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Integral Equations
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BOUNDARY-VALUE PROBLEMS FOR TWO-DIMENSIONAL CANONICAL SYSTEMS

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The two-dimensional canonical system $Jy' = -\ell Hy$ where the nonnegative Hamiltonian matrix function $H(x)$ is trace-normed on $(0, \infty)$ has been studied in a function-theoretic way by L. de Branges in [5]–[8]. We show that the Hamiltonian system induces a closed symmetric relation which can be reduced to a, not necessarily densely defined, symmetric operator by means of Kac' indivisible intervals; cf. [33], [34]. The "formal" defect numbers related to the system are the defect numbers of this reduced minimal symmetric operator. By using de Branges' one-to-one correspondence between the class of Nevanlinna functions and such canonical systems we extend our canonical system from $(0, \infty)$ to a trace-normed system on $\mathbb{R}$, which is in the limit-point case at $\pm\infty$. This allows us to study all possible selfadjoint realizations of the original system by means of a boundary-value problem for the extended canonical system involving an interface condition at 0.

1. Introduction

Consider on $\mathbb{R}$ a two-dimensional canonical system of homogeneous differential equations of the form

$$Jy' = -\ell H(x)y, \quad \text{on } \mathbb{R}. \tag{1.1}$$

It is assumed that $H(x)$ is a real, nonnegative measurable 2×2 matrix function which is trace-normed, i.e. $\operatorname{tr} H(x) = 1$, $x \in \mathbb{R}$, and that J is a 2×2 signature matrix:

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.2}$$

The equation (1.1) gives rise to a selfadjoint realization in the Hilbert space $L^2(H, \mathbb{R})$ provided with the inner product $[f, g] = \int_{\mathbb{R}} g(x)^* H(x) f(x)\, dx$, since the endpoints ∞ and $-\infty$ are in the limit-point case. Let $Q^+(\ell)$ and $Q^-(\ell)$ be the Titchmarsh-Weyl coefficients associated to (1.1) on each of the halflines $\mathbb{R}^+ = (0, \infty)$ and $\mathbb{R}^- = (-\infty, 0)$. These coefficients are uniquely determined by the following property: if the 2×2 matrix function $W(\cdot, \bar{\ell})^*$ is the matrix solution of (1.1) on the halflines $\mathbb{R}^+$ and $\mathbb{R}^-$ with initial values $W(0\pm,\bar{\ell})^* = I$, then

$$W(\cdot,\bar{\ell})^* \begin{pmatrix} 1 \\ -Q^+(\ell) \end{pmatrix}, \quad \text{on } \mathbb{R}^+, \quad W(\cdot,\bar{\ell})^* \begin{pmatrix} 1 \\ Q^-(\ell) \end{pmatrix}, \quad \text{on } \mathbb{R}^-,$$

are square integrable solutions with respect to $H(x)$ near ∞ and $-\infty$, respectively. Now consider the homogeneous system of differential equations (1.1) restricted to the halfline $\mathbb{R}^+$. The selfadjoint realization defined in $L^2(H,\mathbb{R})$ generates on the halfline $\mathbb{R}^+$ a boundary condition at $0+$ of the form

$$(1.3) \qquad \begin{pmatrix} S(\ell) & -1 \end{pmatrix} \begin{pmatrix} f_1(0+) \\ f_2(0+) \end{pmatrix} = 0,$$

where $S(\ell) = Q^-(\ell)$. Conversely, consider the canonical system (1.1) on $\mathbb{R}^+$ with the ℓ-depending boundary condition (1.3), where $S(\ell)$ is any Nevanlinna function (see Section 2). It is well known (see e.g. [42], [43]) that such a boundary value problem gives rise to a generalized resolvent in an exit space determined by $S(\ell)$. Abstract constructions of such exit spaces have been discussed in the literature. However, according to L. de Branges the function $S(\ell)$ is the Titchmarsh-Weyl coefficient $Q^-(\ell)$ of a unique trace-normed canonical system on $\mathbb{R}^-$; see [39], [47]. This means that as a concrete exit space for a selfadjoint realization of the boundary-value problem associated with (1.3) on $\mathbb{R}^+$ one may take the Hilbert space corresponding to the trace-normed canonical system on $\mathbb{R}^-$. Via an orthogonal sum (cf. [19]) this gives rise to a trace-normed canonical system defined on the whole real line $\mathbb{R}$ with an interface condition at 0. Hence, all generalized resolvents associated to the system on the halfline $\mathbb{R}^+$ (equivalently all ℓ-depending boundary value problems of the form (1.3) on $\mathbb{R}^+$) can be described simply by means of a trace-normed canonical system on $\mathbb{R}$. The usual boundary-value problem associated with the canonical system (1.1) on the halfline $\mathbb{R}^+$ is given by

