Journal of Mathematical Sciences, Vol. 99, No. 5, 2000

ON FUNDAMENTAL SOLUTIONS OF THE CAUCHY PROBLEM FOR A CLASS OF DEGENERATE PARABOLIC EQUATIONS

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UDC 517.956.4

We present the results of an investigation and some applications of fundamental solutions of the Cauchy problem for a new class of parabolic equations. In these equations: (i) there exist three groups of spatial variables, one basic and two auxiliary, (ii) different weights of spatial variables from the basic group with respect to the time variable are admitted, (iii) degeneracies in variables from the auxiliary groups are present, (iv) a degeneracy on the initial hyperplane is present.

In the theory of the Cauchy problem for parabolic equations (and systems of equations), one of the most important notions is the notion of fundamental solution. At present, the most precise and complete results for the fundamental solution of the Cauchy problem are obtained in the case of equations uniformly parabolic by Petrovsky (where all the spatial variables have the same status and the same weight $2b$ with respect to the time variable). These results were generalized to the following cases: Eidelman $\overrightarrow{2b}$-parabolic equations, where each spatial variable can have its own vector parabolic weight $\overrightarrow{2b}=(2b_1,\dots,2b_n)$ [7, 11, 13], degenerate parabolic equations which generalize the Kolmogorov classical equation of diffusion with inertia [6, 8–10, 12, 16], equations parabolic by Petrovsky, $\overrightarrow{2b}$-parabolic equations, and degenerate equations of the Kolmogorov type which have certain degeneracies on the initial hypersurface [1–4].

Recently, S. D. Eidelman and one of the authors [5, 15] defined and started the investigation of a new class of parabolic equations, namely, degenerate equations of the Kolmogorov type with $\overrightarrow{2b}$-parabolic part in the main group of variables. In these equations, the definitions of $\overrightarrow{2b}$-parabolicity and the structure of equations of the Kolmogorov type were generalized. Moreover, these equations can also be pseudodifferential.

In this paper, we consider equations from this new class in the case where the coefficients of equations do not depend on the spatial variables and degeneracies are present on the initial hypersurface. We construct the fundamental solution of the Cauchy problem, investigate its properties, and present theorems on well-defined solvability of the Cauchy problem and on the integral representation of solutions for homogeneous equations with weak degeneracy on the initial hypersurface.

1. We use the following notation: $n_1, n_2, n_3, b_1,\dots, b_{n_1}$ are given natural numbers and, furthermore, $n_1\ge n_2\ge n_3$, $N\equiv n_1+n_2+n_3$, $\overrightarrow{2b}\equiv(2b_1,\dots,2b_{n_1})$, $q_j\equiv 2b_j/(2b_j-1)$, $1\le j\le n_1$, $\mathbb{Z}_+^r$ is the set of all r-dimensional multiindices,

$$\|m_1\|\equiv\sum_{j=1}^{n_1}\frac{m_{1j}}{2b_j}\quad\text{if}\quad m_1\equiv(m_{1j},1\le j\le n_1)\in\mathbb{Z}_+^{n_1},$$

$$M_m\equiv\sum_{l=1}^{3}\sum_{j=1}^{n_l}\left(l-1+\frac{1}{2b_j}\right)(m_{lj}+1)\quad\text{if}\quad m\equiv(m_{lj},1\le j\le n_l,\ 1\le l\le 3)\in\mathbb{Z}_+^{N},$$

$$\{X\equiv(x_1,x_2,x_3),\ \Xi\equiv(\xi_1,\xi_2,\xi_3\}\subset\mathbb{R}^N\quad\text{if}\quad\{x_l\equiv(x_{lj},1\le j\le n_l),\ \xi_l\equiv(\xi_{lj},1\le j\le n_l)\}\subset\mathbb{R}^{n_l},\quad 1\le l\le 3,$$

Pidstryhach Institute of Applied Problems in Mechanics and Mathematics, Ukrainian Academy of Sciences, L'viv; Ternopil' Academy of Economics, Ternopil'. Translated from Matematychni Metody ta Fizyko-Mekhanichni Polya, Vol. 41, No. 2, pp. 13–19, April–June, 1998. Original article submitted April 2, 1998.

