HEISENBERG–WEYL OPERATOR ALGEBRAS ASSOCIATED TO THE MODELS OF CALOGERO–SUTHERLAND TYPE AND ISOMORPHISM OF RATIONAL AND TRIGONOMETRIC MODELS

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The object of this article is a construction of the Fock spaces and Weyl algebras associated to different root systems. First, a review of the recent papers devoted to the investigations of the Fock spaces and operator algebras associated to the physical models with groups of symmetries generated by reflections is given. Then the original treatment of the basic notions and operators defined for some vector spaces related to the irreducible root systems is presented. This treatment permits to obtain general constructions of the Fock spaces and the Heisenberg–Weyl operator algebras with symmetric properties for arbitrary root systems.

Let R be a root system, $R \subset V^n$, N the number of roots in R, $|R| = N$. For a given R, Hamiltonians are constructed in the vector space $\mathbb{C}^N$. The inverse images of these Hamiltonians with respect to the map $h : V^n \to \mathbb{C}^N$ are Hamiltonians of Calogero–Sutherland. Representations of these Hamiltonians by means of the universal Dunkl operators associated to the same root system are given. A generalization of the Kakei conjecture about the isomorphism of operator algebras and Fock spaces associated to Hamiltonians of Calogero and Sutherland and corresponding to different root systems is stated.

The research was written in the frame the State Program of Support of Leading Scientific Schools and was supported by Grants RFFI-INTAS 00418 and RFFI-Germany 96-01-00008G.

Nonrelativistic one-dimensional quantum models of rational, trigonometric, and elliptic types with interaction potential proportional to the inverse squares of distances, and with symmetries of different forms, have been investigated in a series of papers of M. A. Olshanetski, A. M. Perelomov, E. M. Opdam, I. V. Cherednik, A. P. Polychronakos, H. Ujino and M. Wadati, A. Veselov, V. M. Buchstaber, J. Felder and A. Veselov, T. Yamamoto, and others.

These papers are mainly devoted to the investigation of the integrability of such models and to the construction of complete sets of their integrals. For this purpose, the Lax method of quantum (L, A)-pairs and the representation of Hamiltonians and integrals by means of Dunkl and Knizhnik–Zamolodchikov operators were exploited.

The ground and exited states of such models were investigated, and corresponding quantum numbers were calculated. However, there are still many open problems concerning the representation of eigenstates of spin systems of particles and many related questions of construction of eigenstates of models of Calogero– Sutherland type and of solutions of the generalized Knizhnik–Zamolodchikov equations and the connection of these spaces. For example, the integral representations of fundamental matrices of solutions of the generalized Knizhnik–Zamolodchikov equations for the most types of root systems either is not known or their construction is not certified by the corresponding verification, i.e., it is not proved that this construction gives the solution.

Translated from Itogi Nauki i Tekhniki, Seriya Sovremennaya Matematika i Ee Prilozheniya. Tematicheskie Obzory. Vol. 54, Functional Analysis–7, 1998.

Models of different types are naturally interrelated. For example, limit processes permit to obtain rational and trigonometric models from the elliptic one by introducing appropriate parameters. But the general geometric and algebraic structures of the models mentioned above are of great interest.

The present paper is devoted to a review of different approaches to construction and comparison of Heisenberg–Weyl algebras and Fock spaces for different models of Calogero–Sutherland type, to the investigation of the eigenstates of the corresponding Hamiltonians and the generalizations of these theories. The basis of eigenstates for Sutherland models was described, for example, in papers of G. J. Heckman [1] and E. M. Opdam [2]. They constructed the orthogonal bases of eigenstates using the W-invariant Jacobi polynomials. A direct generalization of this method to the Calogero model is difficult since there exist the degenerated eigenvalues of Hamiltonians and the Gram–Schmidt orthogonalization procedure applied to the space of solutions of the Calogero equation does not permit to obtain the explicite formulas for the basis.

Calogero–Sutherland models describe one-dimensional dynamics of many-body systems (on line, hyperbola, or circle) with $1/r^2$-type long-range interaction. Translation invariant systems of such type correspond to the root system of A_n-type. The necessity of consideration of the root systems different from A_n is conditioned either by constraints on moving bodies (half-line, segment), or by external forces, and also by effects of many-body interactions. For example, the nonrelativistic dynamics of quantum sine-Gordon solitons in presence of a boundary is described by the Sutherland model of BC_n-type. This model is also related to the physics of the quantum electric transport in mesoscopic systems. The Haldane–Shastry models can be considered as dicrete version of Calogero–Sutherland models (see [2–6]).

Recently, H. Ujino and M. Wadati [7, 8] gave a construction of a basis for the Calogero model associated to the root system A_N with harmonic oscillator, using corresponding basis for the Sutherland model (see also S. Kakei [9]). S. Kakei [10], T. H. Baker and P. J. Forrester [11] extended the results of Ujino and Wadati to other root systems, in particular, to the case of the B_N Calogero systems with the harmonic oscillator.

One of the objects of our paper is a review of recent papers of S. Kakei, P. Forrester, H. Ujino and M. Wadati, A. P. Polychronakos [12], D. Serban [13], K. Takemura and D. Uglov [14] devoted to the investigation of the isomorphisms between the Calogero and Sutherland models associated to identical root systems. These authors considered the root systems A_n, B_n, C_n and D_n.

Another object of the paper is an investigation of general algebraic properties of the Hamiltonians of the models. We will give the construction of Hamiltonians in the most universal form produced by an arbitrary root system corresponding to the finite symmetry group of the model. Further, for any root system the algebras containing the Hamiltonians and complete systems of integrals of the model and the Fock spaces containing all principal and excited states are introduced. By analogy with the results of S. Kakei [9, 10], for root systems A_{n-1}, B_n, C_n, and D_n, the isomorphism of the Fock spaces and the corresponding Heisenberg–Weyl algebras introduced for rational and trigonometric models for arbitrary root system is conjectured.

It is necessary to note that the article abounds with nonstandard, sufficiently hard calculations. This is done for the convenience of the reader.

1. Representation for Sutherland Operators by Means of Dunkl Operators. Ujino–Wadati Approach

The Sutherland model describes a system of n nonrelativistic particles on the circle (or on the hyperbola), interacting with inverse square law, whose Hamiltonian has the following form:

$$H_S = -\sum_{i=1}^{n} \frac{\partial^2}{\partial \theta_i^2} + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}} \frac{\omega^2 \beta(\beta - 1)}{\sin[\omega(\theta_i - \theta_j)/2]}.$$

Different authors investigated the integrability of the Sutherland model. A complete set of n independent integrals, i.e., of n differential operators commuting with each other and with H_S, was found (see [2]). For example, Polychronakos [12], Ujino and Wadati [7, 8], Dunkl [15], and Cherednik [16] gave the following characterization of the operators connected with the Sutherland model.

Let

$$\nabla_j = \frac{\partial}{\partial x_j} - \beta \sum_{\substack{k \\ k \neq j}} \frac{s_{jk} - 1}{x_j - x_k}$$

be the Dunkl operator of rational type. Here s_{ij}, $1 \leq i < j \leq n$, act on a function as operators of permutation of variables. Denote

$$A_j = \sum_{\substack{k \\ k \neq j}} \frac{s_{jk} - 1}{x_j - x_k}.$$

Then we have

$$\nabla_j = \partial_j - \beta A_j.$$

It was shown that the algebra generated by $x^{\pm 1}$, ∇_j and s_{ij} is isomorphic to the double affine Hecke algebra. The structure of this algebra is characterized by the following assertion.

Assertion 1 ([7–10]). *The following commutation relations hold:*

(1) $\quad [\nabla_i, \nabla_j] = 0, \quad i, j = 1, 2, \ldots, n,$

(2) $\quad s_{ij} \nabla_j = \nabla_i s_{ij},$

(3) $\quad s_{ij} \nabla_k = \nabla_k s_{ij}, \quad k \neq i, j,$

(4) $\quad [\nabla_i, x_j] = \delta_{ij} \left(1 + \beta \sum_{k \neq i} s_{ik} \right) - (1 - \delta_{ij}) \beta s_{ij}.$

Proof. Prove the property (1). Let $i \neq j$. We have

$$[\nabla_i, \nabla_j] = [\partial_i - \beta A_i, \partial_j - \beta A_j] = [\partial_i, \partial_j] - \beta([\partial_i, A_j] - [\partial_j, A_i]) + \beta^2 [A_i, A_j],$$

Evidently, $[\partial_i, \partial_j] = 0$. Further we obtain

$$[\partial_i, A_j] = \partial_i A_j - A_j \partial_i = \partial_i(A_j) + \sum_{\substack{k \\ k \neq j}} \frac{\partial_i(s_{jk} - 1)}{x_j - x_k} - A_j \partial_i.$$

Using the relations $s_{ij} x_j = x_i s_{ij}$ and $s_{ij} \partial_j = \partial_i s_{ij}$, we obtain

$$[\partial_i, A_j] = \frac{s_{ji} - 1}{(x_j - x_i)^2} + \sum_{\substack{k \\ k \neq j,i}} \frac{s_{jk} - 1}{x_j - x_k} \partial_i + \frac{s_{ji} - 1}{x_j - x_i} \partial_j - A_j \partial_i = \frac{s_{ji} - 1}{(x_j - x_i)^2} + \frac{s_{ji} - 1}{x_j - x_i} (\partial_j - \partial_i).$$

We see that on the right-hand side we obtain a function symmetric with respect to i and j. Hence, the commutator $[\partial_j, A_i]$ is equal to the above expression. Consequently, the term of $[\nabla_i, \nabla_j]$ in parentheses, following the coefficient β, is equal to zero.

Compute the commutator $[A_i, A_j]$. We have

$$[A_i, A_j] = \left[\sum_{m\neq i} \frac{s_{im}-1}{x_i - x_m}, \sum_{k\neq j} \frac{s_{jk}-1}{x_j - x_k}\right].$$

Keeping in mind that $s_{im} - 1$ and $\dfrac{1}{x_i - x_m}$ commute with the term $\dfrac{s_{jk}-1}{x_j - x_k}$ for $i \neq j, k$ and $m \neq j, k$, we obtain

$$[A_i, A_j] = \sum_{k\neq j,i} \left(\left[\frac{s_{ij}-1}{x_i - x_j}, \frac{s_{jk}-1}{x_j - x_k}\right] + \left[\frac{s_{ik}-1}{x_i - x_k}, \frac{s_{ji}-1}{x_j - x_i}\right] + \left[\frac{s_{ik}-1}{x_i - x_k}, \frac{s_{jk}-1}{x_j - x_k}\right]\right).$$

Obviously, we have $s_{ij} = s_{ji}$. Introducing the notations

$$w = s_{ij}s_{jk} = s_{ik}s_{ij} = s_{jk}s_{ik}, \qquad w_1 = s_{jk}s_{ij} = s_{ji}s_{ik} = s_{ik}s_{jk},$$

we can rewrite the last expression in the form

$$\begin{aligned}
[A_i, A_j] = \sum_{k\neq i,j} \Bigg\{ \Bigg[& \frac{1}{x_i - x_j}\left(\frac{1}{x_i - x_k} - \frac{1}{x_j - x_k}\right) + \frac{1}{x_i - x_k}\left(\frac{1}{x_j - x_k} - \frac{1}{x_j - x_i}\right) \\
& + \frac{1}{x_k - x_j}\left(\frac{1}{x_i - x_j} - \frac{1}{x_i - x_k}\right)\Bigg](w+1) - \Bigg[\frac{1}{x_j - x_k}\left(\frac{1}{x_i - x_k} - \frac{1}{x_i - x_j}\right) \\
& + \frac{1}{x_j - x_i}\left(\frac{1}{x_j - x_k} - \frac{1}{x_i - x_k}\right) + \frac{1}{x_i - x_k}\left(\frac{1}{x_i - x_j} - \frac{1}{x_k - x_j}\right)\Bigg](w_1 + 1)] \\
& + \left[\frac{1}{x_i - x_j}\left(\frac{1}{x_i - x_k} - \frac{1}{x_j - x_k}\right) - \frac{1}{x_j - x_k}\left(\frac{1}{x_i - x_k} - \frac{1}{x_i - x_j}\right)\right](s_{ij} + s_{jk}) \\
& + \left[\frac{1}{x_i - x_k}\left(\frac{1}{x_j - x_k} - \frac{1}{x_j - x_i}\right) - \frac{1}{x_j - x_i}\left(\frac{1}{x_j - x_k} - \frac{1}{x_i - x_k}\right)\right](s_{ik} + s_{ji}) \\
& + \left[\frac{1}{x_k - x_j}\left(\frac{1}{x_i - x_j} - \frac{1}{x_i - x_k}\right) - \frac{1}{x_j - x_k}\left(\frac{1}{x_i - x_j} - \frac{1}{x_j - x_k}\right)\right](s_{ik} + s_{jk}). \Bigg\}
\end{aligned}$$

Direct verification shows that the coefficients at $(w+1)$ and (w_1+1) are equal to zero. Further, simultaneous reduction of similar terms in last three lines shows that the coefficients at s_{ij}, s_{jk}, s_{ik} are equal to zero as well. Hence, we obtain $[A_i, A_j] = 0$ and the proof of (1) is finished.

