_{1+\delta}^{\infty} F_{N^2}(z)e^{-mz}z^{m-1}\,dz + c_1\sum_{k=1}^{\infty}\frac{(-1)^k(1+\delta)^k}{k!}\int_{1+\delta}^{\infty} h_k(z)\frac{e^{-z/2}}{\sqrt{z}}e^{-mz}z^{m-1}\,dz \\
&= \frac{1}{m}\int_{1+\delta}^{\infty}\frac{F_{N^2}(z)}{1-z}\,d(S(z))^m + \frac{c_1}{m+3/2}\sum_{k=1}^{\infty}\frac{(-1)^k(1+\delta)^k}{k!}\int_{1+\delta}^{\infty}\frac{h_k(z)\,d(S(z))^{m+3/2}}{z^2(1-z)} \\
&\geq \frac{-F_{N^2}(1+\delta)}{\delta_m}\int_{1+\delta}^{\infty} d(S(z))^m + c_1\sum_{k=1}^{\infty}\frac{(-1)^k(1+\delta)^k h_k(1+\delta)}{(m+3/2)k!\,(1+\delta)^2(-\delta)}\int_{1+\delta}^{\infty} d(S(z))^{m+3/2} \\
&= \frac{(e^{-(1+\delta)}(1+\delta))^m}{\delta m}\left\{F_{N^2}(1+\delta) + \frac{c_1 m}{m+3/2}\sum_{k=1}^{\infty}\frac{(-1)^k(1+\delta)^k}{k!}h_k(1+\delta)\frac{(e^{-(1+\delta)}(1+\delta))^{3/2}}{(1+\delta)^2}\right\}.
\end{aligned}$$

Since

$$\frac{c_1 e^{-(1+\delta)/2}}{\sqrt{1+\delta}}\sum_{k=1}^{\infty}\frac{(-1)^k(1+\delta)^k h_k(1+\delta)}{k!} = F_{N^2}(0) - F_{N^2}(1+\delta) = -F_{N^2}(1+\delta),$$

we finally obtain

$$J \geq \frac{(e^{-(1+\delta)}(1+\delta))^m}{\delta m}F_{N^2}(1+\delta)\left(1 - \frac{m}{m+3/2}e^{-1-\delta}\right) > \frac{(e^{-(1+\delta)}(1+\delta))^m}{\delta m}F_{N^2}(1+\delta)\left(1-\frac{1}{e}\right).$$

Therefore,

$$F_{N^2}(1)\left(1-\frac{1}{e}\right)\frac{(e^{-(1+\delta)}(1+\delta))^m}{\delta m} < J < \frac{(e^{-(1+\delta)}(1+\delta))^m}{\delta m}$$

or

$$c_0 F_{N^2}(1)\left(1-\frac{1}{e}\right)(1+\varepsilon_m)\frac{(e^{-\delta}(1+\delta))^m}{\delta\sqrt{m}} < \delta < c_0(1+\varepsilon_m)\frac{(e^{-\delta}(1+\delta))^m}{\delta\sqrt{m}}. \tag{29}$$

If $\delta \ll 1$, then $e^{-\delta}(1+\delta) \geq \exp\{-\delta^2/2\}$. On the other hand, $(e^{-\delta}(1+\delta))^m \leq \exp\{-\delta^2 m/2\}\exp\{\delta^3 m/3\} \leq c_3\exp\{-\delta^2 m/2\}$ (we know (Theorem 2) the solution of Eq. (21) should not exceed $c_4(\log n/n)^{1/2}$). Solving the equation

$$\delta = \frac{c_5 e^{-\delta^2 m/2}}{\delta\sqrt{m}}, \qquad c_0 F_{N^2}(1)\left(1-\frac{1}{e}\right) < c_5 \leq 2c_0,$$

we obtain (see [8, p. 57])

$$\frac{\delta^2 m}{2} = \log(\sqrt{m}\,c_5) - \log\log(c_5\sqrt{m}) + \cdots .$$

Hence,

$$\delta_0 = \sqrt{\frac{\log m}{m}}(1+o(1))$$

or (as $m = (n-1)/2$)

$$\delta_0 = \sqrt{2}\sqrt{\frac{\log(n-1)}{n-1}}(1+o(1)).$$

Note that the passage from the random variable $N^2 - \chi^2_{n-1}$ to the random variable $((n-1)/n)(N^2 - \chi^2_{n-1})$ cannot change the order $\sqrt{\log n/n}$, because

$$\sup_x\left|H\left(\frac{xn}{n-1}\right) - H(x)\right| \leq c/\sqrt{n}. \tag{30}$$

This estimate follows, for example, from the Esseen lemma and

$$|\mathbf{E}e^{it((n-1)/n)(N^2-\chi^2_{n-1})} - \mathbf{E}e^{it(N^2-\chi^2_{n-1})}| = \left|\frac{1}{\sqrt{1-2it}}\left(1+\frac{2it}{n}\right)^{-(n-1)/2}\right|$$

$$\times\left|\frac{1}{\sqrt{1-2it/(n(1-2it))}} - \left(\frac{1+2it/n}{1+2it/(n-1)}\right)^{(n-1)/2}\right| \leq \frac{c_6|t|}{n(1+t^2)^{1/4}}, \quad |t| \leq c_7 n.$$

Estimate (30) may also be checked by straightforward calculation, using representation (17).

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