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# HEAT CONDUCTION OF SHELLS REINFORCED BY FIBERS WITH CONSTANT AND VARIABLE CROSS SECTIONS

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We propose a model for heat conduction of a spatially reinforced medium and present its generalization to the case of a polyreinforced layer. We consider the heat-conduction equations for fibrous shells and construct a procedure for reduction of a three-dimensional problem of heat conduction to a two-dimensional one. Analytic solutions of a stationary problem of heat conduction are found for thin conic shells of revolution for various structures of reinforcement, and a graphical comparison of the corresponding results is performed. We study one of the approaches to rational reinforcement of thin shells, according to which the thermal "transparency" of a shell in the transverse direction is taken as a criterion of rational design.

Thin-walled elements of the type of shells and plates are widely used in modern power installations, jet engines of aerospace engineering, and laser and magnetohydrodynamic installations for efficient accumulation or transfer of heat and the assurance of reliable operation of installations and engines at elevated temperatures and high levels of loads. The use of homogeneous structural materials in these installations has, in fact, reached the limit of their modernization. Considerable further progress is possible by way of the creation of composite structures on the basis of synthesis of various materials that ensure discrete, continuous, or discrete-continuous variations in thermal and mechanical characteristics and heat sources. This program can be realized by creating multilayer polyreinforced structures for which the thermal and mechanical properties of the phases of compositions are different and the trajectories of reinforcement are curvilinear (e.g., spiral reinforcement in shells of revolution). From the technological point of view, the creation of such laminated fibrous structures is not very difficult. These structures are widely used as efficient bearing elements of transport and power installations and as elements of aerospace equipment. The active utilization of these structures was promoted by the methods of mathematical simulation and description of the mechanical behavior of these systems, which show a good performance, and by the efficient methods developed for solution of the corresponding boundary-value problems. As usual, in the investigation of the mechanical behavior of structures, the temperature field is considered as known [2, 9, 15]. However, for complicated structures of reinforcement, the equations describing mechanical and thermal fields are coupled and cannot be considered separately [11, 12]. The study of heat conduction of these systems is in an embryonic state and has been restricted, up to the present, to simple models of cylindrically or spherically symmetric laminated bodies [1, 7, 10, 13] and unidirectional fibrous composites [4]. This does not enable one to consider the problem of the search for systems with the most efficient thermal characteristics.

In the most general form, the heat-conduction equation has the form

$$c\rho T_{,t} = (H_1 H_2 H_3)^{-1} \left[ \sum_{i=1}^{3} \left( H_1 H_2 H_3 H_i^{-1} \sum_{j=1}^{3} H_j^{-1} \Lambda_{ij} T_{,j} \right)_{,i} \right] + w \tag{1}$$

in the orthogonal curvilinear coordinate system $x_i$, $i = \overline{1, 3}$, where $t$ is the time, $H_i$ are the Lamé parameters, $T$ is the temperature, $\rho$ and $c$ are, respectively, the density and specific heat capacity of the material, $\Lambda_{ij}$, $i, j = \overline{1, 3}$, are the linear thermal conductivities, and $w$ is the specific intensity of the internal heat sources. In the indices, the symbol after a comma denotes partial differentiation with respect to the variables $t$ or $x_i$. It is usually supposed that the thermal conductivities, heat capacity, and intensity of internal heat sources are known from ex-

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periments. However, in fibrous composite materials, they essentially depend on the structure of reinforcement and vary considerably with variation in this structure. For complicated structures of reinforcement, which are usually realized in the manufacture of reinforced shells by methods of winding and facing, the material displays not only anisotropic but specific inhomogeneous properties that depend on the method of creation of the reinforced structure.

The present work is devoted to the construction and analysis of heat-conduction equations for arbitrary shells manufactured by the most widespread methods of winding and facing on the basis of the assumption that reinforcing fibers preserve or change areas of cross sections along trajectories of reinforcement.

## 1. Heat Conduction of Unidirectionally Reinforced Shells

In solving the problem of heat conduction for structures with complicated geometry such as shells reinforced with fibers whose trajectories are complicated spatial curves, it is quite natural to consider a simple model of heat conduction of a reinforced medium that takes into account, in the first approximation, the main thermal characteristics of phases of the composition. In addition, this model is convenient from the viewpoint of analysis of the corresponding heat-conduction equation for solution of complicated problems, such as rational and optimal reinforcement of structures [11, 12].

It is known that, in the first approximation, the integral coefficient of longitudinal thermal conductivity of a unidirectionally reinforced material is determined by the law of simple mixtures [4]. At the same time, the determination of the integral coefficients of transverse (with respect to the direction of fibers) thermal conductivity is quite a complicated problem which requires a special mathematical apparatus. For example, in [4], this coefficient was determined on the basis of the energy method with regard for a doubly periodic arrangement of fibers in the plane orthogonal to the direction of reinforcement. In structures such as shells, where reinforced layers of elementary thickness have a complicated geometry and the transition from one reinforced layer to another is accompanied by a change in the metric, the notion of doubly periodic arrangement of fibers and parameters of the translation symmetry which determine this periodicity loses its meaning. In addition, upon solution of problems of rational reinforcement, where the structure of reinforcement is determined in the process of solution, the method proposed in [4] for determination of the integral coefficient of transverse thermal conductivity is inefficient because the parameters of the translation symmetry are unknown. For this reason, it is expedient to use an approach that does not take the translation symmetry into account.

It is known that the problem of determination of the actual distribution of thermal flows and the temperature field in a unidirectionally reinforced material is extremely complicated. For this reason, to find any dependences useful for practical purposes in the determination of three independent thermal constants in the form of components of the tensor of linear thermal conductivity, it is necessary to make certain assumptions in the form of initial preconditions, which are similar to those used in [8] for the derivation of components of the tensor of compliance of a unidirectionally reinforced material.

1°. A unidirectionally reinforced material is a continuous macroscopically quasihomogeneous monotropic (transversally isotropic) body.

2°. The base material (binder) is isotropic and homogeneous and the material of reinforcement is homogeneous and transversally isotropic; moreover, the principal axis of anisotropy coincides with the longitudinal axis of a fiber. In both phases of the composition, the relation between the vector of thermal flow and the temperature gradient is the linear Fourier law of heat conduction.

3°. Complete adhesion of the binder and reinforcement occurs, and the surface of contact is not heat insulated.

4°. The increment of the averaged temperature $T$ along an arbitrarily oriented elementary segment of length $\Delta l$ is equal to the sum of increments of temperatures in the phases of the composition that are intersected by this segment.

5°. The averaged thermal flow through an arbitrary oriented elementary area is calculated as a simple superposition of thermal flows in the phases of the composition.

For convenience of further generalization, we assume that the trajectories of reinforcement are parallel to the plane $y_1Oy_2$ of the rectangular Cartesian coordinate system $y_i$, $i = \overline{1, 3}$, their direction is determined by the angle $\alpha$ measured from the direction of the axis $Oy_1$, and the intensity of reinforcement is equal to $\omega$.

The statement of hypothesis 4° can be formally represented as follows:

$$\Delta_l T = \sum_i \Delta_{li} T_b + \sum_j \Delta_{lj} T_r, \tag{2}$$

where $T_b$ and $T_r$ are the temperatures of the binder and reinforcement, respectively. The symbol $\Delta_l$ denotes the increment along the direction of a segment $\Delta l$ given by the unit vector $l$, the symbols $\Delta_{li}$ and $\Delta_{lj}$ denote, respectively, the increments along the direction $l$ on the $i$th section of length $\Delta l_i^b$ of the binder intersected by the segment $\Delta l$ and on the $j$th section of length $\Delta l_j^r$ of the reinforcement intersected by the segment $\Delta l$. The summation is carried out over all sections of the binder (over $i$) and reinforcement (over $j$) intersected by the segment $\Delta l$. Moreover, the following normalizing condition is satisfied:

$$\Delta l = \sum_i \Delta l_i^b + \sum_j \Delta l_j^r. \tag{3}$$

Representation (2) is valid by virtue of hypothesis 3°, according to which the following condition for temperatures of the phases of the composition is valid on the surface of contact $\Gamma$ of the reinforcement with the binder:

$$T_r(\Gamma) = T_b(\Gamma). \tag{4}$$

We rewrite condition (2) in terms of differentials. By virtue of assumption 1°, we have

$$T_{,l}\, dl = \sum_i T_{b,l}\, dl_i^b + \sum_j T_{r,l}\, dl_j^r = T_{b,l} \sum_i dl_i^b + T_{r,l} \sum_j dl_j^r, \tag{5}$$

where $\partial/\partial l$ is the derivative with respect to the direction $l$ and $dl$, $dl_i^b$, and $dl_j^r$ are the differentials corresponding to the lengths of segments $\Delta l$, $\Delta l_i^b$, and $\Delta l_j^r$ in (3). Dividing equality (5) by $dl$, by virtue of the normalizing condition (3), we get

$$l\,\mathrm{grad}\,T = l(a\,\mathrm{grad}\,T_b + \omega\,\mathrm{grad}\,T_r), \qquad a = 1 - \omega, \tag{6}$$

where $a$ is the specific intensity of the binder. According to hypothesis 4°, the first equality in (6) must be valid for an arbitrary unit segment $l$. This yields

$$\mathrm{grad}\,T = a\,\mathrm{grad}\,T_b + \omega\,\mathrm{grad}\,T_r.$$

Thus, assumption 4° can be formally represented as three scalar equalities:

$$T_{,i} = aT_{b,i} + \omega T_{r,i}, \qquad i = \overline{1, 3}. \tag{7}$$

Here and below, the symbol in the subscript after a comma denotes a partial derivative with respect to the corresponding variable $y_i$ or $x_i$ (depending on the meaning).