$$\partial_{x_1}^{m_1} \equiv \prod_{j=1}^{n_1} \partial_{x_{1j}}^{m_{1j}}, \quad \partial_X^m \equiv \prod_{l=1}^{3} \prod_{j=1}^{n_l} \partial_{x_{lj}}^{m_l} \quad \text{if} \quad x_1 \in \mathbb{R}^{n_1}, \ X \in \mathbb{R}^N, \ m_1 \in \mathbb{Z}_+^{n_1}, \ m \in \mathbb{Z}_+^N,$$

$$B(t, \tau) \equiv \int_\tau^t \frac{\beta(\gamma)}{\alpha(\gamma)} d\gamma, \quad X_{lj}(t, \tau) \equiv \sum_{r=1}^{l-1} \frac{1}{r!}(B(t, \tau))^r x_{(l-r)j}, \quad 1 \le j \le n_l, \quad 1 \le l \le 3,$$

$$X(t, \tau) \equiv (X_{lj}(t, \tau), \ 1 \le j \le n_l, \ 1 \le l \le 3),$$

$$E_c(t, X; \tau, \Xi) \equiv \exp\left\{ -c \sum_{l=1}^{3} \sum_{j=1}^{n_l} (B(t, \tau))^{1 - l q_j} \left| X_{lj}(t, \tau) - \xi_{lj} \right|^{q_j} \right\},$$

$$E_c^d(t, X; \tau, \Xi) \equiv E_c(t, X; \tau, \Xi) \exp\left\{ d \int_\tau^t \frac{d\gamma}{\alpha(\gamma)} \right\}, \quad \Pi_H \equiv \left\{ (t, X) \mid t \in H, \ X \in \mathbb{R}^N \right\},$$

T is a given positive number, and i is the imaginary unit.

Consider an equation of the form

$$(Lu)(t, X) \equiv \left(\alpha(t)\partial_t - \beta(t) \left(\sum_{l=2}^{3} \sum_{j=1}^{n_l} x_{(l-1)j} \partial_{x_{lj}} + \sum_{0 < \|m_1\| \le 1} a_{m_1}(t) \partial_{x_1}^{m_1} \right) - a_0(t) \right) u(t, X) = 0, \quad (t, X) \in \Pi_{(0, T]}, \tag{1}$$

where the functions α, β: $[0, T] \to \mathbb{R}$, $a_{m_1}: [0, T] \to \mathbb{C}$, $0 < \|m_1\| \le 1$, a_0: $(0, T] \to \mathbb{C}$, are continuous and such that $\alpha(0)\beta(0) = 0$, $\forall t \in (0, T]$: $\alpha(t) > 0$, $\beta(t) > 0$, where β is monotonically nondecreasing, the differential expression $\partial_t - \sum_{\|m_1\| \le 1} a_{m_1}(t) \partial_{x_1}^{m_1}$, $t \in [0, T]$, is $\overrightarrow{2b}$-parabolic [11, 13], and $\exists A \in \mathbb{R}$ $\forall t \in (0, T]$: $\mathrm{Re}\, a_0(t) \le A$.

2. The fundamental solution of the Cauchy problem for Eq. (1) is defined as the function $Z(t, X; \tau, \Xi)$, $0 < \tau < t \le T$, $\{X, \Xi\} \subset \mathbb{R}^N$, such that the function

$$u(t, X) \equiv \int_{\mathbb{R}^N} Z(t, X; \tau, \Xi) \varphi(\Xi) d\Xi, \quad (t, X) \in \Pi_{(\tau, T]}, \tag{2}$$

is a solution of Eq. (1) which satisfies the condition

$$u(t, X)\big|_{t=\tau} = \varphi(X), \quad X \in \mathbb{R}^N, \tag{3}$$

for any number $\tau \in (0, T)$ and for an arbitrary continuous bounded function φ: $\mathbb{R}^N \to \mathbb{C}$.