Now we prove (4). We have

$$\begin{aligned}
[\nabla_i, x_j] = \nabla_i x_j - x_j \nabla_i &= (\partial_i - \beta \sum_{\substack{k \\ k\neq i}} \frac{s_{ik}-1}{x_i - x_k}) x_j - x_j \nabla_i \\
&= \delta_{ij} + x_i \partial_j - \beta \sum_{\substack{k \\ k\neq i}} \frac{(s_{ik}(x_j))s_{ik} - x_j}{x_i - x_k} - x_j \nabla_i.
\end{aligned}$$

Using in the third term of this sum the following well known formula of reflection,

$$(s_\alpha x)_j = x_j - \frac{2(\alpha, x)}{(\alpha, \alpha)} \alpha_j,$$

we obtain

$$[\nabla_i, x_j] = \delta_{ij} + x_i\partial_j - \beta \sum_{\substack{k\\k\neq i}} \frac{[(x_j - (x_i - x_k)(\delta_{ij} - \delta_{kj})s_{ik} - x_j]}{x_i - x_k} - x_j\nabla_i$$

$$= \delta_{ij}\left(\partial_i - \beta\sum_{\substack{k\\k\neq i}} \frac{s_{ik}-1}{x_i - x_k}\right) + \beta\sum_{\substack{k\\k\neq i}}(\delta_{ij} - \delta_{kj})s_{ik} - x_j\nabla_i$$

$$= \delta_{ij}\left(1 + \beta\sum_{\substack{k\\k\neq i}} s_{ik}\right) - \beta\sum_{\substack{k\\k\neq i}}\delta_{ij}s_{ik} + x_j\nabla_i - x_j\nabla_i$$

$$= \delta_{ij}\left(1 + \beta\sum_{\substack{k\\k\neq i}} s_{ik}\right) - (1-\delta_{ij})\beta s_{ij}.$$

Here we have used the relation

$$\sum_{\substack{k\\k\neq i}} \delta_{kj}s_{ik} = \begin{cases} s_{ij} & \text{if} \quad i\neq j,\\ 0 & \text{if} \quad i = j.\end{cases}$$

The proof is finished. □

Consider the family of commuting operators (see [16])

$$\tilde\nabla_j = x_j\nabla_j + \beta\sum_{k<j} s_{jk}.$$

Remark. $\tilde\nabla_i$ can be represented in the form

$$\tilde\nabla_j = x_j\partial_j + \beta\sum_{k<j}\frac{x_k}{x_j - x_k}(s_{jk}-1) - \beta\sum_{k>j}\frac{x_j}{x_j - x_k}(s_{jk}-1) - \beta(j-1).$$

The following assertion holds.

Assertion 2. *The operators $\tilde\nabla_j$ satisfy the following commutation relations:*

(1) $[\tilde\nabla_i, \tilde\nabla_j] = [x_i, x_j] = 0,$

(2) $s_i^2 = 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},$

(3) $[s_i, s_j] = 0, \quad |i-j| \neq 1,$

(4) $x_i s_{ij} = s_{ij} x_j, \quad x_i s_{jk} = s_{jk} x_i, \quad i\neq j,k,$

(5) $\tilde\nabla_{j+1}s_j - s_j\tilde\nabla_j = \beta, \quad , s_j\tilde\nabla_{j+1} - \tilde\nabla_j s_j = \beta,$

(6) $[s_i, \tilde\nabla_j] = 0, \quad j\neq i, i+1,$

(7) $$[\tilde\nabla_i, x_j] = \begin{cases} -\beta x_j s_{ij}, & i>j,\\ x_i + \beta\left(\sum\limits_{k<i} x_k s_{ik} + \sum\limits_{k>i} x_k s_{ik}\right), & i=j,\\ -\beta x_i s_{ij}, & i<j.\end{cases}$$

Here we use the notation $s_i = s_{i,i+1}$, $i = 1,\dots,n$. The equations (2)–(4) and (6) are either well known or easily verified. The basic relations are (1), (5), and (7). The proof of (7) is very similar to that of (4) in Assertion 1. We verify (1) and (5).

Proof of (1). Let $i < j$. We have

$$[\tilde{\nabla}_i, \tilde{\nabla}_j] = \left[x_i \nabla_i + \beta \sum_{\substack{m \\ m<i}} s_{im}, x_j \nabla_j + \beta \sum_{\substack{k \\ k<j}} s_{jk} \right]$$

$$= [x_i \nabla_i, x_j \nabla_j] + \beta \left[x_i \nabla_i, \sum_{\substack{k \\ k<j}} s_{jk} \right] - \beta \left[x_j \nabla_j, \sum_{\substack{m \\ m<i}} s_{im} \right] + \beta^2 \left[\sum_{\substack{m \\ m<i}} s_{im}, \sum_{\substack{k \\ k<j}} s_{jk} \right].$$

Further, we obtain

$$[x_i \nabla_i, x_j \nabla_j] = x_i \nabla_i x_j \nabla_j - x_j \nabla_j x_i \nabla_i = x_i x_j \nabla_i \nabla_j - x_j x_i \nabla_j \nabla_i + x_i [\nabla_i, x_j] \nabla_j + x_j [\nabla_j, x_i] \nabla_i.$$

The first and the second terms in the last equality cancel out, and we have

$$[x_i \nabla_i, x_j \nabla_j] = x_i(-\beta s_{ij}) \nabla_j - x_j(-\beta s_{ij}) \nabla_i = -\beta x_i \nabla_i s_{ij} + \beta x_j \nabla_j s_{ij} = \beta(x_j \nabla_j - x_i \nabla_i) s_{ij}.$$

Now, all terms of $\sum\limits_{k<j} s_{jk}$ commute with $x_i \nabla_i$ excluding s_{ij}, and all terms of $\sum\limits_{m<i} s_{im}$ commute with $x_j \nabla_j$ since $i < j$. Hence we have

$$\left[x_i \nabla_i, \sum_{k<j} s_{jk} \right] - \left[x_j \nabla_j, \sum_{\substack{m \\ m<i}} s_{im} \right] = [x_i \nabla_i, s_{ij}] = x_i \nabla_i s_{ij} - s_{ji} x_i \nabla_i = (x_i \nabla_i - x_j \nabla_j) s_{ij}.$$

Consequently, the difference of commutators cancels out with the precceding commutator $[x_i \nabla_i, x_j \nabla_j]$. For the last commutator

$$\left[\sum_{\substack{k \\ k<j}} s_{jk}, \sum_{\substack{m \\ m<i}} s_{im} \right]$$

we have

$$\left[\sum_{\substack{k \\ k<j}} s_{jk}, \sum_{\substack{m \\ m<i}} s_{im} \right] = \left[\sum_{\substack{m \\ m<i}} s_{im}, s_{ji} \right] + \left[\sum_{\substack{m \\ m<i}} s_{im}, \sum_{\substack{m \\ m<i}} s_{jm} \right] = \sum_{\substack{m \\ m<i}} [s_{im}, s_{ji}] + \sum_{\substack{m \\ m<i}} [s_{im}, s_{jm}].$$

Taking into account the equations

$$[s_{im}, s_{ji}] = s_{im} s_{ji} - s_{ji} s_{im} = s_{jm} s_{im} - s_{ji} s_{im},$$
$$[s_{im}, s_{jm}] = s_{im} s_{jm} - s_{jm} s_{im} = s_{ji} s_{im} - s_{jm} s_{im},$$

which are the consequences of the relation

$$s_\alpha s_\beta = s_{s_\alpha \beta} s_\alpha,$$

we see that the mth terms of two sums of the computed commutator cancel, and we obtain

$$\left[\sum_{\substack{k \\ k<j}} s_{jk}, \sum_{\substack{m \\ m<i}} s_{im} \right] = 0.$$

The proof of (1) is finished.

Proof of (5). We have to establish

$$\tilde{\nabla}_{j+1} s_j - s_j \tilde{\nabla}_j = \beta,$$

where $s_j = s_{j,j+1}$. We have

$$\begin{aligned}\tilde\nabla_{j+1}s_j - s_j\tilde\nabla_j &= x_{j+1}\nabla_{j+1}s_j - s_jx_j\nabla_j + \beta\left(\left(\sum_{k<j+1} s_{j+1,k}\right)s_j - s_j\sum_{k<j} s_{jk}\right)\\ &= x_{j+1}\nabla_{j+1}s_j - x_{j+1}\nabla_{j+1}s_j + \beta\left(\sum_{k<j} s_{j+1,k}s_j + s_{j+1,j}s_j - \sum_{k<j} s_{j+1,k}s_j\right)\\ &= \beta s_{j,j+1}s_{j+1,j} = \beta s^2_{j,j+1} = \beta.\end{aligned}$$

The proof of the second equation of (5) is similar. □

Let now $\operatorname{Res} P$ be the restriction of the operator P on the space of symmetric functions. Concerning the properties of operators $\tilde\nabla_i$, we have the following important

Assertion 3 ([8, 12]). *The set of commuting operators*

$$I_k = \operatorname{Res}\sum_{i=1}^{n}(\tilde\nabla_i)^k, \quad k = 1, \dots, n,$$

constitute the set of integrals of the Sutherland model.

Remark. It is necessary to note that the operator $2I_2$ is conjugate to the Hamiltonian H_S by means of the ground state function of the Hamiltonian H_S considered in coordinates $x_j = \exp[2i\theta_j]$, $j = 1, \dots, n$.

Consider now the properties of the Sutherland operator H_S. Making the change of variables $x_j = e^{i\theta_j}$ (the trigonometric case) we obtain

$$H_S = \sum_{i=1}^{n}(x_j\partial_j)^2 - \beta(\beta-1)\sum_{j<k}\frac{2x_jx_k}{(x_j-x_k)^2}.$$

Indeed, we have

$$ix_j\partial_j = \frac{\partial}{\partial\theta_j}$$

and

$$\sin^2\frac{\theta_i-\theta_j}{2} = -\frac{1}{4}\frac{(x_i-x_j)^2}{x_ix_j}.$$

Further, let $\tilde H_S$ be the operator of the form

$$\tilde H_S = \operatorname{Res}\sum_{i=1}^{n}\left(\tilde\nabla_j - \beta\frac{(n-1)}{2}\right)^2.$$

Recall that

$$\begin{aligned}\tilde\nabla_j = x_j\nabla_j + \beta\sum_{k<j}s_{jk} &= x_j\partial_j - \beta\sum_{\substack{k\\ k\neq j}}\frac{x_j}{x_j-x_k}(s_{jk}-1) + \beta\sum_{k<j}s_{jk}\\ &= x_j\partial_j - \beta\sum_{k<j}\frac{x_k}{x_j-x_k}(s_{jk}-1) - \beta\sum_{k>j}\frac{x_j}{x_j-x_k}(s_{jk}-1) + \beta(j-1).\end{aligned}$$

The following two assertions describe the properties of the operator $\tilde H_S$.

Assertion 4. *The operator $\tilde H_S$ has the form*

$$\tilde H_S = \sum_{i=1}^{n}(x_j\partial_j)^2 + \beta\sum_{j<k}\frac{x_j+x_k}{x_j-x_k}(x_j\partial_j - x_k\partial_k) - \frac{\beta^2}{12}n(n^2-1).$$

Proof. Introduce the operators

$$B_j = \beta \sum_{k<j} \frac{x_k}{x_j - x_k}(1 - s_{jk}) + \beta \sum_{k>j} \frac{x_j}{x_j - x_k}(1 - s_{jk}),$$

$j = 1, \dots, n$, which vanish on symmetric functions, i.e., $\operatorname{Res} B_j = 0$, $j = 1, \dots, n$.

We have

$$\begin{aligned}
\tilde{H}_S &= \operatorname{Res} \sum_{j=1}^{n} \left(x_j\partial_j + B_j + \beta \frac{2j-n-1}{2} \right)^2 \\
&= \operatorname{Res} \left[\sum_{j=1}^{n} [(x_j\partial_j)^2 + x_j\partial_j B_j + \beta(2j-n-1)B_j + B_j^2 \right. \\
&\qquad \left. + B_j x_j \partial_j + \beta(2j-n-1)x_j\partial_j + \frac{\beta^2}{4}(2j-n-1)^2 \right] \\
&= \sum_{j=1}^{n}(x_j\partial_j)^2 + \sum_{j=1}^{n} \Big(B_j x_j \partial_j + \beta[(j-1) - (n-j)]x_j\partial_j \Big) + \frac{\beta^2}{4}\sum_{j=1}^{n}(2j-n-1)^2 \\
&= \sum_{j=1}^{n}(x_j\partial_j)^2 + \beta \sum_{j=1}^{n} \left[\sum_{k<j} \left(\frac{x_k}{x_j - x_k}(1-s_{jk}) + 1 \right) x_j \partial_j \right. \\
&\qquad \left. + \sum_{k>j} \left(\frac{x_j}{x_j - x_k}(1 - s_{jk}) - 1 \right) x_j\partial_j \right] + \frac{\beta^2}{4} \sum_{j=1}^{n}(2j-n-1)^2.
\end{aligned}$$

Two double sums, over $j = 1, \dots, n$ and over $k < j$ or $j < k$, can be rewritten as a sum in all pairs $1 \le k < j \le n$ or $1 \le j < k \le n$,

$$\begin{aligned}
\tilde{H}_S = \sum_{i=1}^{n}(x_j\partial_j)^2 &+ \beta \sum_{k<j} \frac{x_j}{x_j - x_k} x_j\partial_j - \beta \sum_{k<j} \frac{x_k}{x_j - x_k} x_k \partial_k + \beta \sum_{j<k} \frac{x_k}{x_j - x_k} x_j \partial_j \\
&- \beta \sum_{j<k} \frac{x_j}{x_j - x_k} x_k\partial_k + \frac{\beta^2}{4}\sum_{i=1}^{n}(2j-n-1)^2.
\end{aligned}$$