By virtue of assumption 5°, we have

$$q_i = aq_{bi} + \omega q_{ri}, \quad i = \overline{1,3}, \tag{8}$$

where $q_i$, $q_{bi}$, and $q_{ri}$ are the components of the vector of averaged thermal flow and vectors of thermal flows in the binder and reinforcement, respectively. By virtue of hypothesis 2°, it follows from (8) that

$$q_i = -\lambda_b a T_{b,i} - \omega\left[(\lambda_1 l_i^2 + \lambda_2 l_j^2)T_{r,i} + l_1 l_2(\lambda_1 - \lambda_2)T_{r,j}\right], \quad j = 3 - i, \quad i = 1, 2,$$

$$q_3 = -\lambda_b a T_{b,3} - \lambda_2 \omega T_{r,3}, \tag{9}$$

where $l_1 = \cos\alpha$, $l_2 = \sin\alpha$, $\lambda_b$ is the linear thermal conductivity of the binder, and $\lambda_1$ and $\lambda_2$ are the linear thermal conductivities of the reinforcement in the longitudinal and transverse directions. For an isotropic material of the reinforcement ($\lambda_1 = \lambda_2 = \lambda_r$), equalities (9) can be represented in the following vector form:

$$\boldsymbol{q} = -\lambda_b a \operatorname{grad} T_b - \lambda_r \omega \operatorname{grad} T_r. \tag{10}$$

To determine the thermal conductivities of the reinforced material, it is necessary to find a linear relation between the quantities $q_i$ in (9) and (10) and the derivatives $T_{,i}$ in (7). For this purpose, we express $T_{r,i}$ and $T_{b,i}$ in terms of $T_{,i}$ and exclude them from (9) and (10). First, we express $T_{r,i}$ in terms of $T_{b,i}$. As was mentioned above, on the surface of contact $\Gamma$ of the reinforcement with the binder, the condition of continuity (4) is satisfied. By virtue of this condition and assumption 1°, we obtain

$$l_1 T_{r,1} + l_2 T_{r,2} = l_1 T_{b,1} + l_2 T_{b,2}. \tag{11}$$

In addition, according to hypothesis 3°, the condition of continuity for thermal flows must be satisfied on $\Gamma$, which is equivalent to the two scalar equalities

$$q_{rn} = q_{bn}, \quad q_{r3} = q_{b3}, \tag{12}$$

where $q_{rn}$ and $q_{bn}$ are, respectively, the components of the vectors of thermal flows in the reinforcement and in the binder along the unit vector $\boldsymbol{n} = (-l_2, l_1, 0)$. By virtue of the Fourier law for both phases of the composition, we obtain from relations (12) that

$$\lambda_2(-l_2 T_{r,1} + l_1 T_{r,2}) = \lambda_b(-l_2 T_{b,1} + l_1 T_{b,2}), \tag{13}$$

$$\lambda_2 T_{r,3} = \lambda_b T_{b,3}. \tag{14}$$

By solving system (11), (13), (14), we obtain the following dependence of $T_{r,i}$ on $T_{b,i}$:

$$T_{r,i} = (l_i^2 + \lambda l_j^2)T_{b,i} + l_1 l_2(1 - \lambda)T_{b,j}, \quad j = 3 - i, \quad i = 1, 2,$$

$$T_{r,3} = \lambda T_{b,3}, \tag{15}$$

where $\lambda = \lambda_b \lambda_2^{-1}$ is a dimensionless quantity. By substituting expressions (15) into equalities (7), we obtain the following system of linear algebraic equations for $T_{b,i}$:

$$\left[a + \omega(l_i^2 + \lambda l_j^2)\right] T_{b,i} + \omega l_1 l_2 (1 - \lambda) T_{b,j} = T_{,i}, \quad j = 3 - i, \quad i = 1, 2,$$

$$\Delta T_{b,3} = T_{,3}, \quad \Delta = a + \omega\lambda,$$

the solution of which has the form

$$T_{b,i} = \Delta^{-1}\left\{\left[1 - \omega l_i^2 (1 - \lambda)\right] T_{,i} - \omega l_1 l_2 (1 - \lambda) T_{,j}\right\}, \quad j = 3 - i, \quad i = 1, 2,$$

$$T_{b,3} = \Delta^{-1} T_{,3}. \tag{16}$$

Thus, relations (15) and (16) determine the required dependences of $T_{r,i}$ and $T_{b,i}$ on $T_{,i}$. Upon the substitution of (15) and (16) into (9) and certain transformations, we get

$$q_i = -\Lambda_{i1} T_{,1} - \Lambda_{i2} T_{,2}, \quad i = 1, 2, \quad q_3 = -\Lambda_{33} T_{,3}, \tag{17}$$

where

$$\Lambda_{ij} = l_i l_j (\lambda_1 \omega + \lambda_b a) + (-1)^{i+j} l_p l_r (\omega \lambda_2^{-1} + a \lambda_b^{-1})^{-1}, \quad p = 3 - i, \quad r = 3 - j, \quad i, j = 1, 2,$$

$$\Lambda_{33} = (\omega \lambda_2^{-1} + a \lambda_b^{-1})^{-1}. \tag{18}$$

Thus, within the framework of the accepted assumptions, the integral coefficients of linear thermal conductivity of a unidirectionally reinforced material are determined by relations (18). In the particular case where fibers are oriented along the axis $Oy_1$ $(\alpha = 0)$, relations (18) are simplified to the form

$$\Lambda_{11} = \lambda_1 \omega + \lambda_b a, \quad \Lambda_{22} = \Lambda_{33} = (\omega \lambda_2^{-1} + a \lambda_b^{-1})^{-1}, \quad \Lambda_{ij} = 0, \quad i \neq j. \tag{19}$$

Therefore, the integral coefficient of longitudinal thermal conductivity is determined as a simple superposition, which is in agreement with the well-known fact [4]. The quantity inverse to the integral coefficient of transverse thermal conductivity is determined as a superposition of the quantities inverse to the thermal conductivities of fibers and the binder. It is appropriate to note that the coefficient of transverse thermal conductivity obtained in such a way formally coincides with the effective modulus of transverse shear of a unidirectionally reinforced material given in [8]. In [4], it was shown that the problem on transverse heat conduction is analogous to the problem on transverse shear despite the fact that the model of heat conduction used in [4] differs from the model proposed above.

By rotating the coordinate system about the axis $Oy_3$ by the angle $-\alpha$ and transforming quantities (19) according to the rule of transformation of tensors of the second rank, we arrive at relation (18) for $\Lambda_{ij}$ in a new coordinate system. This means that quantities (18) form a symmetric tensor of the second rank. In addition, if the physical constraint

$$0 < \omega < 1$$

is valid, then the coefficients $\Lambda_{ij}$, $i, j = \overline{1, 3}$, in (19) are nonnegative. Therefore, $\Lambda_{ij}$ in relations (18) and (19) are coefficients of a positive-definite quadratic form, i.e., the differential operator on the right-hand side of the heat-conduction equation analogous to (1) for a unidirectionally reinforced medium satisfies the condition of ellipticity. All these distinctive features confirm the correctness of assumptions 1°–5° and enable one to regard them as physically substantiated.

All calculations made above on the basis of hypotheses 1°–5° can be repeated not only for a unidirectionally reinforced medium but for any elementary volume $dy_1\, dy_2\, dy_3$ as well. For this reason, the integral thermal conductivities of a medium curvilinearly reinforced with one family of fibers can be represented in the form (18) in the orthogonal coordinate system $x_1$, $x_2$, $x_3$ if the reinforcement is carried out along directions parallel to planes that are tangent to the surfaces $x_3 = \text{const}$ and make an angle $\alpha$ with the direction of the coordinate line $x_1$. Note that, as a rule, by virtue of the widespread technological methods of manufacture, this distinctive feature occurs in structures such as shells and plates. For such a reinforcement, it is reasonable to say regarding reinforcing layers that $x_3 = \text{const}$.