One of the basic results of the present paper is the following theorem:

Theorem 1. *The following statements are true:*

(i) there exists the unique fundamental solution $Z(t, X; \tau, \Xi)$, $0 < \tau < t \le T$, $\{X, \Xi\} \subset \mathbb{R}^N$, *of the Cauchy problem for Eq. (1),*

(ii) for the function Z, the formula

$$Z(t, X+iX^*; \tau, \Xi+i\Xi^*) = \Gamma(t, \tau, S)|_{S = V(t, \tau, X, \Xi)+iV(t, \tau, X^*, \Xi^*)},$$

$$0<\tau<t\leq T, \qquad \{X, X^*, \Xi, \Xi^*\} \subset \mathbb{R}^N,$$

is satisfied, where

$$V(t, \tau, X, \Xi) \equiv \left(V_{jk}(t, \tau, X, \Xi),\ 1\leq k\leq n_j,\ 1\leq j\leq 3\right),$$

$$V_{jk}(t, \tau, X, \Xi) \equiv \left(X_{jk}(t, \tau)-\xi_{jk}\right)(B(t, \tau))^{-(j-1)-1/(2b_k)},$$

and $\Gamma(t, \tau, S)$, $S\equiv\left(S_{jk},\ 1\leq k\leq n_j,\ 1\leq j\leq 3\right)\in\mathbb{C}^N$, *with fixed* t *and* τ, *is an entire function of the arguments* S_{jk} *of the orders of growth* $p_{jk}=q_k$ *and of the same orders of decrease for real values of the arguments,*

(iii) the following estimates are correct:

$$\left|\partial_X^m Z(t, X+iX^*; \tau, \Xi+i\Xi^*)\right| + \left|\partial_\Xi^m Z(t, X+iX^*; \tau, \Xi+i\Xi^*)\right|$$

$$\leq C_m (B(t, \tau))^{-M_m} E_c^d(t, X; \tau, \Xi) E_{c_1}(t, X^*; \tau, \Xi^*),$$

$$0<\tau<t\leq T, \qquad \{X, X^*, \Xi, \Xi^*\} \subset \mathbb{R}^N, \qquad m\in\mathbb{Z}_+^N, \tag{4}$$

where $C_m>0$, $c>0$, $c_1>0$, *and* $d\in\mathbb{R}$,

(iv) the fundamental solution Z *has the property of normality, and the following convolution formula is true for this solution:*

$$Z(t, X; \tau, \Xi) = \int_{\mathbb{R}^N} Z(t, X; \gamma, \Lambda) Z(\gamma, \Lambda; \tau, \Xi)\, d\Lambda,$$

$$0<\tau<\gamma<t\leq T, \qquad \{X, \Xi\} \subset \mathbb{R}^N, \tag{5}$$

(v) in the case of weak degeneracy, i.e., if the integral

$$\int_0^T \frac{d\gamma}{\alpha(\gamma)} \tag{6}$$

converges, estimates (4) and equality (5) take place also for $\tau=0$, *and the estimation function* E_c^d *in (4) can be replaced by* E_c.

These results are similar to the results in [4] in the case where $b_1=\ldots=b_n=b$, and also to the results obtained in [11] for $\overrightarrow{2b}$-parabolic equations without degeneracies.

As in the case of degenerate parabolic equations without degeneracy on the initial hypersurface [16, 10], the construction of a fundamental solution of the Cauchy problem for Eq. (1) is performed with the use of the method of Fourier transformation, according to which a solution of problem (1), (3) is sought in the form

$$u(t, X) = \left(F^{-1}[v(t, \Xi)]\right)(t, X), \qquad (t, X) \in \Pi_{(\tau, T]}.$$