Grouping pairwise the sums (the second and the fourth, the third and the fifth), and making the change $j \leftrightarrow k$ in the fourth and fifth sums, we obtain

$$\begin{aligned}
\tilde{H}_S &= \sum_{i=1}^{n}(x_j\partial_j)^2 + \beta \left(\sum_{k<j} \frac{x_j}{x_j - x_k} x_j\partial_j + \sum_{k<j} \frac{x_j}{x_k - x_j} x_k \partial_k \right) \\
&\quad - \beta \left(\sum_{k<j} \frac{x_k}{x_j - x_k} x_k\partial_k + \sum_{k<j} \frac{x_k}{x_k - x_j} x_j\partial_j \right) + \frac{\beta^2}{4}\sum_{i=1}^{n}(2j-n-1)^2 \\
&= \sum_{i=1}^{n}(x_j\partial_j)^2 + +\beta \sum_{k<j} \frac{x_j}{x_j - x_k}(x_j\partial_j - x_k\partial_k) + \beta \sum_{k<j} \frac{x_k}{x_j - x_k}(x_j\partial_j - x_k\partial_k) + \frac{\beta^2}{4}\sum_{i=1}^{n}(2j-n-1)^2 \\
&= \sum_{i=1}^{n}(x_j\partial_j)^2 + \beta \sum_{k<j} \frac{x_j + x_k}{x_j - x_k}(x_j\partial_j - x_k\partial_k) + \frac{\beta^2}{4}\sum_{i=1}^{n}(2j-n-1)^2.
\end{aligned}$$

Using the formula $\sum\limits_{j=1}^{n}(2j-n-1)^2 = \frac{n(n^2-1)}{3}$, we obtain

$$\tilde{H}_S = \sum_{j=1}^{n}(x_j\partial_j)^2 + \beta \sum_{1\le j<k\le n} \frac{x_j+x_k}{x_j-x_k}(x_j\partial_j - x_k\partial_k) + \frac{\beta^2}{12}n(n^2-1).$$

The proof of asserion 4 is finished. □

Let ϕ_S be the function

$$\phi_S = \prod_{j<k}|x_j-x_k|^{\beta}\prod_{j=1}^{n} x_j^{-\beta\frac{n-1}{2}}.$$

The following assertion characterizes the conjugation of the operator $\tilde{H}_S$ by the function ϕ_S.

Assertion 5.

$$\phi_S\tilde{H}_S\phi_S^{-1} = H_S.$$

Proof. Indeed, we have

$$\phi_S\tilde{H}_S\phi_S^{-1} = \phi_S\left(\sum_{j=1}^{n}(x_j\partial_j)^2 + \beta\sum_{j<k}\frac{x_i+x_k}{x_i-x_k}(x_j\partial_j - x_k\partial_k) + \frac{\beta^2}{12}n(n^2-1)\right)\phi_S^{-1}$$

$$= \sum_{j=1}^{n}(x_j\phi_S\partial_j\phi_S^{-1})^2 + \beta\sum_{j<k}\frac{x_i+x_k}{x_i-x_k}\left(x_j(\phi_S\partial_j\phi_S^{-1}) - x_k\phi_S\partial_k\phi_S^{-1}\right) + \frac{\beta^2}{12}n(n^2-1).$$

Since

$$\phi_S\partial_j{\phi_S}^{-1} = \partial_j - \beta\sum_{\substack{k\\k\neq j}}\frac{1}{x_j-x_k} + \beta\frac{n-1}{2}\frac{1}{x_j},$$

we have

$$x_j\phi_S\partial_j\phi_S^{-1} = x_j\partial_j - \beta\sum_{\substack{k\\k\neq j}}\frac{x_j}{x_j-x_k} + \beta\frac{n-1}{2}.$$

Substituting this expression into the first term of the sum $\phi_S\tilde{H}_S\phi_S^{-1}$ and continuing the calculation, we obtain

$$\phi_S\tilde{H}_S\phi_S^{-1} = \sum_{j=1}^{n}\left(x_j\partial_j - \beta\sum_{\substack{k\\k\neq j}}\frac{x_j}{x_j-x_k} + \beta\frac{n-1}{2}\right)^2 + \beta\sum_{j<k}\frac{x_i+x_k}{x_i-x_k}(x_j\partial_j - x_k\partial_k)$$

$$-\beta^2\sum_{j<k}\frac{x_i+x_k}{x_i-x_k}\left(\sum_{\substack{l\\l\neq j}}\frac{x_j}{x_j-x_k} - \sum_{\substack{m\\m\neq k}}\frac{x_k}{x_k-x_m}\right) + \frac{\beta^2}{12}n(n^2-1)$$

$$= \sum_{j=1}^{n}\left[(x_j\partial_j)^2 + \beta^2\left(\sum_{\substack{k\\k\neq j}}\frac{x_j}{x_j-x_k}\right)^2 + \frac{\beta^2}{4}(n-1)^2\right.$$

$$\left.-\beta^2(n-1)\sum_{\substack{k\\k\neq j}}\frac{x_j}{x_j-x_k} - \beta\sum_{\substack{k\\k\neq j}}\frac{x_j}{x_j-x_k}x_j\partial_j - \beta x_j\partial_j\sum_{\substack{k\\k\neq j}}\frac{x_j}{x_j-x_k} + \beta(n-1)x_j\partial_j\right]$$

$$+\beta\sum_{j<k}\frac{x_i+x_k}{x_i-x_k}(x_j\partial_j-x_k\partial_k)-\beta^2\sum_{j<k}\frac{x_i+x_k}{x_i-x_k}\left(\sum_{\substack{l\\ l\neq j}}\frac{x_j}{x_j-x_k}-\sum_{\substack{m\\ m\neq k}}\frac{x_k}{x_k-x_m}\right)+\frac{\beta^2}{12}n(n^2-1)$$

$$=\sum_{j=1}^n(x_j\partial_j)^2+\beta\sum_{j=1}^n\sum_{\substack{k\\ k\neq j}}\frac{x_jx_k}{(x_j-x_k)^2}+\beta^2\sum_{j=1}^n\left(\sum_{\substack{k\\ k\neq j}}\frac{x_j}{x_j-x_k}\right)^2+\frac{\beta^2}{4}n(n-1)^2$$

$$-2\beta\sum_{j=1}^n\sum_{\substack{k\\ k\neq j}}\frac{x_j}{x_j-x_k}x_j\partial_j-\beta^2(n-1)\sum_{j=1}^n\sum_{\substack{k\\ k\neq j}}\frac{x_j}{x_j-x_k}+\beta(n-1)\sum_{j=1}^n x_j\partial_j$$

$$+\beta\sum_{j<k}\frac{x_j+x_k}{x_j-x_k}(x_j\partial_j-x_k\partial_k)-\beta^2\sum_{j<k}\frac{x_j+x_k}{x_j-x_k}\left(\sum_{\substack{l\\ l\neq j}}\frac{x_j}{x_j-x_l}-\sum_{\substack{l\\ l\neq k}}\frac{x_k}{x_k-x_l}\right)+\frac{\beta^2}{12}n(n^2-1).$$

Grouping the terms containing $x_j\partial_j$ and $x_k\partial_k$, we obtain

$$-2\beta\sum_{j=1}^n\sum_{\substack{k\\ k\neq j}}\frac{x_j}{x_j-x_k}x_j\partial_j+\beta(n-1)\sum_{j=1}^n x_j\partial_j+\beta\sum_{j<k}\frac{x_j+x_k}{x_j-x_k}(x_j\partial_j-x_k\partial_k)$$

$$=-\beta\sum_{j=1}^n\left(\sum_{\substack{k\\ k<j}}\frac{x_j}{x_j-x_k}x_j\partial_j+\sum_{\substack{k\\ k>j}}\frac{x_j}{x_j-x_k}x_j\partial_j\right)-\beta\sum_{j=1}^n\sum_{\substack{k\\ k\neq j}}(\frac{x_j}{x_j-x_k}-1)x_j\partial_j$$

$$+\beta\sum_{j<k}\frac{x_j+x_k}{x_j-x_k}(x_j\partial_j-x_k\partial_k)=-\beta\sum_{k<j}\frac{x_j}{x_j-x_k}x_j\partial_j-\beta\sum_{k>j}\frac{x_j}{x_j-x_k}x_j\partial_j$$

$$-\beta\sum_{j=1}^n\left(\sum_{k<j}\frac{x_k}{x_j-x_k}x_j\partial_j+\sum_{k>j}\frac{x_k}{x_j-x_k}x_j\partial_j\right)+\beta\sum_{j<k}\frac{x_j+x_k}{x_j-x_k}(x_j\partial_j-x_k\partial_k)$$

$$=-\beta\sum_{k<j}\frac{x_j}{x_j-x_k}x_j\partial_j-\beta\sum_{k>j}\frac{x_j}{x_j-x_k}x_j\partial_j-\beta\sum_{k<j}\frac{x_k}{x_j-x_k}x_j\partial_j$$

$$-\beta\sum_{k>j}\frac{x_k}{x_j-x_k}x_j\partial_j+\beta\sum_{j<k}\frac{x_j+x_k}{x_j-x_k}(x_j\partial_j-x_k\partial_k).$$

Substituting $j\leftrightarrow k$ in the first and the third sums and grouping the first and the fourth sums, and also the second and the third sums, we obtain

$$-\beta\sum_{\substack{k\\ j<k}}\frac{x_k}{x_j-x_k}(x_j\partial_j-x_k\partial_k)-\beta\sum_{\substack{k\\ j<k}}\frac{x_j}{x_j-x_k}(x_j\partial_j-x_k\partial_k)-\beta\sum_{\substack{k\\ j<k}}\frac{x_j+x_k}{x_j-x_k}(x_j\partial_j-x_k\partial_k)$$

$$=-\beta\sum_{\substack{k\\ j<k}}\frac{x_j+x_k}{x_j-x_k}(x_j\partial_j-x_k\partial_k)+\beta\sum_{\substack{k\\ j<k}}\frac{x_j+x_k}{x_j-x_k}(x_j\partial_j-x_k\partial_k)=0.$$

Hence, we have,

$$\phi_S\tilde{H}_S\phi_S^{-1}=\sum_{j=1}^n(x_j\partial_j)^2+\beta\sum_{j=1}^n\sum_{\substack{k\\ k\neq j}}\frac{x_jx_k}{(x_j-x_k)^2}-\beta^2(n-1)\sum_{j=1}^n\sum_{\substack{k\\ k\neq j}}\frac{x_j}{x_j-x_k}+\beta^2\sum_{j=1}^n\left(\sum_{\substack{k\\ k\neq j}}\frac{x_j}{x_j-x_k}\right)^2$$

$$-\beta^2\sum_{j<k}\frac{x_j+x_k}{x_j-x_k}\left(\sum_{\substack{l\\ l\neq j}}\frac{x_j}{x_j-x_l}-\sum_{\substack{l\\ l\neq k}}\frac{x_k}{x_k-x_l}\right)+\frac{\beta^2}{6}n(n-1)(2n-1)$$

$$=\sum_{j=1}^n(x_j\partial_j)^2+\beta\sum_{j=1}^n\left(\sum_{\substack{k\\ k<j}}\frac{x_jx_k}{(x_j-x_k)^2}+\sum_{\substack{k\\ k>j}}\frac{x_jx_k}{(x_j-x_k)^2}\right)$$

$$-\beta^2(n-1)\sum_{j=1}^n\left(\sum_{\substack{k\\ k<j}}\frac{x_j}{x_j-x_k}+\sum_{\substack{k\\ k>j}}\frac{x_j}{x_j-x_k}\right)+\beta^2\sum_{j=1}^n\left(\sum_{\substack{k\\ k\neq j}}\frac{x_j}{x_j-x_k}\right)^2$$

$$-\beta^2\sum_{j<k}\frac{x_j+x_k}{x_j-x_k}\left(\sum_{\substack{l\\ l\neq j}}\frac{x_j}{x_j-x_l}-\sum_{\substack{l\\ l\neq k}}\frac{x_k}{x_k-x_l}\right)+\frac{\beta^2}{6}n(n-1)(2n-1).$$

Substituting $j\leftrightarrow k$ in the second sum in the bracket at β and in the second sum in the bracket at $-b^2(n-2)$, we obtain

$$\phi_S\tilde{H}_S\phi_S^{-1}=\sum_{j=1}^n(x_j\partial_j)^2+2\beta\sum_{j=1}^n\sum_{\substack{k\\ j<k}}\frac{x_jx_k}{(x_j-x_k)^2}-\beta^2(n-1)\sum_{k<j}\frac{x_j-x_k}{x_j-x_k}$$

$$+\beta^2\sum_{j=1}^n\left(\sum_{\substack{k\\ k\neq j}}\frac{x_j}{x_j-x_k}\right)^2-\beta^2\sum_{j<k}\frac{x_j+x_k}{x_j-x_k}\left(\sum_{\substack{l\\ l\neq j}}\frac{x_j}{x_j-x_l}-\sum_{\substack{l\\ l\neq k}}\frac{x_k}{x_k-x_l}\right)+\frac{\beta^2}{6}n(n-1)(2n-1).$$

Note that the third summand is equal to $-\beta^2\dfrac{n(n-1)^2}{2}$. Hence we have

$$\phi_S\tilde{H}_S\phi_S^{-1}=\sum_{j=1}^n(x_j\partial_j)^2+2\beta\sum_{j<k}\frac{x_jx_k}{(x_j-x_k)^2}+\beta^2\sum_{j=1}^n\left(\sum_{\substack{k\\ k\neq j}}\frac{x_j}{x_j-x_k}\right)^2$$

$$-\beta^2\sum_{j<k}\frac{x_j+x_k}{x_j-x_k}\left(\sum_{\substack{l\\ l\neq j}}\frac{x_j}{x_j-x_l}-\sum_{\substack{l\\ l\neq k}}\frac{x_k}{x_k-x_l}\right)-\frac{\beta^2}{6}n(n-1)(n-2).$$