By using the ordinary scheme of formation of the equation of thermal balance [7], we obtain the following heat-conduction equation for a unidirectionally reinforced body in an orthogonal curvilinear coordinate system:

$$CT_{,t} = (H_1 H_2 H_3)^{-1} \left[ \left( \sum_{i=1}^{3} H_1 H_2 H_3 H_i^{-1} \sum_{j=1}^{3} H_j^{-1} \Lambda_{ij} T_{,j} \right)_{,i} \right] + W, \tag{20}$$

where $C = \omega c_r \rho_r + a c_b \rho_b$ is the reduced heat capacity, $W = \omega w_r + a w_b$ is the reduced intensity of internal heat sources, $c_r$ and $c_b$ are the specific heat capacities of the materials of the reinforcement and the binder, $\rho_r$ and $\rho_b$ are the densities of the materials of the reinforcement and the binder, $w_r$ and $w_b$ are the specific intensities of internal heat sources in the reinforcement and binder, and the quantities $\Lambda_{ij}$ are determined by relations (18).

## 2. Heat Conduction of Polyreinforced Shells

Fibrous composite structures such as the shells and plates used at present are reinforced, as a rule, with not one but a few families of fibers of various orientations. Moreover, in many cases, the physical nature of the fibers is different. First of all, this is due to the strength requirements according to which bearing elements (fibers) in reinforced shells and plates must be subjected to the action of comparable stresses in planes differently oriented in two directions.

Suppose that a shell (plate) of thickness $H$ is made of $N$ layers, unidirectionally reinforced by various families, of thickness $h_i$, $i = \overline{1, N}$, of the same order as $H$. Then the macroscopic mechanical and thermal characteristics of this pack discretely vary across the thickness. Assuming that the thermal conductivities of every reinforced layer are known from relations (18), we can investigate heat conduction of the structure as a whole by using, e.g., the approaches proposed in [1, 7, 13].

However, in laminated shells, for which the thickness $h_i$ of every layer is comparable with the thickness $H$ of the structure and the mechanical and thermal characteristics discretely and considerably vary in passing from one layer to the other, various undesirable effects, such as low thermal stability, stratification of the pack, etc., can appear. For this reason, it is necessary to create polyreinforced shells such that the thickness of a unidirectionally reinforced layer is considerably smaller than the thickness of the pack and every layer is repeated many times and periodically across the thickness of a shell. In this case, regarding a polyreinforced shell, we can say that, across the thickness, its structure is quasiregular and its mechanical and thermal characteristics "quasicontinuously" vary. This raises the natural question on investigation of the law of heat conduction for these polyreinforced structures.

The integral thermal conductivities for a composite material reinforced with $N$ families of fibers in parallel planes can be calculated by using the procedure proposed in [8] for the tensor of compliance. According to this procedure, first, only the reinforcement in one direction $\alpha_1$ with the specific intensity of reinforcement $\omega_1$ is taken into account. Then all thermal conductivities for a unidirectionally reinforced layer are calculated. Only after this, by assuming that the material is anisotropic and macroscopically homogeneous, does one consider its reinforcement with density $\omega_2$ in the direction $\alpha_2$ and determine, respectively, the thermal conductivities just as for a unidirectionally reinforced material with anisotropic binder, etc. However, this approach is inconvenient because the expressions for the effective thermal conductivities of the entire pack are very cumbersome even in the case of two families of fibers, to say nothing of an arbitrary number of families (this leads to considerable difficulties in the analysis of problems of rational and optimal reinforcement). In addition, within the framework of this approach, it is necessary to perform $N$ recalculations of thermal conductivities, which is inconvenient in the case of inhomogeneous reinforcement of structures. For this reason, it is expedient to develop other approaches to determination of the integral thermal conductivities of polyreinforced materials.

In what follows, we distinguish the notions of unidirectionally reinforced layer and elementary unidirectionally reinforced layer. By a unidirectionally reinforced layer, we mean a layer whose thickness $h_i$ is comparable with the thickness $H$ of the pack. By an elementary unidirectionally reinforced layer, we mean a layer whose thickness is much smaller than the thickness of a polyreinforced layer. For example, by the thickness of an elementary unidirectionally reinforced layer, we can mean the thickness of a fiber, etc.

Thus, to derive the integral thermal conductivities of a polyreinforced layer with a structure quasiregular across the thickness, formed by periodic and multiple alternation of an elementary unidirectionally reinforced layer with various families of fibers, we assume the following:

1°°. The material of a layer is obtained by introducing $N$ families of homogeneous and transversally isotropic fibers into the isotropic and homogeneous binder.

2°°. The direction of reinforcement with the $k$th family of fibers in an elementary unidirectionally reinforced layer is parallel to the plane $y_1Oy_2$ in the rectangular Cartesian coordinate system and makes an angle $\alpha_k$, $i = \overline{1, N}$, with the direction of the axis $Oy_1$.

3°°. For each elementary unidirectionally reinforced layer, hypotheses 1°–5° of Sec. 1 remain valid.

4°°. A polyreinforced layer is a continuous macroscopically quasihomogeneous anisotropic body, one of the principle anisotropy axes of which coincides with the direction orthogonal to the elementary unidirectionally reinforced layer (the direction of the axis $Oy_3$).

5°°. Because of the binder, the complete adhesion between elementary unidirectionally reinforced layers occurs.

6°°. Assumptions 4° and 5° of Sec. 1 remain valid for the entire polyreinforced layer, where an elementary unidirectionally reinforced layer should be meant as a phase of the composition.

To determine the effective thermal conductivities of a polyreinforced layer, it is necessary, as in Sec. 1, to find the relation between the vector of averaged thermal flow $\boldsymbol{q}$ and the gradient of averaged temperature $T$. By virtue of assumption 6°°, we obtain the following relations of the type (7) and (8):

$$T_{,i} = \sum_k \Omega_k T_{k,i}, \tag{21}$$

$$q_i = \sum_k \Omega_k q_{ki}, \quad i = \overline{1, 3}, \tag{22}$$

where $\Omega_k = h_k H^{-1}$ is the specific intensity of a unidirectionally reinforced layer with the $k$th family of fibers, $T_k$ and $q_{ki}$ are the temperature and $i$th components of the vector of thermal flow in an elementary unidirectionally reinforced layer with the $k$th family of fibers, respectively, $h_k$ is the total thickness of elementary unidirectionally reinforced layers with the $k$th family of the reinforcement, and $H$ is the thickness of the pack (polyreinforced layer). Moreover, the normalizing condition

$$H = \sum_k h_k, \quad 1 = \sum_k \Omega_k, \quad \Omega_k > 0,$$

is true. Here and below, the summation is carried out over the index mentioned from one to $N$ if the limits are not indicated.

To obtain the required dependence of $\boldsymbol{q}$ on $T_{,i}$, it is necessary to exclude $T_k$ and $q_{ki}$ from (21) and (22). By using the Fourier law for all elementary unidirectionally reinforced layers, with regard for (17) and (18), we obtain the following relations from (22):

$$q_i = -\sum_k \Omega_k \Lambda_{i1}^{(k)} T_{k,1} - \sum_k \Omega_k \Lambda_{i2}^{(k)} T_{k,2}, \quad i = 1, 2, \tag{23}$$

$$q_3 = \sum_k \Omega_k q_{k3} = -\sum_k \Omega_k \Lambda_{33}^{(k)} T_{k,3}, \tag{24}$$

where

$$\Lambda_{ij}^{(k)} = l_{ki} l_{kj} [(\lambda_{1k} - \lambda_b)\overline{\omega}_k + \lambda_b] + (-1)^{i+j} l_{kp} l_{kr} \left[\overline{\omega}_k \lambda_{2k}^{-1} + (1 - \overline{\omega}_k)\lambda_b^{-1}\right]^{-1},$$

$$p = 3 - i, \quad r = 3 - j, \quad i, j = 1, 2,$$

$$\Lambda_{33}^{(k)} = \left[\overline{\omega}_k \lambda_{2k}^{-1} + (1 - \overline{\omega}_k)\lambda_b^{-1}\right]^{-1}, \tag{25}$$

$l_{k1} = \cos\alpha_k$, $l_{k2} = \sin\alpha_k$, $\overline{\omega}_k$ is the specific intensity of the reinforcement of an elementary unidirectionally reinforced layer with the $k$th family of fibers, and $\lambda_{1k}$ and $\lambda_{2k}$ are the linear thermal conductivities of the reinforcement of the $k$th family in the longitudinal and transverse directions, respectively.