$$(1.4) \qquad \begin{pmatrix} \sin\nu & -\cos\nu \end{pmatrix} \begin{pmatrix} f_1(0+) \\ f_2(0+) \end{pmatrix} = 0,$$

where $-\pi/2 < \nu \leq \pi/2$. In fact, (1.4) gives rise to all (canonical) selfadjoint realizations of the boundary-value problem on $\mathbb{R}^+$. A similar situation holds for the restriction of the canonical system to the interval $\mathbb{R}^-$. A combination of the two boundary-value problems on $\mathbb{R}^-$ and on $\mathbb{R}^+$ yields a boundary-value problem on $\mathbb{R}$ alluded to above with interface conditions at 0 involving the boundary-values

$$\begin{pmatrix} f_1(0+) \\ f_1(0-) \end{pmatrix}, \quad \begin{pmatrix} f_2(0+) \\ -f_2(0-) \end{pmatrix}.$$

The interface conditions are described via Nevanlinna pairs of 2×2 matrices (see Section 6). When the corresponding selfadjoint realizations defined in $L^2(H,\mathbb{R})$ are compressed to $L^2(H,\mathbb{R}^+)$ they generate on the halfline $\mathbb{R}^+$ boundary conditions at $0+$ of the form (1.3), where the function $S(\ell)$ involves the function $Q^-(\ell)$ and the data of the interface conditions. In fact, the interface conditions fall into two categories: one class is parametrized with 2×2

symmetric matrices (t_{ij}), in which case

$$S(\ell) = t_{11} - \frac{|t_{12}|^2}{Q^-(\ell) + t_{22}}, \tag{1.5}$$

and the other class is parametrized with $\tau \in \mathbb{R}$ and $|r_1|^2 + |r_2|^2 = 1$, $r_1 \neq 0$, in which case

$$S(\ell) = \tau(1 + |r_2/r_1|^2) + |r_2/r_1|^2 Q^-(\ell). \tag{1.6}$$

For $|r_2/r_1| = 1$ and $\tau = 0$ in (1.6) the interface conditions provide continuity at 0 and produce the boundary condition (1.3) with $S(\ell) = Q^-(\ell)$ for the canonical system on $\mathbb{R}^+$.

The system (1.1) on the halfline $\mathbb{R}^+$ has been studied by L. de Branges in connection with Hilbert spaces of entire functions [5]–[8]; see also [3], [17]. An operator-theoretic point of view was taken up by B.C. Orcutt [40], by I.S. Kac [33], [34], and later by M.G. Kreĭn and H. Langer [37]. Further results in this direction were obtained by H. Winkler [46]–[49]; see also [39]. The theory of strings as given by I.S. Kac and M.G. Kreĭn [35] is included in the theory of canonical systems. For strings there is an application oriented approach due to H. Dym and H.P. McKean [15], where the theory of de Branges is connected with operator-theoretic methods. Our aim is to give a full operator-theoretic treatment of the system (1.1) on $\mathbb{R}^+$, completing the work of I.S. Kac. We will introduce a closed symmetric relation with the equation on the halfline $\mathbb{R}^+$. In the degenerate case, when $\mathbb{R}^+$ is an H-indivisible interval (see Section 3), this symmetric relation is a selfadjoint purely multivalued relation. In the non-degenerate case, the closed symmetric relation has defect numbers $(1, 1)$, but in general it is multivalued. Reducing the Hilbert space $L^2(H, \mathbb{R}^+)$ by means of this multivalued part we obtain a closed symmetric operator which is completely nonselfadjoint. Its defect numbers coincide with the "formal" defect numbers of (1.1) and its selfadjoint extensions correspond to the boundary-value problem (1.4). The Titchmarsh-Weyl coefficient is a Nevanlinna function; in fact it is the so-called Weyl function in the sense of [10] or a Q-function of the completely nonselfadjoint symmetric operator and a selfadjoint extension determined by the boundary condition $f_1(0+) = 0$. The symmetric operator is nondensely defined precisely when the interval $\mathbb{R}^+$ begins with an H-indivisible interval; in this case the generalized Friedrichs extension, which is the only selfadjoint extension which is not an operator, will be characterized. The boundary conditions (1.4) lead to selfadjoint extensions in $L^2(H, \mathbb{R}^+)$, while boundary conditions of the form (1.3) lead to selfadjoint extensions in $L^2(H, \mathbb{R})$, so that $L^2(H, \mathbb{R}^-)$ acts as an exit space. The corresponding abstract theory is due to A.V. Štrauss; see [42], [43]. In particular, the Nevanlinna pairs alluded to above, induce symmetric extensions of the orthogonal sum of the symmetric relations in $L^2(H, \mathbb{R}^+)$ and $L^2(H, \mathbb{R}^-)$ (see [19], [20], [21]), which lead to the classification in (1.5) and (1.6). The Titchmarsh-Weyl coefficients were originally introduced in the context of Sturm-Liouville problems [44], [45]; an important contribution to the theory can be found in [36]. For the inclusion of these problems in the theory of canonical systems, see [4]. Boundary conditions involving the eigenvalue parameter have been studied by many authors; for further references see [12], [13], [14], [26], [27], [28], [29]. Spectral properties of canonical systems, or equivalently of Nevanlinna functions, can be described in terms of subordinate solutions. An extension of the results of D. J. Gilbert and D. B. Pearson originally stated for Schrödinger operators can be found for canonical systems in [24].