Compute the following difference:

$$\sum_{j=1}^n\left(\sum_{\substack{k\\ k\neq j}}\frac{x_j}{x_j-x_k}\right)^2-\sum_{j<k}\frac{x_j+x_k}{x_j-x_k}\left(\sum_{\substack{l\\ l\neq j}}\frac{x_j}{x_j-x_l}-\sum_{\substack{l\\ l\neq k}}\frac{x_k}{x_k-x_l}\right)$$

$$=\sum_{j=1}^n\sum_{\substack{k,m\\ k,m\neq j}}\frac{x_j^2}{(x_j-x_k)(x_j-x_m)}\left(-\sum_{j<k}\sum_{\substack{l\\ l\neq j}}\frac{x_j^2}{(x_j-x_k)(x_j-x_l)}+\sum_{j<k}\sum_{\substack{l\\ l\neq k}}\frac{x_k^2}{(x_j-x_k)(x_k-x_l)}\right)$$

$$-\left(\sum_{j<k}\sum_{\substack{l\\ l\neq j}}\frac{x_jx_k}{(x_j-x_k)(x_j-x_l)}-\sum_{j<k}\sum_{\substack{l\\ l\neq k}}\frac{x_jx_k}{(x_j-x_k)(x_k-x_l)}\right).$$

Substituting $j \leftrightarrow k$ in the third and the fifth sums, we obtain

$$\sum_{j=1}^{n} \sum_{\substack{k,m \\ k,m\neq j}} \frac{x_j^2}{(x_j - x_k)(x_j - x_m)} - \sum_{j<k} \sum_{\substack{l \\ l\neq j}} \frac{x_j^2}{(x_j - x_k)(x_j - x_l)} + \sum_{k<j} \sum_{\substack{l \\ l\neq j}} \frac{x_j^2}{(x_j - x_k)(x_j - x_l)}$$

$$- \left(\sum_{j<k} \sum_{\substack{l \\ l\neq j}} \frac{x_j x_k}{(x_j - x_k)(x_j - x_l)} + \sum_{k<j} \sum_{\substack{l \\ l\neq j}} \frac{x_j x_k}{(x_j - x_k)(x_j - x_l)} \right).$$

It is easy to see that the sum of the first, the second, and the third summands is equal to zero. Hence we have

$$- \left(\sum_{j=1}^{n} \sum_{\substack{k \\ k>j}} \sum_{\substack{l \\ l\neq j}} \frac{x_j x_k}{(x_j - x_k)(x_j - x_l)} + \sum_{j=1}^{n} \sum_{\substack{k \\ k<j}} \sum_{\substack{l \\ l\neq j}} \frac{x_j x_k}{(x_j - x_k)(x_j - x_l)} \right)$$

$$= - \left(\sum_{j=1}^{n} \sum_{\substack{k \\ k>j}} \sum_{\substack{l \\ l\neq j,k}} \frac{x_j x_k}{(x_j - x_k)(x_j - x_l)} + \sum_{j=1}^{n} \sum_{\substack{k \\ k<j}} \sum_{\substack{l \\ l\neq j,k}} \frac{x_j x_k}{(x_j - x_k)(x_j - x_l)} \right)$$

$$- \left(\sum_{j=1}^{n} \sum_{\substack{k \\ k>j}} \frac{x_j x_k}{(x_j - x_k)^2} + \sum_{j=1}^{n} \sum_{\substack{k \\ k<j}} \frac{x_j x_k}{(x_j - x_k)^2} \right)$$

$$= - \sum_{\substack{j,k,l \\ j\neq k\neq l}} \frac{x_j x_k}{(x_j - x_k)(x_j - x_l)} - 2 \sum_{j<k} \frac{x_j x_k}{(x_j - x_k)^2}.$$

Note that the summation in the first sum is done over all ordered triples (j, k, l) of pairwise distinct natural numbers $(j, k, l = 1, \dots, n)$. Grouping by six summands corresponding to all permutations of the triples (j, k, l), and using for any given triple (j, k, l) the identity

$$x_j x_k (x_j - x_k) + x_k x_l (x_k - x_l) + x_l x_j (x_l - x_j) = -(x_j - x_k)(x_k - x_l)(x_l - x_j),$$

we obtain $C_n^3 = \frac{n(n-1)(n-2)}{6}$ terms of the sum, each of them equal to -1. Substituting this expression into the last result for $\phi_S \tilde{H}_S \phi_S^{-1}$, we obtain

$$\phi_S \tilde{H}_S \phi_S^{-1} = \sum_{j=1}^{n} (x_j \partial_j)^2 + \beta \sum_{j<k} \frac{2x_j x_k}{(x_j - x_k)^2} - \beta^2 \sum_{j<k} \frac{2x_j x_k}{(x_j - x_k)^2}$$

$$-\beta^2 \frac{n(n-1)(n-2)}{6} + \beta^2 \frac{n(n-1)(n-2)}{6}$$

$$= \sum_{j=1}^{n} (x_j \partial_j)^2 + \beta \sum_{j<k} \frac{2x_j x_k}{(x_j - x_k)^2} - \beta^2 \sum_{j<k} \frac{2x_j x_k}{(x_j - x_k)^2}$$

$$= \sum_{j=1}^{n} (x_j \partial_j)^2 - \beta(\beta - 1) \sum_{j<k} \frac{2x_j x_k}{(x_j - x_k)^2}.$$

Assertion 5 is proved. □

Consider the Calogero operator in the field of a harmonic oscillator

$$H_C^h = H_C + \frac{1}{2} \sum_{i=1}^{n} x_i^2,$$

where

$$H_C = -\frac{1}{2}\sum_{i=1}^{n} \partial_i^2 + \sum_{i<j} \frac{\beta(\beta-1)}{(x_j - x_k)^2}.$$

Introduce the operators

$$\nabla_j^- = \frac{1}{\sqrt{2}}(-\nabla_j + x_j), \qquad \nabla_j^+ = \frac{1}{\sqrt{2}}(\nabla_j + x_j).$$

The commutator of ∇_j^- and ∇_j^+ has the following form:

$$[\nabla_j^-, \nabla_j^+] = \delta_{ij}\left(1 + \beta\sum_{k\neq i} s_{ik}\right) - (1-\delta_{ij})\beta s_{ij}.$$

Let

$$\nabla_j^\pm(c) = \phi_c \nabla_j^\pm \phi_c^{-1},$$

where

$$\phi(c) = \prod_{i<j} |x_i - x_j|^\beta.$$

It is easy to see that

$$\nabla_j^\pm(c) = \pm\left(\partial_j - \beta\sum_{\substack{k\\k\neq j}} \frac{s_{jk}}{x_j - x_k}\right) + x_j. = \pm\nabla_j(c) + x_j,$$

where

$$\nabla_j(c) = \partial_j - \beta\sum_{\substack{k\\k\neq j}} \frac{s_{jk}}{x_j - x_k}.$$

Now we shall show that for $\tilde{H}_C$ we have the following representation:

$$H_c^h = \sum_{j=1}^{n} \left(\nabla_j^+(c)\nabla_j^-(c) + \nabla_j^-(c)\nabla_j^+(c)\right).$$

Indeed,

$$\begin{aligned}
&\sum_{j=1}^{n}\Big[\big(\nabla_j(c) + x_j\big)\big(-\nabla_j(c) + x_j\big) - \big(-\nabla_j(c) + x_j\big)\big(\nabla_j(c) + x_j\big)\Big] \\
&= -\sum_{j=1}^{n}\nabla_j^2(c) + \frac{1}{2}\sum_{j=1}^{n}[\nabla_j(c), x_j] - \frac{1}{2}\sum_{j=1}^{n}[\nabla_j(c), x_j] + \sum_{j=1}^{n} x_j^2 \\
&= -2\sum_{j=1}^{n}\nabla_j^2(c) + 2\sum_{j=1}^{n} x_j^2.
\end{aligned}$$

Hence the equation $\sum_{j=1}^{n} \nabla_j^2 = H_C$ implies the desired assertion.

2. Fock Spaces of Calogero and Sutherland Models

Consider now the spaces of eigenstates of the Sutherland model and the Calogero model with harmonic oscillator. More exactly, we describe the eigenfunctions of the Hamiltonian

$$H_S = -\sum_{i=1}^{n} \frac{\partial^2}{\partial\theta_i^2} + \frac{1}{2} \sum_{\substack{i,j=1 \\ i\neq j}} \frac{\omega^2\beta(\beta-1)}{\sinh[\omega(\theta_i-\theta_j)/2]}.$$

We have defined the algebra $\mathfrak{A}_S = \mathbb{C}\,[\nabla_j, x_j, s_{ij},\ i,j=1,\dots,n]$, associated to the Sutherland model. Let ∇^-, ∇^+ be the operators

$$\nabla_j^- = \frac{1}{\sqrt{2}}(-\nabla_j + x_j), \qquad \nabla_j^+ = \frac{1}{\sqrt{2}}(\nabla_j + x_j).$$

Introduce the algebra $\mathfrak{A}_C = \mathbb{C}\,[\nabla_j^-, \nabla_j^-, s_{ij},\ i,j=1,\dots,n]$. The algebras $\mathfrak{A}_S$ and $\mathfrak{A}_C$ are called Heisenberg–Weyl algebras.

It is easy to show that the operators ∇^-, ∇^+, s_{ij} satisfy the same commutation relations as the operators ∇_j, x_j, s_{ij}, i.e.,

$$\begin{aligned}
&(1) \qquad [\nabla_i^\varepsilon, \nabla_i^\varepsilon] = 0, \quad \varepsilon = \pm, \quad i,j=1,\dots,n,\\
&(2) \qquad s_{ij}\nabla_j^\varepsilon = \nabla_i^\varepsilon s_{ij}; \quad s_{ij}\cdot\nabla_k^\varepsilon = \nabla_k^\varepsilon\cdot s_{ij}, \quad \varepsilon = \pm 1,\\
&(3) \qquad [\nabla_i^\varepsilon, x_j] = \frac{\varepsilon}{\sqrt{2}}[\delta_{ij}(1+\beta\sum_{k\neq j} s_{jk}) - (1-\delta_{ij})\beta s_{ij}],\\
&(4) \qquad [\nabla_i^-, \nabla_i^+] = [\nabla_i, x_j].
\end{aligned}$$

Let $\hat{\mathfrak{A}}_C$ be the algebra isomorphic to $\mathfrak{A}_C$ and generated by the operators conjugate to the operators ∇_j^ε, s_{ij} through the operator of multiplication by the function $\hat\phi_C = \prod_{i=1}^{n} \exp(-x_i^2/2)$. In other words, $\hat{\mathfrak{A}}_C$ is generated by the operators

$$\hat\nabla_j^\varepsilon = \hat\phi_C^{-1}\nabla_j^\varepsilon\hat\phi_C, \qquad \hat s_{ij} = \hat\phi_C^{-1} s_{ij}\hat\phi_C = s_{ij}.$$

The homomorphism $\rho : \mathfrak{A}_S \to \hat{\mathfrak{A}}_C$ is defined by the following rules:

$$\rho(\nabla_j) = \hat\nabla_j^+, \qquad \rho(x_j) = \hat\nabla_j^-, \qquad \rho(s_{ij}) = s_{ij}.$$

Consider now the images of operators $\tilde\nabla_j = x_j\nabla_j + \beta\sum_{k<j} s_{ik}$, $i=1,\dots,n$, under homomorphism ρ. The resulting operators will be denoted by $\hat h_j$,

$$\hat h_j = \rho(\tilde\nabla_j) = \nabla_j^-\nabla_j^+ + \beta\sum_{k<j} s_{ik}, \qquad i=1,\dots,n.$$

The operators $h_j, j=1,\dots,n$ commute pairwise. This correspondence and the commuting set of operators $\hat h_j$ was considered for the first time by A. Polichronakos [12].

In algebra $\hat{\mathfrak{A}}_C$ consider the operator

$$\hat H_C = \operatorname{Res}\left(\sum_{j=1}^{n} \hat h_j\right) - \frac{1}{2}n(n-1).$$

It has the form

$$\hat H_C = \frac{1}{2}\sum_{j=1}^{n}(\partial_j^2 + 2x_j\partial_j) - \beta\sum_{j<k}^{n}\frac{1}{x_j-x_k}(\partial_j-\partial_k).$$

Consider the operator

$$\hat{H}'_C = \phi_C \circ \hat{H}_C \circ \phi_C^{-1},$$

conjugate to the operator $\hat{H}_C$ through the operator of multiplication by the symmetric function

$$\phi_C = \prod_{j<k} |x_j - x_k|^\beta \prod_{j=1}^{n} \exp(-x_j^2/2).$$

The operator $\hat{H}'_C$ coincides with the Hamiltonian of the Calogero model in the field of the harmonic oscillator H_C,

$$H_C^h = -\frac{1}{2}\sum_{j=1}^{n} \partial_j^2 + \beta(\beta-1)\sum_{j<k}\frac{1}{(x_j-x_k)^2} + \sum_{j=1}^{n} x_j^2.$$

The expressions for $\tilde{H}_C$ and the coincidence $H'_C = H_C^h$ can be obtained as above (for $\tilde{H}_S$ and $H'_S = \phi_S \tilde{H}_S \phi_S^{-1} = H_S$).

Following Kakei [9, 10], define the Fock spaces for the Calogero operator with harmonic potential and the Sutherland operator. These spaces define linear representations of the algebras $\mathfrak{A}_S$ and $\mathfrak{A}_C$.