To find a function v, we obtain the following Cauchy problem with first-order partial derivatives:

$$\left(\alpha(t)\partial_t + \beta(t)\sum_{l=2}^{3}\sum_{j=1}^{n_l} \xi_{lj}\,\partial_{\xi_{(l-1)j}} - A(t, \xi_1)\right)v(t, \Xi) = 0, \qquad (t, \Xi) \in \Pi_{(\tau, T]},$$

$$v(t, \Xi)\big|_{t=\tau} = F^{-1}[\varphi](\Xi), \qquad \Xi \in \mathbb{R}^N,$$

where

$$A(t, \xi_1) \equiv \beta(t) \sum_{0 < \|m_1\| \le 1} a_k(t)(i\xi_1)^{m_1} + a_0(t).$$

By using the method of characteristics for solving this problem and by performing the necessary transformations, we obtain formula (2) for the solution of problem (1), (3). In this case, the fundamental solution Z is determined by the equality

$$Z(t, X; \tau, \Xi) \equiv (B(t, \tau))^{-M_0}\left(F^{-1}[Q(t, \tau, \Lambda)]\right)(t, \tau, V(t, \tau, X, \Xi)),$$

$$0 < \tau < t \le T, \qquad \{X, \Xi\} \subset \mathbb{R}^N,$$

where

$$Q(t, \tau, \Lambda) \equiv \exp\left\{\int_0^1 \sum_{0 < \|m_1\| \le 1} a_{m_1}(P^{-1}[B(t, \tau)\gamma])\, i^{|m_1|}[B(t, \tau)]^{1 - \|m_1\|}\right.$$

$$\left.\times\left(\lambda_1' + \gamma\lambda_2' + \frac{\gamma^2}{2}\lambda_3\right)^{m_1'}(\lambda_1'' + \gamma\lambda_2'')^{m_1''}(\lambda_1''')^{m_1'''}d\gamma\right\}\exp\left\{\int_\tau^t \frac{a_0(\gamma)}{\alpha(\gamma)}d\gamma\right\},$$

$P(t) \equiv B(t, \tau)$, P^{-1} is the function inverse to P, $\Lambda \equiv (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^N$, $\lambda_l \equiv (\lambda_{lj},\ 1 \le j \le n_l)$, $1 \le l \le 3$, $\lambda_l' \equiv (\lambda_{lj}, 1 \le j \le n_3)$, $\lambda_l'' \equiv (\lambda_{lj},\ n_3 + 1 \le j \le n_2)$, $l = 1, 2$, $\lambda_1''' \equiv (\lambda_{1j}, n_2 + 1 \le j \le n_1)$, $m_1 \equiv (m_1', m_1'', m_1''')$, $|m_1| \equiv m_{11} + \ldots + m_{1n_1}$.

Hence, the investigation of properties of the fundamental solution Z reduces to the investigation of properties of the function Q. By carrying out the latter and using Lemma 1.1 in [14] on the Fourier transformation of entire functions, we prove statements (i)–(iii) of Theorem 1. The proof of statement (iv) is carried out by the standard method [14, 11, 10] with the use of the appropriate Green–Ostrogradskii formula.

3. Properties of the fundamental solution Z allow one to investigate the issue of well-defined solvability of the inhomogeneous equation $Lu = f$ with the initial condition

$$u(t, X)\big|_{t=0} = \varphi(X), \qquad X \in \mathbb{R}^N,$$

in the case of weak degeneracy and without the initial condition if strong degeneracy takes place, i.e., if integral (6) diverges. Similar results for equations parabolic by Petrovsky are presented in [3].

If the degeneracy is weak, then, for Eq. (1), one can obtain results similar to those presented in [2, 4, 9], which refer to integral representations and the description of sets of the initial values of solutions. In order to formulate them, we first present the definitions of the necessary norms and spaces.