For further transformations of relations (23) and (24), we use the thermal conditions of continuity of two elementary unidirectionally reinforced layers with the $k$th and $m$th families of fibers on the surface of contact $\Gamma_{km}$. By virtue of assumptions 2°° and 5°°, the surface $\Gamma_{km}$ is parallel to the plane $y_1 O y_2$, and the conditions

$$T_k(\Gamma_{km}) = T_m(\Gamma_{km}), \tag{26}$$

$$q_{k3}(\Gamma_{km}) = q_{m3}(\Gamma_{km}), \quad k, m = \overline{1, N}, \quad k \neq m, \tag{27}$$

are satisfied on it. By virtue of assumption 4°° and the fact that an elementary unidirectionally reinforced layer has an elementary thickness, system (26), (27) can be replaced by the equivalent system of equations

$$T_k = T_1, \tag{28}$$

$$q_{k3} = q_{13}, \quad k = \overline{2, N}. \tag{29}$$

By differentiating (26) and (28) with respect to $y_1$ and $y_2$, which is admissible because $\Gamma_{km}$ is parallel to the plane $y_1Oy_2$, we obtain, in the first approximation, the equalities

$$T_{k,i} = T_{1,i}, \quad i = 1, 2, \quad k = \overline{2, N}. \tag{30}$$

By virtue of the Fourier law, relations (29) yield

$$T_{k,3} = \Lambda_{33}^{(1)} (\Lambda_{33}^{(k)})^{-1} T_{1,3}, \quad k = \overline{2, N}. \tag{31}$$

By substituting equalities (29)–(31) into (21), (23), and (24), we obtain the relations for $T_{,i}$ and $q_i$ in terms of $T_{1,i}$:

$$T_{,i} = \sum_k \Omega_k T_{1,i} = T_{1,i}, \quad i = 1, 2, \qquad T_{,3} = \sum_k \Omega_k \Lambda_{33}^{(1)} (\Lambda_{33}^{(k)})^{-1} T_{1,3},$$

$$q_i = -\sum_k \Omega_k (\Lambda_{i1}^{(k)} T_{1,i} + \Lambda_{i2}^{(k)} T_{1,2}), \quad i = 1, 2, \qquad q_3 = \sum_k \Omega_k q_{13} = -\Lambda_{33}^{(1)} T_{1,3}.$$

After the elimination of $T_{1,j}$, $j = \overline{1, 3}$, from this system, we obtain the final dependence of $q$ on grad$dT$ in the form

$$q_i = -\Lambda_{i1} T_{,i} - \Lambda_{i2} T_{,2}, \quad i = 1, 2, \qquad q_3 = -\Lambda_{33} T_{,3},$$

where

$$\Lambda_{ij} = \sum_k \Omega_k \Lambda_{ij}^{(k)}, \quad i, j = 1, 2, \qquad \Lambda_{33} = \left[ \sum_k \Omega_k (\Lambda_{33}^{(1)})^{-1} \right]^{-1}, \tag{32}$$

$\Lambda_{ij}^{(k)}$, $i, j = 1, 2$, and $\Lambda_{33}^{(k)}$ are given by relations (25).

Relations (32) show that effective longitudinal (along elementary unidirectionally reinforced layers) thermal conductivities are obtained by averaging the corresponding coefficients of elementary unidirectionally reinforced layers proportionally to their percentage $(\Omega_k)$ in the pack, and the quantity inverse to the effective coefficient of transverse (to elementary unidirectionally reinforced layers) thermal conductivity is obtained by averaging the quantities inverse to the transverse thermal conductivities of elementary unidirectionally reinforced layers proportionally to the content of the corresponding elementary unidirectionally reinforced layers in the pack.

Since thermal conductivities of every elementary unidirectionally reinforced layer form the coefficients of a positive-definite quadratic form, the thermal conductivities of the entire pack obtained by the method of averaging described above also form coefficients of a positive-definite quadratic form. This means that the differential operator on the right-hand side of the heat-conduction equation, analogous to (20), for a polyreinforced layer satisfies the condition of ellipticity

By virtue of the fact that the manipulations performed above can be repeated for any elementary volume $dy_1\, dy_2\, dy_3$ by using assumptions $1^{\circ\circ}$–$6^{\circ\circ}$, the effective thermal conductivities of a polyreinforced layer in the curvilinear coordinate system $x_1$, $x_2$, $x_3$ can be presented in the form (32) if the reinforcement of every elementary unidirectionally reinforced layer is carried out along directions parallel to planes which are tangent to surfaces $x_3 = \text{const}$ and make an angle $\alpha_k$ with the direction of the coordinate line $x_1$.

By deducing the equation of thermal balance according to an ordinary procedure, we obtain the heat-conduction equation for a polyreinforced layer, which is similar to Eq. (20) with

$$C = c_b \rho_b (1-\Omega) + \sum_k \omega_k c_k \rho_k, \quad W = w_b (1-\Omega) + \sum_k \omega_k w_k, \quad \Omega = \sum_k \omega_k, \tag{33}$$

where $C$ and $W$ are, respectively, the reduced thermal conductivity and intensity of internal heat sources in the polyreinforced layer, $c_b$ and $c_k$ are the specific heat capacities of the materials of the binder and reinforcement of the $k$th family, $\rho_b$ and $\rho_k$ are the densities of the materials of the binder and reinforcement of the $k$th family, $w_b$ and $w_k$ are the specific intensities of internal heat sources in the binder and reinforcement of the $k$th family, $\omega_k$ is the specific intensity of reinforcement with fibers of the $k$th family in the polyreinforced layer defined as

$$\omega_k = \Omega_k \overline{\omega}_k, \tag{34}$$

where $\overline{\omega}_k$ is the specific intensity of reinforcement of an elementary unidirectionally reinforced layer with the $k$th family of fibers, and the quantities $\Lambda_{ij}$ are determined by relations (25) and (32).

However, relations (25) and (32) for integral thermal conductivities of the entire polyreinforced layer can be inefficient for solution of problems of rational and optimal reinforcement of shells. Indeed, in the solution of these problems, in the general case, the specific intensity $\overline{\omega}_k$ of reinforcement of an elementary unidirectionally reinforced layer with the $k$th family of fibers, the angle of reinforcement $\alpha_k$, and the specific intensity $\Omega_k$ of a unidirectionally reinforced layer in the pack with the $k$th family of fibers are unknown. For this reason, in problems of rational reinforcement, it makes sense to use a procedure of averaging the thermal conductivities of the pack that uses $N$ parameters $\omega_k$, $k = \overline{1, N}$, instead of $2N$ parameters $\overline{\omega}_k$ and $\Omega_k$.

Assume that the specific intensities $\omega_k$ of reinforcement of the entire polyreinforced layer with fibers of the $k$th family are known and satisfy the physical constraints

$$\omega_k > 0, \quad k = \overline{1, N}, \quad \Omega < 1. \tag{35}$$

By virtue of assumptions 1°° and 5°°, the adhesion between elementary unidirectionally reinforced layers is realized at the expense of the binder, the material of which is the same in all elementary unidirectionally reinforced layers. Therefore, the boundary between two elementary unidirectionally reinforced layers is, to a certain extent, conditional. This enables us to distribute the reinforcement intensity of the binder in the pack, $a = 1 - \Omega$, over all unidirectionally reinforced layers proportionally to $\omega_k$:

$$a_k = \omega_k \Omega^{-1} a = \omega_k \Omega^{-1} - \omega_k.$$

Thus, we associate the following part of the reinforcement intensity of a polyreinforced layer with every unidirectionally reinforced layer including the $k$th family of the reinforcement:

$$\Omega_k = a_k + \omega_k = \omega_k \Omega^{-1}, \quad k = \overline{1, N}. \tag{36}$$

To determine the specific intensity $\overline{\omega}_k$ of reinforcement of an elementary unidirectionally reinforced layer with the $k$th family of fibers, it is necessary to exclude $\Omega_k$ from (34) and (36). This yields

$$\overline{\omega}_k = \Omega. \tag{37}$$

By virtue of inequality (35), the physical constraints $0 < \overline{\omega}_k = \Omega < 1$ are true. According to relation (37), for the mentioned distribution of the binder over reinforced layers, the quantity $\overline{\omega}_k$ is the same for all families of fibers.

By substituting (36) and (37) into (25) and (32), we obtain the following relations for the effective thermal conductivities for the entire polyreinforced layer:

$$\Lambda_{ij} = \sum_k \omega_k \Omega^{-1}\Big\{l_{ki} l_{kj}[(\lambda_{1k} - \lambda_b)\Omega + \lambda_b] + (-1)^{i+j} l_{kp} l_{kr}\big[\Omega\lambda_{2k}^{-1} + (1-\Omega)\lambda_b^{-1}\big]^{-1}\Big\},$$

$$p = 3-i, \quad r = 3-j, \quad i,j = 1,2,$$

$$\Lambda_{33} = \Big[\sum_k \omega_k \lambda_{2k}^{-1} + (1-\Omega)\lambda_b^{-1}\Big]^{-1}. \tag{38}$$

Note that quantities (38) form coefficients of a positive-definite quadratic form.