Definition. The Fock space $\mathfrak{F}_S$ of the Sutherland model is the linear space over $\mathbb{C}$ which is a one-dimensional module over the algebra $\mathfrak{A}_S$ with the generating "vacuum vector" v_S^0 identically equal to 1.

Note that we have

$$\nabla_j v_S^0 = 0, \qquad s_{ij} v_S^0 = v_S^0.$$

Similarly define the Fock space $\mathfrak{F}_C$ for Calogero model as a one-dimensional module over the algebra $\mathfrak{A}_C$ with generating "vacuum vector"

$$v_C^0 = \prod_{j=1}^{n} \exp(-x_j^2/2).$$

Evidently, we have

$$\nabla_j^+ v_C^0 = 0, \qquad s_{ij} v_C^0 = v_C^0.$$

Define a linear map

$$\rho^{\mathfrak{F}} : \mathfrak{F}_S \to \mathfrak{F}_C$$

as a map of one-dimensional modules satisfying the following rules:

$$\rho^{\mathfrak{F}}(v_S^0) = v_C^0,$$
$$\forall a \in \mathfrak{A}_S, \quad \rho^{\mathfrak{F}}(a v_S^0) = \rho(a)\rho^{\mathfrak{F}}(v_S^0).$$

It is easy to verify that for any elements $v \in \mathfrak{F}_S$ and $a \in \mathfrak{A}_S$ we have

$$\rho^{\mathfrak{F}}(av) = \rho(a)\rho^{\mathfrak{F}}(v).$$

Considering the actions of algebras $\mathfrak{A}_S$ and $\mathfrak{A}_C$ (and their conjugates by means of the function ϕ) we can, roughly speaking, consider that the Hamiltonian H_S transforms under the homomorphism ρ into the operator H_C^h, and the map $\rho^{\mathfrak{F}}$ converts the eigenfunctions of the first Hamiltonian to the eigenfunctions of the second one. The spaces $\mathfrak{F}_S$ and $\mathfrak{F}_C$ are spaces of polynomials in variables $x_1, \dots, x_n$, investigated in sufficiently broad classes of models. It is nesessary to note that the action of Hamiltonians H_S and H_C^h can be transferred to the spaces of polynomials $\mathfrak{F}_S$ and $\mathfrak{F}_C$ by conjugation through functions of their "ground states."

Define now nonsymmetric Jack polynomials $E_i(x) \in \mathbb{C}[x]$ by following two conditions:

(1) $E_i(x)$ are eigenfunctions of commuting operators $\tilde{\nabla}_j$, $j = 1, \dots, n$;
(2) the set of $E_i(x)$ constitute the basis in $\mathbb{C}[x]$.

More exactly, Jack polynomials are indexed by partitions $\lambda = (\lambda_1, \dots, \lambda_n)$, that is, by the sets of nonincreasing nonpositive numbers $(\lambda_1 \geq \dots \geq \lambda_n)$ and by the elements of the permutation group $w \in \Sigma_n$. On the pairs (λ, w) the lexicographical order is introduced [14]. We say that $(\mu, w') \prec (\lambda, w)$ if $\mu < \lambda$ with respect to the order of domination of partitions, and in case $\mu = \lambda$, if $w' <_B w$ with respect to the Bruhat order.

Definition ([1, 2, 17]). Nonsymmetric Jack polynomial $E_w^\lambda(x)$, where $\lambda = (\lambda_1, \dots, \lambda_n)$ is a partition and $w \in \Sigma_n$, is defined by following two properties:

(1) $E_w^\lambda(x) = x_w^\lambda + \sum\limits_{(\mu,w') \prec (\lambda,w)} u_{ww'}^{\lambda\mu} x_{w'}^\mu$, where $x_w^\lambda = x_{w(1)}^{\lambda_1}, \dots, x_w^{\lambda_n}(n)$;

(2) $E_w^\lambda(x)$ is the common eigenfunction of operators $\tilde{\nabla}_j$.

Consider now the symmetric Jack polynomials.

Definition. Symmetric Jack polynomials $J_\lambda(x)$ are defined by the two following conditions:

(1) $J_\lambda(x) = m_\lambda(x) + \sum\limits_{(\lambda,\mu)} u_{\lambda,\mu} m_\mu(x)$, where $m_\lambda(x)$ is the symmetrization of the monomial $x^\lambda = x^{\lambda_1}$, $\dots$, x^{λ_n};

(2) $J_\lambda(x)$ is the eigenfunction of the operator $\tilde{H}_S$.

The fact that the operator $\tilde{H}_S$ is conjugate to the operator H_S (through the principal state ϕ_S of H_S) has a consequence that the symmetric Jack polynomials describe the exited states of theoperator H_S. It was shown that the symmetrical Jack polynomials define the orthogonal eigenbasis in the space of states of the operator $\tilde{H}_S$ [1, 14] with respect to the scalar product

$$\langle f(x), g(x) \rangle_J = \int\limits_{T^n} f(x) g(x^{-1}) \phi_S(x) \phi_S(x^{-1}) \frac{dx_1 \dots x_n}{(2\pi i)^n \prod\limits_{i=1}^{n} x_i},$$

where $T^n = S^1 \times \dots \times S^1$ is the n-dimensional torus.

The isomorphism ρ of Fock spaces defined above transforms Jack polynomials to the eigenfunctions of Calogero operator which are called hidden Jack polynomials or Hermite polynomials. S. Kakei has shown that these polynomials generate orthogonal basis with respect to scalar product

$$\langle f(x), g(x) \rangle_H = \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} f(x) g(x) (\phi_C)^2 dx_1 \dots dx_n.$$

S. Kakei constructed the analogs of homomorphism ρ for Calogero and Sutherland models, associated to the root systems B_n, C_n, D_n [10], and defined the analogs of Laguerre polynomials which, as Hermite polynomials, are the images of Jack polynomials.

We shall not give the corresponding Kakei construction for classical root systems and consider further technical propositions for construction of a ρ-homomorphism which is suitable for arbitrary root system.

3. Universal Spaces, Quadratic Forms, and Laplasians Associated to the Root Systems

The aim of this section is to recall principal definitions and to investigate properties of differential operators defined by means of a nondegenerate positive symmetric bilinear form (see Bourbaki) associated to the root system.

Let R be a reduced and irreducible root system in a n-dimensional real vector space V. Let R_+ be the set of positive roots, $R_0 = \{\alpha_1, \dots, \alpha_m\}$ the set of simple roots, $R_0 \subset R_+$, $W(R)$ the Weyl group of R. For a given root system, a unique nondegenerate positive symmetric bilinear form $F_R(x, y)$ on V, invariant under $W(R)$, can be constructed. The form satisfies the condition

$$F_R(x,y) = \sum_{\alpha \in R} F_R(x,\alpha) F_R(\alpha, y).$$

From now on we will denote by the same characters $V, F, s_\alpha \in W(R)$, the complexifications $V \otimes \mathbb{C}$ and the natural extensions of F, s_α to $V \otimes \mathbb{C}$.

Let $e_1, \dots, e_n$ be an orthonormal basis in V with respect to F. Any vector $x \in V$ can be represented in the form

$$x = \sum_{i=1}^{n} F(x, e_i) e_i.$$

Then, by the bilinearity of F, we have

$$F(x,y) = \sum_{i=1}^{n} F(x, e_i) F(e_i, y).$$

Let $|R| = N$ be the number of roots in the root system R, $\mathbb{C}^N$ be the complex space associated with R. Let $\{u_\alpha, \alpha \in R\}$ be coordinates in $\mathbb{C}^N$, ordered by some order chosen on R. For example, $\alpha \succ \beta$ if $\alpha - \beta \in P^+$, where P^+ is the positive part of the root lattice. Define on $\mathbb{C}^N$ the quadratic form

$$Q(u) = \sum_{\alpha, \beta \in R} F(\alpha, \beta) u_\alpha u_\beta,$$

and its polar symmetric bilinear form

$$Q(u,v) = \sum_{\alpha, \beta \in R} F(\alpha, \beta) u_\alpha v_\beta.$$

The following lemma holds.

Lemma 1. *Let*

$$L_\gamma(u) = \sum_{\alpha \in R} F(\gamma, \beta) u_\alpha.$$

Then we have

$$Q(u) = \sum_{\gamma \in R} (L_\gamma(u))^2, \qquad Q(u,v) = \sum_{\gamma \in R} L_\gamma(u) L_\gamma(v).$$

Corollary. *The restriction of Q to $\mathbb{R}^N \subset \mathbb{C}^N$ is a nonnegative form.*

The proof is immediate.

Define the action of the Weyl group $W(R)$ on $\mathbb{C}^N$ by the rule

$$w u_\alpha = u_{w\alpha}, \qquad \alpha \in R, \quad w \in W(R).$$

Define an action of $W(R)$ on the space of complex-valued functions on $\mathbb{C}^N$,

$${}^w f(u) = f(wu) = f(u_{w\alpha}), \quad \alpha \in R.$$

We have

$$^{w}L_{\gamma}(u) = L_{w\gamma}(u).$$

Indeed,

$$\begin{aligned} ^{w}L_{\gamma}(u) &= \sum_{\alpha\in R} F(\gamma,\alpha)u_{w\alpha} = \sum_{\alpha'\in R} F(\gamma, w^{-1}\alpha')u_{a'} \\ &= \sum_{\alpha'\in R} F(w\gamma,\alpha')u_{\alpha'} = L_{w\gamma}(u). \end{aligned}$$

Introduce now the Laplacians and define some maps for rational, trigonometric, and elliptic cases. Denote $\partial_\alpha = \dfrac{\partial}{\partial u_\alpha}$. Consider the differential operator of the first order (momentum operator)

$$D_\gamma = \sum_{\alpha\in R} F(\gamma,\alpha)\partial_\alpha.$$

For the differential operator of the second order

$$\Delta_1 = \sum_{\alpha,\beta\in R} F(\alpha,\beta)\partial_\alpha\partial_\beta,$$

using Lemma 1 and commutativity of ∂_α and ∂_β, we obtain

$$\widehat{\Delta}_1 = \sum_{\gamma\in R} D_\gamma^2.$$

This operator, naturally, is called Laplacian for the rational case.

Similarly, for trigonometric case we have the momentum operator

$$\widetilde{D}_\gamma = \sum_{\alpha\in R} F(\gamma,\alpha)u_\alpha\partial_\alpha.$$

Using Lemma 1 and commutativity of ∂_α and ∂_β, we obtain for the differential operator of the second order

$$\widehat{\Delta}_2 = \sum_{\alpha,\beta\in R} F(\alpha,\beta)(u_\alpha\partial_\alpha)(u_\beta\partial_\beta) = \sum_{\gamma\in R} \widetilde{D}_\gamma^2.$$

This operator is called Laplacian for trigonometric case.

For the elliptic case we have the momentum operator

$$\widetilde{\widetilde{D}}_\gamma = \sum_{\alpha\in R} F(\gamma,\beta)u_\alpha u_{-\alpha}\partial_\alpha,$$

and the Laplacian

$$\widehat{\Delta}_3 = \sum_{\alpha,\beta\in R} F(\alpha,\beta)(u_\alpha u_{-\alpha}\partial_\alpha)(u_\beta u_{-\beta}\partial_\beta) = \sum_{\gamma\in R} \widetilde{\widetilde{D}}_\gamma^2.$$

The following assertion is an easy consequence of the $W(R)$-invariancy of $F(\alpha,\beta)$ and of the equation $w\circ\partial\gamma = \partial_{w\gamma}\circ w$.

Proposition 1. *The families of operators $\partial_\gamma, D_\gamma$ and $\tilde{D}_\gamma$ are equivariant with respect to the action of the Weyl group $W(R)$, i.e.,*

$$w\partial_\gamma = \partial_{w\gamma}w, \quad wD_\gamma = D_{w\gamma}w, \quad w\tilde{D}_\gamma = \tilde{D}_{w\gamma}w,$$

and the operators Δ_1 and Δ_2 are invariant with respect to $W(R)$,

$$w\circ\Delta_1 = \Delta_1\circ w, w\circ\Delta_2 = \Delta_2\circ w.$$

The proof of the first group of equations follows easily from the fact that $F(\alpha, \beta)$ is $W(R)$-invariant. The proof of the second group of equations is sufficient to carry out for the generators s_α, $\alpha \in R$, of $W(R)$. Indeed, if $s_\alpha \gamma = \delta$, then $s_\alpha \delta = \gamma$ (since $s_\alpha^2 = 1$), and we have $s_\alpha \circ \partial_\gamma = \partial_\delta \circ s_\alpha$.

$$
\begin{aligned}
w \circ \Delta_1 = w \circ \sum_{\alpha,\beta \in R} F(\alpha,\beta)\partial_\alpha \partial_\beta &= \sum_{\alpha,\beta \in R} F(\alpha,\beta) w \circ \partial_\alpha \partial_\beta \\
&= \sum_{\alpha,\beta \in R} F(\alpha,\beta)\partial_{w\alpha} \circ w \circ \partial_\beta = \sum_{\alpha,\beta \in R} F(\alpha,\beta)\partial_{w\alpha} \circ \partial_{w\beta} \circ w \\
&= \left(\sum_{\alpha',\beta' \in R} F(\alpha',\beta')\partial_{\alpha'}\partial_{\beta'} \right) \circ w = \Delta_1 \circ w.
\end{aligned}
$$

Let $h : M \to N$ be the map of smooth manifolds, and f_M and F_N be the spaces of functions on M and N respectively. Let, further, D be a differential operator on F_N.

Definition. The differential operator on F_M, which makes the following diagram commutative:

$$
\begin{array}{ccc}
F_M & \xleftarrow{h^*} & F_N \\
{\scriptstyle h^*D}\downarrow & & \downarrow{\scriptstyle D} \\
F_M & \xleftarrow{h^*} & F_N,
\end{array}
$$

is called the inverse image h^*D of the differential operator D.