Let $0 < c_0 < c$, $\boldsymbol{a} \equiv (a_{lj},\ 1 \le j \le n_l,\ 1 \le l \le 3)$, where c is the constant from estimates (4), and the numbers a_{lj}, $1 \le j \le n_l$, $1 \le l \le 3$, are such that $0 \le a_{lj} < c_0 T^{(2b_j(l-1)+1)/(2b_j-1)}$, $1 \le j \le n_l$, $1 \le l \le 3$,

$$k_{lj}(t, a_{lj}) \equiv c_0 a_{lj}(c_0^{2b_j-1} - a_{lj}^{2b_j-1}(T - B(T,t))^{2b_j(l-1)+1})^{1-q_j}, \quad 0 \le t \le T, \quad 1 \le j \le n_l, \quad 1 \le l \le 3,$$

$$\boldsymbol{k}(t, \boldsymbol{a}) \equiv (k_{lj}(t, a_{lj}),\ 1 \le j \le n_l,\ 1 \le l \le 3).$$

We note that $k_{lj}(t, a_{lj}) \ge k_{lj}(0, a_{lj})$, $t \in [0, T]$, $1 \le j \le n_l$, $1 \le l \le 3$, and the following inequality is satisfied:

$$E_{c_0}(t, X; 0, \Xi)\Psi(0, \Xi) \le \Psi(t, X), \qquad t \in [0, T], \qquad \{X, \Xi\} \in \mathbb{R}^N,$$

where

$$\Psi(t, X) \equiv \exp\left\{ \sum_{l=1}^{3} \sum_{j=1}^{n_l} k_{lj}(t, a_{lj}) \left| X_{lj}(t, 0) \right|^{q_j} \right\}$$

Let $1 \le p \le \infty$ and let $u(t, X)$, $(t, X) \in \Pi_{[0,T]}$, be a given complex-valued function which is measurable in X for every $t \in [0, T]$. For $t \in [0, T]$, we define the norms $\|u(t,\cdot)\|_p^{\boldsymbol{k}(t,\boldsymbol{a})} \equiv \left\| u(t,\cdot)(\Psi(t,\cdot))^{-1} \right\|_{L_p(\mathbb{R}^N)}$ and denote by $L_p^{\boldsymbol{k}(0,\boldsymbol{a})}$, $1 \le p \le \infty$, the space of all measurable functions $\varphi\colon \mathbb{R}^N \to \mathbb{C}$ whose norm $\|\varphi\|_p^{\boldsymbol{k}(0,\boldsymbol{a})}$ is finite. Let $M^{\boldsymbol{k}(0,\boldsymbol{a})}$ be the space of all complex-valued generalized measures μ which are defined on the σ-algebra of Borel sets of the space $\mathbb{R}^N$ and satisfy the condition

$$\|\mu\|^{\boldsymbol{k}(0,\boldsymbol{a})} \equiv \int_{\mathbb{R}^N} (\Psi(0, X))^{-1} \, d|\mu|(X) < \infty,$$

where $|\mu|$ is the total variation of μ.

We also set

$$s_{lj}(t, \boldsymbol{a}) = \sum_{r=l}^{3} \frac{r^{q_j-1}}{(r-l)^{q_j}} \theta(n_r - j)(B(t,0))^{(r-l)q_j} k_{rj}(t, a_{rj}), \qquad 1 \le j \le n_l, \qquad 1 \le l \le 3,$$

$$s(t, \boldsymbol{a}) \equiv (s_{lj}(t, \boldsymbol{a}),\ 1 \le j \le n_l,\ 1 \le l \le 3),$$

where

$$\theta(\tau) = \begin{cases} 1, & \tau \ge 0, \\ 0, & \tau < 0, \end{cases}$$

$$\|u(t,\cdot)\|_p^{s(t,a)} \equiv \left\| u(t,X)\exp\left\{-\sum_{l=1}^{3}\sum_{j=1}^{n_l} s_{lj}(t,a)\left|x_{lj}\right|^{q_j}\right\}\right\|_{L_p(\mathbb{R}^N)}.$$

In the following theorems, we assume that integral (6) converges.