Thus, representations (38) are the most convenient for solution of problems of rational and optimal reinforcement of shells. The heat-conduction equation for a polyreinforced layer has the previous form (20). For $N = 1$ in (38), $\Omega = \omega_1$ and relations (38) are immediately reduced to relations (18) for a unidirectionally reinforced layer.

## 3. Reduction of the Heat-Conduction Equation for Thin Fibrous Shells and Plates to a Two-Dimensional Equation

To integrate the initial boundary-value problem corresponding to the heat-conduction equation (20), we can use various approximate methods, e.g., the method of straight lines [6]. The main difficulty arising in the course of integration lies in the fact that Eq. (20) is three-dimensional. For this reason, in the solution of the problem of heat conduction in structures such as shells and plates, one usually tries to reduce the problem to a two-dimensional one. This can be done by using, e.g., the Bubnov–Galerkin method. Indeed, let the temperatures $T_+(t, x_1, x_2)$ and $T_-(t, x_1, x_2)$ be given, respectively, on the lateral "external" $(x_3 = x_3^0 > 0)$ and "internal" $(x_3 = -x_3^0 < 0)$ surfaces of the shell. Then, following the main idea of the Bubnov–Galerkin method, we represent the temperature in the form

$$T = (2x_3^0)^{-1}\big[(x_3 + x_3^0)T_+ - (x_3 - x_3^0)T_-\big]$$

$$+ \sum_{N=0}^{\infty}\Big\{T_{cn}(t, x_1, x_2)\cos\big[(2n+1)\pi x_3(2x_3^0)^{-1}\big] + T_{sn}(t, x_1, x_2)\sin\big[2n\pi x_3(2x_3^0)^{-1}\big]\Big\} \tag{39}$$

and rewrite Eq. (20) in the operator form

$$L(T) = 0, \tag{40}$$

where $L$ is the parabolic differential operator appearing at the expense of the difference of the left- and right-hand sides of Eq. (20). We substitute relation (39) into Eq. (40) and require that

$$\int_{-x_3^0}^{x_3^0} L(T)\cos\frac{(2n+1)\pi x_3}{2x_3^0}\,dx_3 = 0, \qquad \int_{-x_3^0}^{x_3^0} L(T)\sin\frac{n\pi x_3}{x_3^0}\,dx_3 = 0, \qquad n = 0,1,2,\ldots. \tag{41}$$

By integrating (41), we obtain systems of differential equations for the functions $T_{cn}$ and $T_{sn}$ which depend only on two spatial coordinates $x_1$ and $x_2$ and time $t$.

Suppose that heat transfer with the environment is realized through the lateral surfaces of the shell by the Newton law

$$q_3^+ = \mu_+(T_+ - T_{+\infty}), \qquad -q_3^- = \mu_-(T_- - T_{-\infty}), \tag{42}$$

where $q_3^\pm = q_3(t, x_1, x_2, \pm x_3^0)$, $T_\pm = T(t, x_1, x_2, \pm x_3^0)$, $T_{\pm\infty}$ is the temperature of the environment from the side of the "external" (+) and "internal" (–) lateral surfaces, and $\mu_\pm$ are the thermal transfer coefficients. Then it is reasonable to present a solution of Eq. (20) as follows:

$$T = \sum_{n=0}^{\infty} T_n(t, x_1, x_2)(x_3^0)^n. \tag{43}$$

We again substitute relation (43) into (40) and require that

$$\int_{-x_3^0}^{x_3^0} L(T)(x_3^0)^n \, dx_3 = 0, \qquad n = 0, 1, 2, \dots . \tag{44}$$

Note that this system should be supplemented with two boundary conditions (42). By virtue of the Fourier law, these conditions can be represented in the form

$$-(\pm)\Lambda_{33}^\pm \sum_{n=1}^{\infty} T_n(t, x_1, x_2) n(\pm x_3^0)^{n-1} = \mu_\pm\left[\sum_{n=0}^{\infty} T_n(t, x_1, x_2)(\pm x_3^0)^n - T_{\pm\infty}\right], \tag{45}$$

where $\Lambda_{33}^\pm = \Lambda_{33}(x_1, x_2, \pm x_3^0)$. The system of equations (44) and (45) determines the functions $T_n$ which depend only on the time and two spatial coordinates.

Equations (44) and (45) have the most simple form for thin shells because, in this case, the Lamé parameters $H_i$, $i = \overline{1,3}$, can be approximately considered as independent of $x_3$, and it suffices to take only two or three terms in expansion (43).

Consider a thin shell of constant thickness $H = 2h$ $(h = x_3^0)$ reinforced along equidistant surfaces. Let $x_3$ be the distance from the reference surface $(x_3 = 0)$ of the shell to a reinforced layer. Then $H_3 = 1$. Let $H_1$ and $H_2$ and the reinforcement parameters be independent of $x_3$. We take only three terms in expansion (43). Then, for $n = 0$, conditions (45) and Eq. (44) take the form

$$\mp \Lambda_{33}(T_1 \pm 2hT_2) = \mu_\pm(T_0 \pm hT_1 + h^2 T_2 - T_{\pm\infty}), \tag{46}$$

$$C\Theta_{,t} = (H_1 H_2)^{-1} \sum_{i=1,2} \left[H_1 H_2 H_i^{-1} \sum_{j=1,2} (H_j^{-1} \Lambda_{ij} \Theta_{,j})\right]_{,i} - (q_3^+ - q_3^-) + HW, \tag{47}$$

where

$$\Theta = \int_{-h}^{h} T \, dx_3 = HT_0 + \frac{2}{3}h^3 T_2.$$

Since $q_3^{\pm}$ are given in (42), Eq. (47) can be rewritten as follows:

$$C\Theta_{,t} = (H_1 H_2)^{-1} \sum_{i=1,2} \left[ H_1 H_2 H_i^{-1} \sum_{j=1,2} (H_j^{-1} \Lambda_{ij} \Theta_{,j}) \right]_{,i}$$

$$- \mu_-(T_0 - hT_1 + h^2 T_2 - T_{-\infty}) - \mu_+(T_0 + hT_1 + h^2 T_2 - T_{+\infty}) + HW. \qquad (48)$$

Note that Eq. (47) can be immediately derived from the equation of thermal balance for an element of a thin shell of finite height $H H_1 H_2 dx_1 dx_2$.

Thus, the closed system of equations (46), (48) determines three functions $T_i$, $i = \overline{1,3}$, independent of the variable $x_3$.

Restricting ourselves to two terms in expansion (43), we obtain

$$-\Lambda_{33} T_1 = \mu_+(T_0 + hT_1 - T_{+\infty}), \qquad \Lambda_{33} T_1 = \mu_-(T_0 - hT_1 - T_{-\infty}) \qquad (49)$$

instead of (46). The left-hand sides of these equalities indicate that $q_3^+ = q_3^-$. In view of this fact, Eq. (47) is reduced to the form

$$CT_{0,t} = (H_1 H_2)^{-1} \sum_{i=1,2} \left[ H_1 H_2 H_i^{-1} \sum_{j=1,2} (H_j^{-1} \Lambda_{ij} T_{0,j}) \right]_{,i} + W. \qquad (50)$$

From Eq. (50), we can determine the quantity $T_0$. However, in this case, system (49) is overdetermined with respect to $T_1$ and compatible only in exceptional cases, e.g., if $\mu_+ = \mu_-$, $T_{+\infty} = -T_{-\infty}$, and $T_0 = 0$ This means that, in the general case, in the solution of the problem of heat conduction in thin shells, it is reasonable to take three terms in expansion (43).

Nevertheless, if $\mu_+ = \mu_-$ and $T_{+\infty} = T_{-\infty}$, then, neglecting effects of the second order of smallness in the thickness of a thin shell, we can assume that the temperature $T$ is independent of $x_3$. In this case, the heat-conduction equation can be derived from Eq. (47) as

$$CT_{,t} = (H_1 H_2)^{-1} \sum_{i=1,2} \left[ H_1 H_2 H_i^{-1} \sum_{j=1,2} (H_j^{-1} \Lambda_{ij} T_{,j}) \right]_{,i} - 2H^{-1}\mu_+(T - T_{+\infty}) + W,$$

where $T_{,3} = 0$. If the lateral surfaces of the thin shell are thermally isolated, then we can again assume that $T_{,3} = 0$. In this case, the heat-conduction equation has form (50), where $T_0 = T$.