Define the following maps from $V = V \otimes \mathbb{C}$ to $\mathbb{C}^N$:

$$
\begin{aligned}
U : V \to \mathbb{C}^N, &\quad x \to U(x) = \{u_\alpha(x) = F(\alpha, x), \alpha \in R\}, \\
E : V \to \mathbb{C}^N, &\quad x \to E(x) = \{u_\alpha(x) = \exp F(\alpha, x), \alpha \in R\}.
\end{aligned}
$$

We prove the following proposition.

Proposition 2. *The Laplace operator on V $\Delta = \sum_{i=1}^{n} \partial_i^2$ is the inverse image of the operators Δ_1 and Δ_2 under the maps U and E, respectively.*

Proof. We have

$$
\begin{aligned}
\partial_i f(U(x)) &= \sum_{\alpha \in R} \partial_\alpha f \frac{\partial U_\alpha(x)}{\partial x_i} = \sum_{\alpha \in R} F(\alpha, e_i)\partial_\alpha f, \\
\partial_i^2 f(U(x)) &= \partial_i \left(\sum_{\alpha \in R} F(\alpha, e_i)\partial_\alpha f \right) = \sum_{\alpha \in R} F(\alpha, e_i) \sum_{\beta \in R} F(\beta, e_i)\partial_\alpha \partial_\beta f \\
&= \sum_{\alpha,\beta \in R} F(\alpha, e_i)F(\beta, e_i)\partial_\alpha \partial_\beta f(U(x)),
\end{aligned}
$$

$$
\sum_{i=1}^{n} \partial_i^2 f(U(x)) = \sum_{\alpha,\beta \in R} \sum_{i=1}^{n} F(\alpha, e_i)F(\beta, e_i)\partial_\alpha \partial_\beta f(U(x)) = \sum_{\alpha,\beta \in R} F(\alpha,\beta)\partial_\alpha \partial_\beta f(U(x)) = \Delta_1 f(U(x)).
$$

The equation $\Delta = E^*\Delta_2$ can be proved similarly.

Hint: use the equations

$$
\frac{\partial U_\alpha(x)}{\partial x_i} = \frac{\partial \exp F(\alpha, x)}{\partial x_i} = F(\alpha, e_i) \exp F(\alpha, x) = F(\alpha, e_i) u_\alpha(x).
$$

4. Universal Dunkl Operators and Hamiltonians. The Bethe–Dunkl Varieties

Let $\alpha \to k_\alpha, \alpha \in R$, be the $W(R)$-invariant function on R and A_γ and B_γ, $\gamma \in R$, the operators of the form

$$A_\gamma = \sum_{\alpha \in R_+} \frac{F(\gamma,\alpha)k_\alpha}{u_\alpha - u_{-\alpha}} s_\alpha, \qquad B_\gamma = \sum_{\alpha \in R_+} F(\gamma,\alpha)k_\alpha \frac{u_\alpha + u_{-\alpha}}{u_\alpha - u_{-\alpha}} s_\alpha.$$

These operators generate equivariant families of operators, that is, for A_γ we have

$$w \circ A_\gamma = A_{w\gamma} \circ w.$$

Indeed,

$$wA_\gamma = w \circ \sum_{\alpha \in R_+} \frac{F(\gamma,\alpha)k_\alpha}{u_\alpha - u_{-\alpha}} s_\alpha = \sum_{\alpha \in R_+} \left(\frac{F(\gamma,\alpha)k_\alpha}{u_{w\alpha} - u_{-w\alpha}} s_{w\alpha} \right) w$$

$$= \sum_{\alpha' \in R_+} \left(\frac{F(\gamma, w^{-1}\alpha')k_{w^{-1}\alpha'}}{u_{\alpha'} - u_{-\alpha'}} s_{\alpha'} \right) w = \sum_{\alpha' \in R_+} \left(\frac{F(w\gamma,\alpha')k_{\alpha'}}{u_{\alpha'} - u_{-\alpha'}} s_{\alpha'} \right) w = A_{w\gamma} w.$$

The case of B_γ is considered similarly.

Introduce the "universal" Dunkl operators

$$\nabla_\gamma = -D_\gamma + A_\gamma, \quad \tilde{\nabla}_\gamma = -\tilde{D}_\gamma + B_\gamma, \quad \nabla^\pm_\gamma = \nabla_\gamma \pm L_\gamma(u)$$

for $\gamma \in R$. Each of the introduced families of operators is equivariant.

Proposition 3. *The following commutation relations hold:*

(1) $$[\nabla_\gamma, \nabla_\delta] = \sum_{w \in W(R)} \left\{ \sum_{\substack{\alpha,\beta \in R_+ \\ s_\alpha s_\beta = w}} \frac{k_\alpha k_\beta \big(F(\gamma,\alpha)F(\delta,\beta) - F(\gamma,\beta)F(\delta,\alpha)\big)}{(u_\alpha - u_{-\alpha})(u_\beta - u_{-\beta})} \right\} w,$$

(2) $$[\tilde{\nabla}_\gamma, \tilde{\nabla}_\delta] = \sum_{w \in W(R)} \left\{ \sum_{\substack{\alpha,\beta \in R_+ \\ s_\alpha s_\beta = w}} k_\alpha k_\beta \big(F(\gamma,\alpha)F(\delta,\beta) - F(\gamma,\beta)F(\delta,\alpha)\big) \frac{u_\alpha + u_{-\alpha}}{u_\alpha - u_{-\alpha}} \cdot \frac{u_\beta + u_{-\beta}}{u_\beta - u_{-\beta}} \right\} w,$$

(3) $$[D_\gamma, L_\delta] = F(\gamma,\delta),$$

(4) $$[\nabla_\gamma, L_\delta] = -F(\gamma,\delta) - 2\sum_{\alpha \in R_+} \frac{F(\gamma,\alpha)F(\delta,\alpha)k_\alpha L_\alpha(u) s_\alpha}{F(\alpha,\alpha)(u_\alpha - u_{-\alpha})},$$

(5) $$[L_\gamma \nabla_\gamma, L_\delta \nabla_\delta] = -K(\gamma,\delta)\Big(L_\gamma \nabla_\delta - L_\delta \nabla_\gamma\Big)$$
$$- 4 \sum_{\alpha \in R_+} \frac{F(\gamma,\alpha)F(\delta,\alpha)L^2_\alpha(u)s_\alpha}{F^2(\alpha,\alpha)(u_\alpha - u_{-\alpha})}\Big(F(\gamma,\alpha)\nabla_\delta - F(\delta,\alpha)\nabla_\gamma\Big),$$

where

$$K(\gamma,\delta) = F(\gamma,\delta) + 2\sum_{\alpha \in R_+} \frac{F(\gamma,\alpha)F(\delta,\alpha)k_\alpha L_\alpha(u)s_\alpha}{F(\alpha,\alpha)(u_\alpha - u_{-\alpha})}.$$

In the equations above, the terms related to $w \in W(R)$ that cannot be represented in the form of the product $s_\alpha s_\beta$ are supposed to be zero.

Proof. For the first equation we have

$$[\nabla_\gamma, \nabla_\delta] = [-D_\gamma + A_\gamma, -D_\delta + A_\delta] = [D_\gamma, D_\delta] - ([D_\gamma, A_\delta] - [D_\delta, A_\gamma]) + [A_\gamma, A_\delta].$$

Evidently the first bracket is equal to zero since the operators D_δ and D_γ are sums of commuting differential operators ∂_α and, hence, commute.

Further we have

$$[D_\gamma, A_\delta] = D_\gamma \circ A_\delta - A_\delta \circ D_\gamma = D_\gamma(A_\delta) + \sum_{\alpha\in R_+} \frac{k_\alpha F(\delta,\alpha) s_\alpha}{u_\alpha - u_{-\alpha})} D_{s_\alpha \gamma} - A_\delta D_\gamma.$$

Using the equation

$$s_\alpha \gamma = \gamma - 2\frac{F(\gamma,\alpha)}{F(\alpha,\alpha)}\alpha,$$

we obtain

$$\begin{aligned}
[D_\gamma, A_\delta] &= D_\gamma(A_\delta) + \sum_{\alpha\in R_+}\left(\frac{k_\alpha F(\delta,\alpha)s_\alpha}{u_\alpha - u_{-\alpha}}\right) D_\gamma - 2\sum_{\alpha\in R_+}\frac{k_\alpha F(\gamma,\alpha)F(\delta,\alpha)}{F(\alpha,\alpha)(u_\alpha - u_{-\alpha})} D_\alpha - A_\delta D_\gamma \\
&= D_\gamma(A_\delta) + A_\delta D_\gamma - 2\sum_{\alpha\in R_+}\frac{F(\gamma,\alpha)F(\delta,\alpha)k_\alpha D_\alpha}{F(\alpha,\alpha)(u_\alpha - u_{-\alpha})} - A_\delta D_\gamma \\
&= -2\sum_{\alpha\in R_+}\frac{k_\alpha F(\gamma,\alpha)F(\delta,\alpha)s_\alpha}{(u_\alpha - u_{-\alpha})^2} - 2\sum_{\alpha\in R_+}\frac{F(\gamma,\alpha)F(\delta,\alpha)k_\alpha D_\alpha}{F(\alpha,\alpha)(u_\alpha - u_{-\alpha})}.
\end{aligned}$$

It is easy to see that this expression is symmetric with respect to γ and δ. Hence, the commutator $[D_\delta, A_\gamma]$ is equal to the same expression, and we obtain

$$[\nabla_\gamma, \nabla_\delta] = [A_\gamma, A_\delta].$$

Computation of the last commutator gives

$$[A_\gamma, A_\delta] = \sum_{\alpha,\beta\in R_+}\frac{k_\alpha k_\beta F(\gamma,\alpha)F(\delta,\beta)}{(u_\alpha - u_{-\alpha})(u_{s_\alpha\beta} - u_{-s_\alpha\beta})} s_\alpha s_\beta.$$

Let $\beta' = \varepsilon_\alpha(s_\beta)s_\alpha\beta = \pm s_\alpha\beta, \quad \beta'\in R_+$. We obtain, further,

$$\begin{aligned}
[A_\gamma, A_\delta] &= \sum_{\alpha,\beta\in R_+}\frac{k_\alpha k_{\beta'} F(\gamma,\alpha)F(s_\alpha\delta,\beta)}{(u_\alpha - u_{-\alpha})(u_{s_\alpha\beta} - u_{-s_\alpha\beta})} s_\alpha s_\beta \\
&\quad - \sum_{\alpha,\beta'\in R_+}\frac{k_\alpha k_{\beta'} F(s_\alpha\gamma,\beta)F(\delta,\alpha)}{(u_\alpha - u_{-\alpha})(u_{\beta'} - u_{-\beta'})} s_\alpha s_{\beta'} \\
&= \sum_{w\in W(R)}\sum_{\substack{\alpha,\beta\in R_+\\ s_\alpha s_\beta = w}}\frac{k_\alpha k_\beta\big(F(\gamma,\alpha)F(\delta,\beta) - F(\gamma,\beta)F(\delta,\alpha)\big)}{(u_\alpha - u_{-\alpha})(u_\beta - u_{-\beta})} w.
\end{aligned}$$

The first equation is proved.

The other equations are proved similarly. □

Now we will prove the analogs of Proposition 2 for trigonometric case. We have the following operators:

$$\tilde{D}_\gamma = \sum_{\alpha\in R} F(\gamma,\alpha)u_\alpha\partial_\alpha = \sum_{\alpha\in R_+} F(\gamma,\alpha)(u_\alpha + u_{-\alpha})\partial_\alpha,$$

$$B_\gamma = \sum_{\alpha\in R_+} k_\alpha F(\gamma,\alpha)\frac{(u_\alpha + u_{-\alpha})}{(u_\alpha - u_{-\alpha})} s_\alpha,$$

$$\tilde{\nabla}_\gamma = -\tilde{D}_\gamma + B_\gamma.$$

Proposition 4. *The following commutation relations hold:*

$$[\tilde{\nabla}_\gamma, \tilde{\nabla}_\delta] = \sum_{\substack{\alpha\in R_+\\ \alpha\neq\beta}} G_{\gamma,\delta}(\alpha,\beta)\frac{(u_\alpha+u_{-\alpha})}{(u_\beta-u_{-\beta})}\frac{(u_\beta+u_{-\beta})}{(u_\beta-u_{-\beta})}s_\alpha s_\beta,$$

where

$$G_{\gamma,\delta}(\alpha,\beta) = k_\alpha k_\beta(F(\gamma,\alpha)F(\delta,\beta) - F(\gamma,\beta)F(\delta,\alpha)).$$

Proof. In the equation

$$[\tilde{\nabla}_\gamma, \tilde{\nabla}_\delta] = [\tilde{D}_\gamma, \tilde{D}_\delta] - ([\tilde{D}_\gamma, B_\delta] - [\tilde{D}_\delta, B_\gamma]) + [B_\gamma, B_\delta]$$

we have $[\tilde{D}_\gamma, \tilde{D}_\delta] = 0$.