Theorem 2. *For any functions* $\varphi \in L_p^{k(0,a)}$, $1 \le p \le \infty$, *and for any generalized measure* $\mu \in M^{k(0,a)}$, *the formulas*

$$u(t,X) \equiv \int_{\mathbb{R}^N} Z(t,X;0,\Xi)\varphi(\Xi)d\Xi, \tag{7}$$

$$u_0(t,X) \equiv \int_{\mathbb{R}^N} Z(t,X;0,\Xi)d\mu(\Xi), \quad (t,X)\in\Pi_{(0,T]}, \tag{8}$$

define the unique solutions of Eq. (1) in the ball $\Pi_{(0,T]}$ *with the following properties: there exists a constant* $C>0$ *independent of* φ *and* μ *and such that*

$$\forall\, t\in(0,T]:\ \|u(t,\cdot)\|_p^{k(t,a)} \le C\|\varphi\|_p^{k(0,a)},$$

$$\forall\, t\in(0,T]:\ \|u_0(t,\cdot)\|_1^{k(t,a)} \le C\|\mu\|^{k(0,a)},$$

for $1 \le p < \infty$,

$$\lim_{t\to 0}\|u(t,\cdot)-\varphi(\cdot)\|_p^{s(t,a)} = 0,$$

and, for $p=\infty$, $u(t,\cdot)\to\varphi$ *and* $u_0(t,\cdot)\to\mu$ *weakly as* $t\to 0$ *also for function (8), i.e., for any* ψ *from the spaces* $L_1^{-s(T,a)}$ *and* $C_0^{-s(T,a)}$, *respectively, the relations*

$$\lim_{t\to 0}\int_{\mathbb{R}^N}\psi(X)u(t,X)dX = \int_{\mathbb{R}^N}\psi(X)\varphi(X)dX,$$

$$\lim_{t\to 0}\int_{\mathbb{R}^N}\psi(X)u_0(t,X)dX = \int_{\mathbb{R}^N}\psi(X)d\mu(X),$$

are true, where $L_1^{-s(T,a)}$ *is the space of measurable functions* $\psi\colon \mathbb{R}^N\to\mathbb{C}$ *for which the norm*

$$\left\|\psi(X)\exp\left\{\sum_{l=1}^{3}\sum_{j=1}^{n_l} s_{lj}(T,a)\left|x_{lj}\right|^{q_j}\right\}\right\|_{L_1(\mathbb{R}^N)}$$

is bounded, and $C_0^{-s(T,a)}$ *is the space of continuous functions* $\psi\colon \mathbb{R}^N\to\mathbb{C}$ *such that, for* $|X|\to\infty$,

$$|\psi(X)|\exp\left\{\sum_{l=1}^{3}\sum_{j=1}^{n_l} s_{lj}(T,a)\left|x_{lj}\right|^{q_j}\right\}\to 0.$$

Theorem 3. *Let* u *be a solution of Eq. (1) in the ball* $\Pi_{(0,T]}$ *which satisfies the condition*

$$\forall\, t \in (0, T]\colon \|u(t,\cdot)\|_p^{k(t,a)} \le C \tag{9}$$

for some $C > 0$ *and* $1 \le p \le \infty$. *Then, for* $1 < p \le \infty$, *there exists a unique function* $\varphi \in L_p^{k(0,a)}$ *and, for* $p = 1$, *a unique generalized measure* $\mu \in M^{k(0,a)}$ *such that the solution* u *is represented in the form (7) and (8), respectively.*

Corollary 1. *Theorems 2 and 3 imply the following statements:*

(*i*) *the spaces* $L_p^{k(0,a)}$, $1 < p \le \infty$, *and* $M^{k(0,a)}$ *are the sets of initial values of solutions of Eq. (1) if and only if the solutions satisfy condition (9) for* $1 < p \le \infty$ *and* $p = 1$, *respectively,*

(*ii*) *in order that solutions of Eq. (1) be representable in the form (7) or (8) with* $\varphi \in L_p^{k(0,a)}$, $1 < p \le \infty$, *and* $\mu \in M^{k(0,a)}$, *it is necessary and sufficient that condition (9) be satisfied.*

The proofs of Theorems 2 and 3 are rather cumbersome and require the development of the methods formulated in [2, 7, 9]. Considerable additional difficulties in the reasoning and in estimations are caused by the complicated anisotropy of the functions under investigation.

This work was supported by the State Foundation of Fundamental Research of the State Committee of Science and Technology of Ukraine.

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