## 4. Solution of the Stationary Heat-Conduction Problem for a Thin Conic Shell for Various Structures of Reinforcement

For a more visual demonstration of the influence of the structure of reinforcement on the distribution of the temperature in a shell, we consider some examples for which the heat-conduction problem can be solved analytically. We consider a stationary axially symmetric heat-conduction problem (in the absence of internal heat sources) for a thin conic shell of revolution reinforced along equidistant surfaces with fibers of various orientations (circular, meridional, or spiral).

The surface of a conic shell with conicity angle $\psi$ is described by the equalities

$$y_1 = (x_1 \tan\psi + x_3 \cos\psi)\cos x_2, \qquad y_2 = (x_1 \tan\psi + x_3 \cos\psi)\sin x_2,$$

$$y_3 = x_1 - x_3 \sin\psi, \tag{51}$$

where $x_i$ and $y_i$ are, respectively, the curvilinear orthogonal coordinates and rectangular Cartesian ones, $Oy_3$ is the axis of revolution of the shell, $x_1$ is the distance from points of the midsurface of the shell ($x_3 = 0$) to the plane $y_3 = 0$ and varies in the interval $0 < y_3^0 \le x_1 \le y_3^1$, the values $x_1 = y_3^0$ and $x_1 = y_3^1$ specify the lower and upper edges of the shell, respectively, $x_2$ is the polar angle, $0 \le x_2 < 2\pi$, and $x_3$ is the distance between the reinforced layer and the midsurface, $-h \le x_3 \le h$. With regard for equalities (51), the Lamé parameters have the form

$$H_1 = \cos^{-1}\psi, \quad H_2 = x_1 \tan\psi + x_3 \cos\psi \approx x_1 \tan\psi, \quad H_3 = 1, \quad \psi = \text{const}, \tag{52}$$

where the approximate equality is admissible by virtue of the assumption on the thinness of the shell.

In the absence of internal heat sources, the equation of the stationary axially symmetric problem of heat conduction for an axially symmetrically reinforced shell with thermally isolated lateral surfaces has the form [see (50)] $(H_1 H_2)^{-1}(H_2 H_1^{-1} \Lambda_{11} T')' = 0$. Whence, by using relations (52), we obtain

$$H_2 H_1^{-1} \Lambda_{11} T' = q_* = \text{const} \quad \text{or} \quad x_1 \Lambda_{11} T' = Q_0 = q_* \sin^{-1}\psi = \text{const}, \tag{53}$$

where $q_*$ is the constant of integration that has the meaning of the meridian component of the vector of thermal flow multiplied by $H_2$ and is determined from the boundary conditions. The prime denotes differentiation with respect to $x_1$.

We consider that the shell is meridionally symmetrically reinforced with two families of fibers ($\alpha_1 = -\alpha_2 = \alpha$) made of the same isotropic material ($\lambda_{11} = \lambda_{21} = \lambda_{12} = \lambda_{22} = \lambda_1$) and wound with the same density ($\omega_1 = \omega_2 = \omega$). In this case, by virtue of (38), the thermal conductivity $\Lambda_{11}$ in (53) has the form

$$\Lambda_{11} = \cos^2\alpha[2\omega\lambda_1 + (1-2\omega)\lambda_b] + \sin^2\alpha\left[2\omega\lambda_1^{-1} + (1-2\omega)\lambda_b^{-1}\right]^{-1}. \tag{54}$$

By taking into account the most widespread modern methods of winding or placement of the reinforcement in composite shells, we require that fibers preserve areas of cross sections along their trajectories. The condition of constancy of cross sections of fibers for thin shells of revolution with axially symmetric structure of the reinforcement is [5]

$$H_2 \omega \cos\alpha = \omega_* = \text{const}, \tag{55}$$

where $\omega_*$ determines, up to a factor, the total area of cross sections of fibers of the chosen family. For the edge $x_1 = y_3^0$, where fibers enter the shell, condition (55) is reduced to

$$H_2(x_1)\omega(x_1)\cos\alpha(x_1) = H_2(y_3^0)\omega_0 \cos\alpha(y_3^0) = \text{const}, \tag{56}$$

where $\omega_0$ is the initial value of $\omega$ at the edge $x_1 = y_3^0$. In what follows, we suppose that the shell is wound at a constant angle of reinforcement $\alpha = \text{const}$. By using relations (52), we obtain from (56) that

$$\omega(x_1) = y_3^0 x_1^{-1} \omega_0 \quad \text{or} \quad \omega(z) = z^{-1} \omega_0. \tag{57}$$

Here, $z = x_1 (y_3^0)^{-1}$ is a new dimensionless variable. In this notation, the edges of the shell are defined as $z = z_0 = y_3^0 (y_3^0)^{-1} = 1$ and $z = z_1 = y_3^1 (y_3^0)^{-1} > 1$. Thus, if the angle of reinforcement is not equal to $\pi/2$, then the intensity of reinforcement in a thin conic shell is determined by equality (57). For a circular winding, $\alpha = \pi/2$. Therefore, $\cos\alpha = 0$ and Eq. (55) has no sense. In this case, the function $\omega(z)$ satisfying the physical constraints (35) should be specified arbitrarily.

We construct solutions of the heat-conduction problem for various types of winding. Let the reinforcement of a shell be circular ($\alpha = \pi/2$), and let the function $\omega$ have a linear distribution in $z$:

$$\omega = Az + B, \quad A = (\overline{\omega}_1 - \overline{\omega}_0)(z_1 - z_0)^{-1}, \quad B = (z_1 \overline{\omega}_0 - z_0 \overline{\omega}_1)(z_1 - z_0)^{-1}, \tag{58}$$

where $\overline{\omega}_0$ and $\overline{\omega}_1$ are the values of the function $\omega$ on the edges $z = z_0, z_1$, respectively. In this case, with regard for (54) and (58), a solution of the heat-conduction equation

$$z \Lambda_{11} T' = q_0 = \text{const} \tag{59}$$

(the prime denotes differentiation with respect to $z$) has the form

$$T = \frac{2A(\lambda_b - \lambda_1)(z-1) + [\lambda_1 + 2B(\lambda_b - \lambda_1)]\ln z}{2A(\lambda_b - \lambda_1)(z_1 - 1) + [\lambda_1 + 2B(\lambda_b - \lambda_1)]\ln z_1} (\Theta_1 - \Theta_0) + \Theta_0, \tag{60}$$

where $\Theta_0$ and $\Theta_1$ are the boundary values of the temperature on the edges $z = z_0, z_1$, respectively. If the density of reinforcement $\omega$ is uniformly distributed over the shell ($\omega = \overline{\omega}_0 = \overline{\omega}_1 = \text{const}$), then (58) implies that $A = 0$, and relation (60) takes the form

$$T = (\Theta_1 - \Theta_0)\ln^{-1} z_1 \ln z + \Theta_0.$$

That is, in the case of a circular winding for $\omega = \text{const}$, the temperature is independent of the thermal characteristics of the phases of the composition and volume content of the reinforcement as in the case of the isotropic material of the shell.

We consider the shell reinforced along meridional directions ($\alpha = 0$). In this case, with regard for (54) and (57), a solution of the heat-conduction equation (59) has the form

$$T = \ln\left(\frac{\lambda_b z + 2(\lambda_1 - \lambda_b)\omega_0}{\lambda_b + 2(\lambda_1 - \lambda_b)\omega_0}\right) \ln^{-1}\left(\frac{\lambda_b z_1 + 2(\lambda_1 - \lambda_b)\omega_0}{\lambda_b + 2(\lambda_1 - \lambda_b)\omega_0}\right)(\Theta_1 - \Theta_0) + \Theta_0. \tag{61}$$

For the shell with spiral winding at an angle $\alpha = \pi/4$, by using (54) and (57), we obtain the following solution of Eq. (59):

$$T = Q_0 \Bigg\{ \ln\left[2\lambda_1 \lambda_b z^2 + 2(\lambda_1 - \lambda_b)^2 \omega_0 z - 4(\lambda_1 - \lambda_b)^2 \omega_0^2\right] - \frac{\lambda_1 - 3\lambda_b}{\sqrt{(\lambda_1 - \lambda_b)^2 + 8\lambda_1 \lambda_b}} \ln\left|\frac{4\lambda_1 \lambda_b z + 2(\lambda_1 - \lambda_b)^2 \omega_0 - \sqrt{D}}{4\lambda_1 \lambda_b z + 2(\lambda_1 - \lambda_b)^2 \omega_0 + \sqrt{D}}\right| \Bigg\} + Q_1, \tag{62}$$

where $D = 4\omega_0^2(\lambda_1 - \lambda_b)^2[(\lambda_1 - \lambda_b)^2 + 8\lambda_1\lambda_b] > 0$, $Q_0$ and $Q_1$ are the constants of integration determined from the thermal boundary conditions: by using the temperatures $T(z_0) = \Theta_0$ and $T(z_1) = \Theta_1$ on the edges or the thermal flow $q(z_0)$ and temperature $T(z_0)$ (the expressions for $Q_0$ and $Q_1$ are omitted here in view of their awkwardness).