Find the commutator $[\tilde{D}_\gamma, B_\delta]$. We obtain

$$[\tilde{D}_\gamma, B_\delta] = \tilde{D}_\gamma \circ B_\delta - B_\delta\tilde{D}_\gamma = \tilde{D}_\gamma(B_\delta) + \sum_{\alpha\in R_+} F(\delta,\alpha)k_\alpha\frac{(u_\alpha+u_{-\alpha})}{(u_\alpha-u_{-\alpha})}s_\alpha D_{s_\alpha\gamma}$$

$$-B_\delta\tilde{D}_\gamma = \sum_{\alpha\in R_+} F(\gamma,\alpha)F(\delta,\alpha)k_\alpha\frac{(-4u_\alpha u_{-\alpha})}{(u_\alpha-u_{-\alpha})}s_\alpha + \sum_{\alpha\in R_+} -2\frac{F(\gamma,\alpha)F(\delta,\alpha)}{F(\alpha,\alpha)}\frac{(-4u_\alpha u_{-\alpha})}{(u_\alpha-u_{-\alpha})}s_\alpha F_\alpha.$$

We see that the indices γ and δ are contained in the commutator $[\tilde{D}_\gamma, B_\delta]$ in a symmetric way. This gives

$$([\tilde{D}_\gamma, B_\delta] - [\tilde{D}_\delta, B_\gamma]) = 0.$$

Hence,

$$[\tilde{\nabla}_\gamma, \tilde{\nabla}_\delta] = [B_\gamma, B_\delta].$$

Compute now the commutator $[B_\gamma, B_\delta]$.

Let $c(u_\alpha) = \frac{(u_\alpha+u_{-\alpha})}{(u_\alpha-u_{-\alpha})}$. We obtain

$$[B_\gamma, B_\delta] = \sum_{\alpha,\beta\in R_+} k_\alpha k_\beta F(\gamma,\alpha)F(\delta,\beta)[c(u_\alpha)s_\alpha, c(u_\beta)s_\beta].$$

Further we have

$$\begin{aligned}[c(u_\alpha)s_\alpha, c(u_\beta))s_\beta] &= c(u_\alpha))s_\alpha c(u_\beta))s_\beta - c(u_\beta))s_\beta c(u_\alpha))s_\alpha\\ &= c(u_\alpha)c(u_{s_\alpha\beta})s_\alpha s_\beta - c(u_\beta)c(u_{s_\beta\alpha})s_\beta s_\alpha;\\ [B_\gamma, B_\delta] &= \sum_{\alpha,\beta\in R_+} k_\alpha k_\beta F(\gamma,\alpha)F(\delta,\beta)c(u_\alpha)c(u_{s_\alpha\beta})s_\alpha s_\beta\\ &\quad - \sum_{\alpha,\beta\in R_+} k_\alpha k_\beta F(\gamma,\alpha)F(\delta,\beta)c(u_\beta)c(u_{s_\beta\alpha})s_\beta s_\alpha.\end{aligned}$$

Making the change $\beta' = \pm s_\alpha\beta$ and $\alpha' = \pm s_\beta\alpha$, supposing that k_α are W-invariant, and using the properties of the form $F(\alpha,\beta)$ with respect to the reflections s_α, we obtain

$$\begin{aligned}[B_\gamma, B_\delta] &= \sum_{\alpha,\beta'\in R_+} k_\alpha k_{\beta'} F(\gamma,\alpha)F(s_\alpha\delta,\beta)c(u_\alpha)c(u_{\beta'})s_{\beta'}s_\alpha\\ &\quad - \sum_{\alpha',\beta\in R_+} k_{\alpha'} k_\beta F(s_\beta\gamma,\alpha')F(\delta,\beta)c(u_{\alpha'})c(u_\beta)s_{\alpha'}s_\beta\\ &= \sum_{\alpha,\beta\in R_+} k_\alpha k_\beta F(\gamma,\alpha)F(\delta,\beta)c(u_\alpha)c(u_\beta)s_\beta s_\alpha\end{aligned}$$

$$
\begin{aligned}
&- \sum_{\alpha,\beta\in R_+} k_\alpha k_\beta F(\gamma,\alpha)F(\delta,\beta)c(u_\alpha)c(u_\beta)s_\alpha s_\beta \\
&- 2\sum_{\alpha,\beta\in R_+} k_\alpha k_\beta \frac{F(\gamma,\alpha)F(\alpha,\beta)F(\delta,\alpha)}{F(\alpha,\alpha)} c(u_\alpha)c(u_\beta)s_\beta s_\alpha \\
&+ 2\sum_{\alpha,\beta\in R_+} k_\alpha k_\beta \frac{F(\gamma,\beta)F(\beta,\alpha)F(\delta,\beta)}{F(\beta,\beta)} c(u_\alpha)c(u_\beta)s_\alpha s_\beta .
\end{aligned}
$$

Substituting the indices $\alpha \leftrightarrow \beta$, we se that the third and the fourth summands cancel out, and the first two give the following expression for the commutator $[B_\gamma, B_\delta]$:

$$
\begin{aligned}
[B_\gamma, B_\delta] &= \sum_{\alpha,\beta\in R_+} k_\alpha k_\beta \Big(F(\gamma,\alpha)F(\delta,\beta) - F(\gamma,\beta)F(\delta,\alpha)\Big) c(u_\alpha)c(u_\beta)s_\alpha s_\beta \\
&= \sum_{w\in W(R)} \sum_{\substack{\alpha,\beta\in R_+\\ s_\alpha s_\beta = w}} G_{\gamma,\delta}(\alpha,\beta) \frac{u_\alpha + u_{-\alpha}}{u_\alpha - u_{-\alpha}} \frac{u_\beta + u_{-\beta}}{u_\beta - u_{-\beta}} w.
\end{aligned}
$$

Introduce the "universal" Hamiltonians of Calogero–Sutherland type

$$
\begin{aligned}
H_C &= -\Delta_1 + \sum_{\alpha\in R_+} \frac{F(\alpha,\alpha)({k_\alpha}^2 - 2k_\alpha s_\alpha)}{(u_\alpha - u_{-\alpha})^2}, \\
H_S &= -\Delta_2 + \sum_{\alpha\in R_+} \frac{F(\alpha,\alpha)4u_\alpha u_{-\alpha}({k_\alpha}^2 - k_\alpha s_\alpha)}{(u_\alpha - u_{-\alpha})^2}, \\
H_C^h &= H_C + Q(u).
\end{aligned}
$$

It is easy to verify that these Hamiltonians are W-invariant, i.e.,

$$
wH_C = H_C w, \quad wH_C^h = H_C^h w \quad \forall w \in W(R).
$$

□

Proposition 5. *The following representations hold:*

$$
\sum_{\gamma\in R} \nabla_\gamma^2 = -H_C - \sum_{w\in W(R)} \left\{ \sum_{\substack{\alpha,\beta\in R_+\\ s_\alpha s_\beta = w}} \frac{k_\alpha k_\beta F(\alpha,\beta)}{(u_\alpha - u_{-\alpha})(u_\beta - u_{-\beta})} \right\} w,
$$

$$
\sum_{\gamma\in R} \tilde{\nabla}_\gamma^2 = -H_S - \sum_{w\in W(R)} \left\{ \sum_{\substack{\alpha,\beta\in R_+\\ \alpha\neq\beta\\ s_\alpha s_\beta = w}} k_\alpha k_\beta F(\alpha,\beta) \frac{u_\alpha + u_{-\alpha}}{u_\alpha - u_{-\alpha}} \cdot \frac{u_\beta + u_{-\beta}}{u_\beta - u_{-\beta}} \right\} w - \sum_{\alpha\in R_+} k_\alpha^2 F(\alpha,\alpha),
$$

$$
\sum_{\gamma\in R} \nabla_\gamma^- \nabla_\gamma^+ = -H_C^h - \left\{ \sum_{\gamma\in R} F(\gamma,\gamma) + 2\sum_{\alpha\in R_+} \frac{k_\alpha L_\alpha(u) s_\alpha}{u_\alpha - u_{-\alpha}} \right\}.
$$

Proof. Prove of the first of these equations. We have

$$
\begin{aligned}
\sum_{\gamma\in R} \nabla_\gamma^2 &= \sum_{\gamma\in R} (-D_\gamma + A_\gamma)^2 = \sum_{\gamma\in R} ({D_\gamma}^2 - (D_\gamma \circ A_\gamma + A_\gamma D_\gamma) + {A_\gamma}^2) \\
&= \sum_{\gamma\in R} D_\gamma^2 - \sum_{\gamma\in R} (D_\gamma \circ A_\gamma + A_\gamma D_\gamma) + \sum_{\gamma\in R} A_\gamma^2.
\end{aligned}
$$

Earlier it was shown that $\Delta_1 = \sum_{\gamma\in R} D_\gamma^2$. Furter we compute $\sum_{\gamma\in R} D_\gamma \circ A_\gamma$. We have

$$\begin{aligned}\sum_{\gamma\in R} D_\gamma \circ A_\gamma &= \sum_{\gamma\in R}\sum_{\alpha\in R_+} D_\gamma(A_\gamma) + \sum_{\gamma\in R} \frac{k_\alpha F(\gamma,\alpha) s_\alpha D_{s_\alpha\gamma}}{u_\alpha - u_{-\alpha}} \\ &= -\sum_{\gamma\in R}\sum_{\alpha\in R_+} \frac{2k_\alpha F(\gamma,\alpha)F(\gamma,\alpha)s_\alpha}{(u_\alpha - u_{-\alpha})^2} + \sum_{\alpha\in R_+} \frac{k_\alpha s_\alpha}{u_\alpha - u_{-\alpha}} \sum_{\gamma\in R} F(\gamma,\alpha) D_{s_\alpha\gamma} \\ &= -\sum_{\alpha\in R_+} 2\frac{F(\alpha,\alpha)k_\alpha s_\alpha}{(u_\alpha - u_{-\alpha})^2} + \sum_{\alpha\in R_+} \frac{k_\alpha s_\alpha D_{s_\alpha\alpha}}{u_\alpha - u_{-\alpha}} \\ &= -\sum_{\alpha\in R_+} 2\frac{F(\alpha,\alpha)k_\alpha s_\alpha}{(u_\alpha - u_{-\alpha})^2} - \sum_{\alpha\in R_+} \frac{k_\alpha s_\alpha D_\alpha}{u_\alpha - u_{-\alpha}}.\end{aligned}$$

Further,

$$\begin{aligned}\sum_{\gamma\in R} A_\gamma D_\gamma &= \sum_{\gamma\in R}\sum_{\alpha\in R_+} \frac{F(\gamma,\alpha)k_\alpha s_\alpha}{u_\alpha - u_{-\alpha}} D_\gamma \\ &= \sum_{\alpha\in R_+} \frac{k_\alpha s_\alpha}{u_\alpha - u_{-\alpha}} \sum_{\gamma\in R} F(\alpha,\gamma) D_\gamma = \sum_{\alpha\in R_+} \frac{k_\alpha s_\alpha}{u_\alpha - u_{-\alpha}} D_\alpha.\end{aligned}$$

We see that

$$\sum_{\gamma\in R} D_\gamma \circ A_\gamma + \sum_{\gamma\in R} A_\gamma D_\gamma = -\sum_{\alpha\in R_+} 2\frac{F(\alpha,\alpha)k_\alpha s_\alpha}{(u_\alpha - u_{-\alpha})^2}.$$

Compute now $\sum_{\gamma\in R} A_\gamma^2$. We have

$$\begin{aligned}\sum_{\gamma\in R} A_\gamma^2 &= \sum_{\gamma\in R}\sum_{\alpha\in R_+} \frac{k_\alpha F(\gamma,\alpha)s_\alpha}{u_\alpha - u_{-\alpha}} \sum_{\beta\in R_+} \frac{k_\beta F(\gamma,\beta)s_\beta}{u_\beta - u_{-\beta}} \\ &= \sum_{\alpha\in R_+} \frac{k_\alpha^2 s_\alpha^2}{(u_\alpha - u_{-\alpha})(u_{-\alpha} - u_\alpha)} \sum_{\gamma\in R} F(\gamma,\alpha)F(\gamma,\alpha) \\ &\quad + \sum_{\substack{\alpha,\beta\in R_+\\ \alpha\neq\beta}} \frac{k_\alpha k_\beta s_\alpha s_\beta}{(u_\alpha - u_{-\alpha})(u_{s_\alpha\beta} - u_{-s_\alpha\beta})} \sum_{\gamma\in R} F(\alpha,\gamma)F(\beta,\delta) \\ &= -\sum_{\alpha\in R_+} \frac{k_\alpha^2 F(\alpha,\alpha)}{(u_\alpha - u_{-\alpha})^2} + \sum_{\substack{\alpha,\beta\in R_+\\ \alpha\neq\beta'}} \frac{k_\alpha k'_\beta F(\alpha, s_\alpha\beta')}{(u_\alpha - u_{-\alpha})(u'_\beta - u_{-\beta'})} s'_\beta s_\alpha \\ &= -\sum_{\alpha\in R_+} \frac{k_\alpha^2 F(\alpha,\alpha)}{(u_\alpha - u_{-\alpha})^2} - \sum_{\substack{\alpha,\beta\in R_+\\ \alpha\neq\beta}} \frac{k_\alpha k_\beta F(\alpha,\beta)}{(u_\alpha - u_{-\alpha})(u_\beta - u_{-\beta})} s_\alpha s_\beta \\ &= -\sum_{\alpha\in R_+} \frac{k_\alpha^2 F(\alpha,\alpha)}{(u_\alpha - u_{-\alpha})^2} - \sum_{w\in W} \left\{ \sum_{\alpha,\beta\in R_+} \frac{k_\alpha k_\beta F(\alpha,\beta)}{(u_\alpha - u_{-\alpha})(u_\beta - u_{-\beta})} \right\} w.\end{aligned}$$

The proof of the first assertion of Proposition 3 is finished.