Distributions of the temperature in a thin metal-composite conic shell made of copper ($\lambda_b = 400$ W/m · deg, $c_b = 419$ J/kg · deg, and $\rho_b = 8940$ kg/m$^3$ [14]) reinforced with steel fibers ($\lambda_1 = 45$ W/m · deg, $c_1 = 568$ J/kg · deg, and $\rho_1 = 7780$ kg/m$^3$) are shown in Fig. 1. The faces of the shell are thermally isolated, and the thermal flow $q_0$ characterized by the value $y_3^0 \cos^{-1}\psi\, q_0 = 14{,}000$ W/m and the temperature $T(z_0) = 300°C$ are given on the edge $z_0 = 1$. Curve 1 corresponds to the structure with circular ($\alpha = \pi/2$) and uniform windings of the reinforcement ($\omega = 14.4/99$). Curve 2 corresponds to a circular winding with linear distribution of the intensity of the reinforcement: moreover, $2\overline{\omega}_0 = 0.015$ and $2\overline{\omega}_1 = 0.22$. Curves 3–5 describe distributions of the temperature in the shell with circular ($\alpha = \pi/2$), meridional ($\alpha = 0$), and spiral ($\alpha = \pi/4$) structures of reinforcement, respectively, for the reinforcement density defined by (57) with the initial condition $2\omega_0 = 0.8$. The condition of equal total consumption of the reinforcement in these shells defined by the relation

$$\Omega = 2h \int_0^{2\pi} dx_2 \int_{y_3^0}^{y_3^1} 2\omega(x_1) H_1 H_2\, dx_1 = 8\pi h (y_3^0)^2 \tan\psi \cos^{-1}\psi \int_{z_0}^{z_1} \omega(z) z\, dz$$

is taken as a criterion of comparability for these projects.

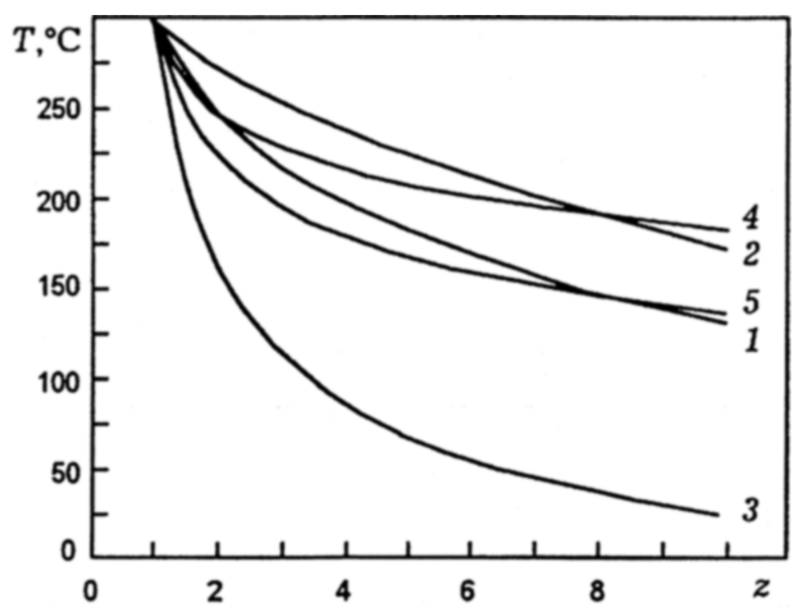


Fig. 1

Comparison of curves 1–3 shows that the intensity of reinforcement has a considerable effect on the temperature distribution for the same direction of winding ($\alpha = \pi/2$). Comparison of curves 4 and 5 with curves 1 and 2 is useful for the visualization of the behavior of $T$ with variation in the entire structure of reinforcement (the direction and density of reinforcement). Comparison of curves 3–5 illustrates the variation in $T$ versus the angle of reinforcement for the same density (57). If we calculate the total amount of heat accumulated by a shell with various structures of reinforcement by the formula

$$Q = 2h \int_0^{2\pi} dx_2 \int_{y_3^0}^{y_3^1} C T H_1 H_2 dx_1 = 4\pi h (y_3^0)^2 \tan\psi \cos^{-1}\psi \int_{z_0}^{z_1} C(z) T(z) z dz,$$

where the quantity $C$ is known from (33), we make sure that the structures corresponding to curves 2 and 3 accumulate, respectively, the maximum and minimum amount of heat $Q$, which is well seen in Fig. 1.

Thus, the solutions of the stationary heat-conduction problem (60)–(62) for a thin conic shell and the curves displayed in Fig. 1 show that the temperature field in the shell considerably depends on the structure of reinforcement ($\alpha$ and $\omega$) and thermal characteristics of the composition phases ($\lambda_1$ and $\lambda_b$). Therefore, by varying the structure of reinforcement, we can formulate problems of purposeful control by using various thermal-physical criteria. Below, we consider one of these problems of rational reinforcement.

### 5. Design of Thermally "Transparent" Thin Shells

We consider the stationary variant of Eq. (50),

$$\sum_{i=1,2} \left[ H_1 H_2 H_i^{-1} \sum_{j=1,2} (H_j^{-1} \Lambda_{ij} T_{0,j}) \right]_{,i} = -H_1 H_2 W, \tag{63}$$

under conditions (49). This case of the distribution of temperature is of interest because the thermal flow $q_3$ in the transverse direction does not change across the thickness of a shell ($q_{3,3} = 0$). In addition, the amount of heat entering the shell through one lateral surface completely leaves the shell through another lateral surface of it [as demonstrated by the left-hand sides of relations (49)]. That is, the shell is as if "transparent" for the thermal flow in the transverse direction. However, the specific feature of system (49), (63) lies in the fact that three equations contain only two unknown functions, $T_0$ and $T_1$, which determine the temperature in the shell. This means that the system is overdetermined. To close this system, we suppose that the thermal conductivity $\Lambda_{33}$ is an unknown function that must be found in the process of solution. Since $\Lambda_{33}$ is expressed in terms of the intensities of reinforcement $\omega_k$ according to relation (38), we can speak in this case about a solution of the problem of rational reinforcement. Moreover, the condition of "transparency" of a thin shell with respect to the thermal flow in the transverse direction serves as a criterion of rational design.

To obtain the system of solving equations for the problem on rational reinforcement, we transform system (49) to the form

$$\Lambda_{33}(\mu_+ - \mu_-) T_1 = \mu_+ \mu_- (2T_0 - T_{+\infty} - T_{-\infty}),$$

$$\Lambda_{33}(\mu_+ + \mu_-) T_1 = \mu_+ \mu_- (-2h T_1 + T_{+\infty} - T_{-\infty}). \tag{64}$$

First, we assume that $\mu_+ = \mu_- = \mu$. Then, from the first equation of system (64), we obtain the following expression for $T_0$:

$$T_0 = \frac{(T_{+\infty} + T_{-\infty})}{2}. \tag{65}$$

If $T_0 \neq \text{const}$, then the substitution of (65) into (63) implies the equation of the first order with respect to the reinforcement parameters $\omega_k$ and $\alpha_k$:

$$\sum_{i=1,2}\left[H_1 H_2 H_i^{-1} \sum_{j=1,2}(H_j^{-1}\Lambda_{ij}(T_{+\infty}+T_{-\infty})_{,j}\right]_{,i} = -2H_1 H_2 W, \tag{66}$$

where $\Lambda_{ij}$ are expressed in terms of $\omega_k$ and $\alpha_k$ by relations (38). To close Eq. (66), it is necessary to impose additional constraints on the reinforcement parameters, for example, the condition of constancy of cross sections of fibers. It has the form $\operatorname{div}(\omega_k l_k) = 0$ [3] in the general case, where $l_k$ is the unit vector setting the direction of reinforcement with fibers of the $k$ th family. For thin shells, this condition acquires the form

$$(H_2 \omega_k \cos\alpha_k)_{,1} + (H_1 \omega_k \sin\alpha_k)_{,2} = 0, \qquad k = \overline{1, N}. \tag{67}$$

For $N = 1$, the system of two equations of the first order (66) and (67) is closed with respect to two functions $\omega_1$ and $\alpha_1$. For $N = 2$, this system is underdetermined because it consists of three equations and contains four reinforcement parameters $\omega_k$ and $\alpha_k$, where $k = 1, 2$. In this case, we can close the system by assuming that one of the reinforcement parameters, e.g., $\alpha_1$, is a known function or by specifying additional conditions that relate the reinforcement parameters. For example, one can place fibers along symmetric trajectories

$$\alpha_2 = -\alpha_1 \tag{68}$$

or along orthogonal directions

$$\alpha_2 = \alpha_1 + \frac{\pi}{2}. \tag{69}$$

For $N \geq 3$, to close system (66), (67), it is necessary to use conditions of the type (68) and (69) and to arbitrarily specify a part of the reinforcement parameters. For fibers with the area of cross sections varying with length, we can impose constraints of the type (68) and (69) on the trajectories of reinforcement and quite arbitrarily set the intensities of reinforcement provided that they satisfy the physical constraints (35) and Eq. (66).