Now we prove the equation for sum of squares of trigonometric Dunkl operators $\sum_{\gamma\in R} \tilde\nabla_\gamma^2$. We have

$$\sum_{\gamma\in R} \tilde\nabla_\gamma^2 = \sum_{\gamma\in R} \tilde D_\gamma^2 - \sum_{\gamma\in R} (\tilde D_\gamma B_\gamma + B_\gamma \tilde D_\gamma) + \sum_{\gamma\in R} {B_\gamma}^2.$$

It was shown above that $\Delta_2 = \sum_{\gamma\in R} \tilde{D}_\gamma^2$. Let, as above, $c(u_\alpha) = \frac{(u_\alpha + u_{-\alpha})}{(u_\alpha - u_{-\alpha})}$. We obtain

$$\tilde{D}_\gamma \circ B_\gamma = \tilde{D}_\gamma(B_\gamma) + \sum_{\alpha\in R_+} k_\alpha F(\gamma,\alpha)c(u_\alpha)s_\alpha \tilde{D}_{s_\alpha\gamma}$$

$$= \sum_{\alpha\in R_+} k_\alpha F(\gamma,\alpha)F(\gamma,\alpha)\frac{(-4u_\alpha u_{-\alpha})}{(u_\alpha - u_{-\alpha})^2}s_\alpha + \sum_{\alpha\in R_+} k_\alpha c(u_\alpha)F(\alpha,\gamma)\tilde{D}_{s_\alpha\gamma}.$$

Make the change $\gamma' = s_\alpha\gamma$. We obtain

$$\tilde{D}_\gamma B_\gamma = -\sum_{\alpha\in R_+} k_\alpha F(\alpha,\alpha)\frac{(-4u_\alpha u_{-\alpha})}{(u_\alpha - u_{-\alpha})^2}s_\alpha + \sum_{\alpha\in R_+} k_\alpha c(u_\alpha)\sum_{\gamma\in R} F(s_\alpha\alpha,\gamma')\tilde{D}'_\gamma$$

$$= -\sum_{\alpha\in R_+} k_\alpha F(\alpha,\alpha)\frac{(-4u_\alpha u_{-\alpha})}{(u_\alpha - u_{-\alpha})^2}s_\alpha + \sum_{\alpha\in R_+} k_\alpha c(u_\alpha)\tilde{D}_{s_\alpha\alpha}$$

$$= -\sum_{\alpha\in R_+} k_\alpha F(\alpha,\alpha)\frac{(-4u_\alpha u_{-\alpha})}{(u_\alpha - u_{-\alpha})^2}s_\alpha - \sum_{\alpha\in R_+} k_\alpha c(u_\alpha)\tilde{D}_\alpha.$$

Since

$$\sum_{\gamma\in R} B_\gamma\tilde{D}_\gamma = \sum_{\alpha\in R_+} k_\alpha c(u_\alpha)s_\alpha \sum_{\alpha\in R_+} F(\alpha,\gamma)\tilde{D}_\gamma = \sum_{\alpha\in R_+} k_\alpha c(u_\alpha)\tilde{D}_\alpha$$

we have

$$\sum_{\gamma\in R}\tilde{D}_\gamma B_\gamma + \sum_{\gamma\in R} B_\gamma\tilde{D}_\gamma = -\sum_{\alpha\in R_+} k_\alpha F(\alpha,\alpha)\frac{(-4u_\alpha u_{-\alpha})}{(u_\alpha - u_{-\alpha})^2}s_\alpha.$$

Compute now $\sum_{\gamma\in R} B_\gamma^2$. We have

$$\sum_{\gamma\in R} B_\gamma^2 = -\sum_{\alpha\in R_+} {k_\alpha}^2 c(u_\alpha)^2 \sum_{\gamma\in R} F(\alpha,\gamma)F(\gamma,\alpha)$$

$$+ \sum_{\substack{\alpha,\beta\in R_+\\ \alpha\neq\beta}} k_\alpha k_\beta \sum_{\gamma\in R} F(\alpha,\gamma)F(\gamma,\beta)c(u_\alpha)s_\alpha c(u_\beta)s_\beta$$

$$= -\sum_{\alpha\in R_+} {k_\alpha}^2 F(\alpha,\alpha)c(u_\alpha)^2 + \sum_{\substack{\alpha,\beta\in R_+\\ \alpha\neq\beta}} k_\alpha k_\beta F(\alpha,\beta)c(u_\alpha)c(u_{s_\alpha\beta})s_\alpha s_\beta.$$

Using the equality $c(u_\alpha)^2 - 1 = \frac{(-4u_\alpha u_{-\alpha})}{(u_\alpha - u_{-\alpha})^2}$, we obtain

$$\sum_{\gamma\in R} B_\gamma^2 = -\sum_{\alpha\in R_+} {k_\alpha}^2 F(\alpha,\alpha)\frac{(-4u_\alpha u_{-\alpha})}{(u_\alpha - u_{-\alpha})^2} - \sum_{\alpha\in R_+} {k_\alpha}^2 F(\alpha,\alpha)$$

$$- \sum_{\substack{\alpha,\beta'\in R_+\\ \alpha\neq\beta'}} k_\alpha k_{\beta'} F(\alpha,\beta')c(u_\alpha)c(u_{\beta'})s_{\beta'}s_\alpha.$$

Substituting $\beta = \beta'$ and $\alpha \leftrightarrow \beta$, we have

$$\sum_{\gamma\in R} B_\gamma^2 = -\sum_{\alpha\in R_+} {k_\alpha}^2 F(\alpha,\alpha)\frac{(-4u_\alpha u_{-\alpha})}{(u_\alpha - u_{-\alpha})^2} - \sum_{\alpha\in R_+} {k_\alpha}^2 F(\alpha,\alpha) - \sum_{\substack{\alpha,\beta\in R_+\\ \alpha\neq\beta}} k_\alpha k_\beta F(\alpha,\beta)c(u_\alpha)c(u_\beta)s_\alpha s_\beta.$$

Hence,

$$\sum_{\gamma\in R}\tilde{\nabla}_\gamma^2 = \Delta_2 - \sum_{\alpha\in R_+} F(\alpha,\alpha)\frac{(-4u_\alpha u_{-\alpha})}{(u_\alpha - u_{-\alpha})^2}k_\alpha(k_\alpha - s_\alpha)$$

$$-\sum_{\substack{\alpha,\beta\in R_+\\ \alpha\neq\beta}} k_\alpha k_\beta F(\alpha,\beta)\frac{u_\alpha + u_{-\alpha}}{u_\alpha - u_{-\alpha}}\cdot\frac{u_\beta + u_{-\beta}}{u_\beta - u_{-\beta}}s_\alpha s_\beta - \sum_{\alpha\in R_+} {k_\alpha}^2 F(\alpha,\alpha)$$

$$= -H_S - \sum_{\substack{\alpha,\beta\in R_+\\ \alpha\neq\beta}} k_\alpha k_\beta F(\alpha,\beta)\frac{u_\alpha + u_{-\alpha}}{u_\alpha - u_{-\alpha}}\cdot\frac{u_\beta + u_{-\beta}}{u_\beta - u_{-\beta}}s_\alpha s_\beta - \sum_{\alpha\in R_+} {k_\alpha}^2 F(\alpha,\alpha).$$

The proof is finished. □

Previous calculations naturally lead to the following definitions.

Definition 1. Let $\gamma,\delta\in R$ and $\alpha,\beta\in R_+$ are such that $s_\alpha s_\beta = w\in W$. Then, for all $w\in W$, we can define an algebraic variety by the equations

$$M_D(R) = \left\{ u_\alpha \in C^N \;\middle|\; \sum_{\substack{\alpha,\beta\in R_+\\ \alpha\neq\beta\\ s_\alpha s_\beta = w}} k_\alpha k_\beta \left\{\frac{F(\gamma,\alpha)F(\delta,\beta) - F(\gamma,\beta)F(\delta,\alpha)}{(u_\alpha - u_{-\alpha})(u_\beta - u_{-\beta})}\right\} = 0. \right\}.$$

This variety will be called the Dunkl variety for the Calogero model.

Definition 2. The algebraic subvariety in $\mathbb{C}^N$ defined by the equations

$$\sum_{\substack{\alpha,\beta\in R_+\\ \alpha\neq\alpha\\ \sigma_\alpha s_\beta = w}} k_\alpha k_\beta \frac{F(\alpha,\beta)}{(u_\alpha - u_{-\alpha})(u_\beta - u_{-\beta})} = 0$$

will be called the Bethe variety for Calogero model.

Definition 3. The intersection of the Dunkl variety and Bethe variety will be called the Bethe–Dunkl variety.

It is posible to give similar definitions for the Sutherland models, but in this case the modification of operators $\tilde{\nabla}_i$ is necessary. On these varieties the preceeding equations are simplified considerably and we obtain following assertions.

Theorem 1. (1) *On the Dunkl variety the operators $\nabla_\gamma, \nabla_\delta$, $\gamma,\delta\in R$ commute;*

(2) *on the Bethe variety we have the following representation of the Hamiltonian H_C:*

$$\sum_{\gamma\in R}\nabla_\gamma^2 = -H_C;$$

(3) *On the Bethe–Dunkl variety, the quantum problem with the spin Hamiltonian H_C is integrable and the set of algebraically independent integrals is given by*

$$I_s = \sum_{\gamma\in R}\nabla_\gamma^s.$$

The proof follows easily from Propositions 3 and 4.

Let ∇_γ be the operator

$$\nabla_\gamma = -D_\gamma + \sum_{\alpha\in R_+}\frac{k_\alpha F(\gamma,\alpha)s_\alpha}{u_\alpha - u_{-\alpha}}.$$

Define the operators

$$\nabla_\gamma{}^+ = \nabla_\gamma + L_\gamma(u), \nabla_\gamma{}^- = \nabla_\gamma - L_\gamma(u).$$

The following proposition holds.

Proposition 6. *On the Bethe variety the following equations holds:*

$$\frac{1}{2}\sum_{\gamma\in R}(\nabla_\gamma{}^-\nabla_\gamma{}^+ + \nabla_\gamma{}^+\nabla_\gamma{}^-)$$

$$= \Delta_1 - \sum_{\alpha\in R_+}\frac{F(\alpha,\alpha)}{(u_\alpha - u_{-\alpha})^2}k_\alpha(k_\alpha - 2s_\alpha) - Q(u) = -H_C^h.$$

Proof. Indeed,

$$\nabla_\gamma{}^-\nabla_\gamma{}^+ + \nabla_\gamma{}^+\nabla_\gamma{}^- = 2\nabla_\gamma^2 - L_\gamma\nabla_\gamma - \nabla_g L_\gamma + L_\gamma\nabla_g - 2L_\gamma{}^2 = 2(\nabla_\gamma{}^2 - L_\gamma^2);$$

$$\frac{1}{2}\sum_{\gamma\in R}(\nabla_\gamma{}^-\nabla_\gamma{}^+ + \nabla_\gamma{}^+\nabla_\gamma{}^-) = \sum_{\gamma\in R}\nabla_\gamma{}^2 - \sum_{\gamma\in R}L_\gamma{}^2 = H_C - Q(u) = -(H_C + Q(u)) = -H_C^h.$$

5. Algebras of Dunkl Operators and Fock Spaces for Arbitrary Root Systems

Let P_α be the operators of multiplication by the generators $s_\alpha \in W(R)$ in $\mathbb{C}[W(R)]$. Let $\Phi_M^{W(R)}$ be the space of W-invariant functions on $\mathbb{C}^{|R|}$ restricted to the Dunkl variety $M_D(R)$. Define the algebra

$$A_S = \mathbb{C}[\nabla_\gamma, L_\gamma, s_\gamma], \qquad \gamma \in R.$$

Consider an A_S-module F_S (Fock space) generated by the vacuum vector $|0\rangle_S = 1$. The operators ∇_γ annihilate the vacuum vector, and s_γ preserve it,

$$\nabla_\gamma|0\rangle_S = 0, \qquad s_\gamma|0\rangle_S = |0\rangle_S.$$

Let, as above, $L_\gamma = \sum\limits_{\alpha\in R} F(\gamma,\alpha)u_\alpha$. Define the algebra

$$A_C = \mathbb{C}[\nabla_\gamma^+, \nabla_\gamma^-, s_\gamma], \qquad \gamma \in R,$$

where

$$\nabla_\gamma^+ = \nabla_\gamma + L_\gamma(u), \quad \nabla_\gamma^- = \nabla_\gamma - L_\gamma(u).$$

Introduce an A_C-module F_C (Fock space) generated by the vacuum vector,

$$|0\rangle_C = \exp^- 1/2Q(u)|0\rangle_S.$$

The operators ∇_γ^- annihilate $|0\rangle_C$.

Define the maps

$$\rho_A : A_S \to A_C, \qquad \rho_F : F_S \to F_C,$$

such that we have

$$\rho_A(\nabla_\gamma) = \nabla_\gamma^+, \qquad \rho_A(L_\gamma) = \nabla_\gamma^-,$$

$$\rho_A(s_\gamma) = s_\gamma, \qquad \rho_F(a|v\rangle_S) = \rho_A(a)\rho_F(|v\rangle_S).$$

It is easy to see that ρ_A is an epimorphism.

The following lemma holds.

Lemma 2. *The Hamiltonian H_C and its integrals I_C^k belong to the algebra A_C.*

The proof follows from the propositions above.

In conclusion we propose some conjectures.

Lemma. *The Hamiltonian H_S and its integrals $I_S^k = \sum_{\gamma \in R}$ belong to the algebra A_S.*

The proof of this lemma is known only in some particular cases.

Theorem. *Epimorphisms ρ_A and ρ_F are isomorphisms.*

At the present time we have no complete proof of this theorem. For the root systems A, B, C, and D similar result was announced in short communications by S. Kakei.

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