Suppose that system (66), (67) is already integrated with respect to $\omega_k$ and $\alpha_k$. Then (38) implies the expression for $\Lambda_{33}$, and the second equation in (64) gives

$$T_1 = \mu^2 (T_{+\infty} - T_{-\infty})(2\mu\Lambda_{33} + \mu^2 H)^{-1}. \tag{70}$$

Thus, for $\mu_+ = \mu_- = \mu$ and $T_0 \neq \text{const}$, the problem of rational reinforcement is reduced to the system of equations (65)–(67), (38), and (70).

For $T_0 = \text{const}$, Eq. (66) is identically true only for $W = 0$. In this case, an arbitrary reinforcement satisfies the conditions of rational design and $T_1$ is determined from (70) for arbitrary $\Lambda_{33}$ depending only on the structure of reinforcement.

***Remark 1.*** In the above-mentioned case, $\mu_+ = \mu_- = \mu$, and the function $T_0$ is uniquely determined from relation (65); moreover, the boundary conditions for it on the edges of the shell are not considered. This means that local effects revealed near the edges should actually be neglected in this case, which is admissible if $T_0$ determined by relation (65) slightly differs on the edges from the boundary values. The mentioned specific features are absent if the thermal flows are given on the edges or thermal exchange is realized by the Newton law because, in this case, the thermal boundary conditions pass into the boundary conditions for reinforcement parameters.

Now let $\mu_+ \neq \mu_-$. In this case, by using the second equation in (64), we obtain the following dependence of $T_1$ on $\Lambda_{33}$:

$$T_1 = \mu_+\mu_-(T_{+\infty} - T_{-\infty})[\Lambda_{33}(\mu_+ + \mu_-) + \mu_+\mu_- H]^{-1}.$$

To determine the dependence of $T_0$ on $\Lambda_{33}$, we use the first equation in (64) and the equation

$$\mu_+(T_0 + T_1 h - T_{+\infty}) = -\mu_-(T_0 - T_1 h - T_{-\infty}),$$

which follows from (49). As a result, we obtain

$$(\mu_+ - \mu_-)hT_1 = \mu_+ T_{+\infty} + \mu_- T_{-\infty} - (\mu_+ + \mu_-)T_0. \tag{71}$$

By using Eq. (71) and the first equation in (64), we obtain

$$(\mu_+ - \mu_-)T_1 = \Lambda_{33}^{-1}\mu_+\mu_-(2T_0 - T_{+\infty} - T_{-\infty}) = h^{-1}\left[\mu_+ T_{+\infty} + \mu_- T_{-\infty} - (\mu_+ + \mu_-)T_0\right].$$

This implies that

$$T_0 = \frac{\Lambda_{33}(\mu_+ T_{+\infty} + \mu_- T_{-\infty}) + h\mu_+\mu_-(T_{+\infty} + T_{-\infty})}{H\mu_+\mu_- + (\mu_+ + \mu_-)\Lambda_{33}}, \qquad H = 2h, \tag{72}$$

where $\Lambda_{33}$ has the form (38). By substituting the last relation into the heat-conduction equation (66), we obtain the differential equation of the second order with respect to $\omega_k$ and of the first order with respect to $\alpha_k$:

$$\sum_{i=1,2}\left\{H_1 H_2 H_i^{-1} \sum_{j=1,2} H_j^{-1}\Lambda_{ij}\left[\frac{\Lambda_{33}(\mu_+ T_{+\infty} + \mu_- T_{-\infty}) + h\mu_+\mu_-(T_{+\infty} + T_{-\infty})}{H\mu_+\mu_- + (\mu_+ + \mu_-)\Lambda_{33}}\right]_{,j}\right\}_{,i} = -H_1 H_2 W,$$

$$\Lambda_{33} = \left(\lambda_b^{-1} a + \sum_k \lambda_k^{-1}\omega_k\right)^{-1}. \tag{73}$$

By supplementing Eq. (73) with the conditions of constancy of cross sections of fibers (67), we obtain the system of equations for the parameters of rational reinforcement. However, unlike the case $\mu_+ = \mu_-$, the boundary conditions for $T_0$ on the edges of the shell must be taken into account in this case. Owing to relation (72), these conditions determine the boundary values for $\Lambda_{33}$ and, what is the same, for $\omega_k$.

Now we consider a solution of the problem of rational reinforcement (73), (67) for thin shells of revolution with axially symmetric structure of reinforcement and axially symmetric distribution of temperature. Let the variable $x_1$ specify the meridional direction. Then Eq. (73) has the form

$$\frac{d}{dx_1}\left(\frac{h\mu_+\mu_-(T_{+\infty} + T_{-\infty}) + (\mu_+ T_{+\infty} + \mu_- T_{-\infty})[\lambda_b^{-1}(1 - 2\omega_1) + \lambda_{12}^{-1}2\omega_1]^{-1}}{H\mu_+\mu_- + (\mu_+ + \mu_-)[\lambda_b^{-1}(1 - 2\omega_1) + \lambda_{12}^{-1}2\omega_1]^{-1}}\right) = \Lambda_{11}^{-1}R^{-1}(x_1)H_1(x_1)Q_0 \tag{74}$$

for $N = 2$ $W = 0$, $\alpha_2 = -\alpha_1$, $\omega_1 = \omega_2$, and $\lambda_{k1} = \lambda_{k2}$, $k = 1, 2$, where

$$\Lambda_{11} = [\lambda_b(1 - 2\omega_1) + \lambda_{11}2\omega_1]\cos^2\alpha_1 + \sin^2\alpha_1\left[\lambda_b^{-1}(1 - 2\omega_1) + \lambda_{12}^{-1}2\omega_1\right]^{-1}, \tag{75}$$

$R(x_1)$ is the distance from points of the reference surface of the shell to the axis of revolution, and $Q_0$ is the constant of integration that has the meaning of a thermal flow multiplied by $R(x_1)$. We supplement Eq. (74) with the condition of constancy of cross sections of fibers:

$$R(x_1)\omega_1 \cos\alpha_1 = \omega_{*1} = \text{const}. \tag{76}$$

By using (75) and (76), we obtain the following relation for $\Lambda_{11}$:

$$\Lambda_{11} = [\lambda_b(1-2\omega_1) + \lambda_{11} 2\omega_1]\omega_{*1}^2 (R\omega_1)^{-2} + [1 - \omega_{*1}^2 (R\omega_1)^{-2}][\lambda_b^{-1}(1-2\omega_1) + \lambda_{12}^{-1} 2\omega_1]^{-1}. \tag{77}$$

With regard for (77), the differential equation (74) contains only one unknown function $\omega_1$, and the two-point boundary-value problem for it can be approximately solved by the shooting method. If the thermal flow $q_0$ and temperature $T_0(x_1^0) = T_{in}$ are given on the edge $x_1 = x_1^0$, then $Q_0 = -q_0 R^{-1}(x_1^0)$ in Eq. (74) and the initial condition for $\omega_1$ is defined by the equality following from (72):

$$(\lambda_{12}^{-1} - \lambda_b^{-1})\omega_1 = \frac{\mu_+ T_{+\infty} + \mu_- T_{-\infty} - (\mu_+ + \mu_-)T_0}{h\mu_+\mu_-(2T_0 - T_{+\infty} - T_{-\infty})} - \lambda_b^{-1}, \tag{78}$$

where $x_1 = x_1^0$ and $T_0 = T_{in}$. If $T_0$ has the same value $T_0(x_1^0) = T_0(x_1^1) = T_{in}$ on both edges of the shell ($x_1 = x_1^0$ and $x_1 = x_1^1 > x_1^0$), then, in the absence of internal heat sources ($W = 0$), the solution of the heat-conduction equation (63) is a constant function, $T_0 = T_{in} = \text{const}$. This follows from the maximum principle for elliptic differential equations of the second order and the fact that the operator on the left-hand side of Eq. (63) is elliptic for arbitrary reinforcement parameters satisfying the physical constraints (35). In this case, $T_0$ is known, and, by using relation (78), we can determine $\omega_1$ for the entire shell. The expression for the angles of reinforcement $\alpha_1$ follows from (76). Thus, for $T_0 = \text{const}$, the problem of rational reinforcement is solved analytically in the axially symmetric statement.

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