We outline the contents of this paper. Some necessary preliminaries are collected in Section 2. Maximal and minimal relations are introduced in Section 3, see also [13], [14], [40], and the multivalued part of the symmetric minimal relation and the reduction are described by means of Kac' indivisible intervals. The selfadjoint extensions of the minimal relation are given in Section 4; here is also a connection with Q-functions and a description of what happens in the nondensely defined case. In Section 5 the subdivision of the Nevanlinna class as in [25] is characterized in terms of the canonical system. In Section 6 the canonical system is extended from the halfline to a system defined on the whole real line; all selfadjoint realizations and the resulting interface conditions will be determined. To make the paper self-contained proofs of the fundamental results concerning (1.1) are presented in the last section; our considerations lead to a direct elementary proof of the fact that the trace-normalization gives rise to the limit-point case at ∞.

2. Preliminaries

In this section some facts concerning the canonical system of the form

$$Jy' = -\ell H(x)y, \quad \text{on } \mathbb{R}^+, \tag{2.1}$$

will be reviewed. In (2.1) $H(x)$ is a real, nonnegative measurable 2×2 matrix function and J is the signature matrix (1.2). With a solution of (2.1) one means a vector function

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix},$$

whose entries are locally absolutely continuous on $\mathbb{R}^+$, such that

$$Jf'(x) = -\ell H(x)f(x), \quad \text{for a.a. } x > 0.$$

The entries of H are assumed to be locally integrable functions such that

$$\operatorname{tr} H(x) = 1, \quad x \in \mathbb{R}^+. \tag{2.2}$$

This condition is essentially equivalent to the condition $\int_0^\infty \operatorname{tr} H(x)dx = +\infty$, and it gives rise to Weyl's limit-point case at ∞; cf. [6]. Two matrix functions H_1 and H_2 are considered to be equivalent if $H_1(x) = H_2(x)$ a.e. with respect to the Lebesgue measure. Let $L^2(H, \mathbb{R}^+)$ be the Hilbert space of all (equivalence classes of) measurable, almost everywhere finite vector functions $f(x) = (f_1(x)\ f_2(x))^\top$ on $\mathbb{R}^+$ such that

$$\int_0^\infty f(x)^* H(x) f(x)\, dx < \infty,$$

provided with the corresponding inner product. Clearly, $f \in L^2(H, \mathbb{R}^+)$ is equivalent to the null-element if and only if $Hf = 0$ almost everywhere on $\mathbb{R}^+$. With the usual corresponding inner product $L^2(H, \mathbb{R}^+)$ is a Hilbert space. The construction of the space $L^2(H, \mathbb{R}^+)$ and the proof of the completeness of this space in a more general situation was originally given by Kac in [30]. Let the 2×2 matrix function $W(\cdot, \ell)$ be the solution of the initial value problem

$$\frac{dW(x,\ell)}{dx} J = \ell W(x,\ell)H(x), \quad \text{for a.a. } x > 0, \quad W(0+,\ell) = I, \tag{2.3}$$

so that the 2×2 matrix function $W(\cdot, \bar{\ell})^*$ is the solution of the initial value problem

$$J\frac{dW(x,\bar{\ell})^*}{dx} = -\ell H(x)W(x,\bar{\ell})^*, \quad \text{for a.a. } x > 0, \quad W(0+,\bar{\ell})^* = I. \tag{2.4}$$

It follows that for $\ell, \lambda \in \mathbb{C}$

$$W(x,\ell)JW(x,\lambda)^* - J = (\ell - \bar{\lambda})\textstyle\int_0^x W(t,\ell)H(t)W(t,\lambda)^*\,dt, \quad x > 0, \tag{2.5}$$

and in particular, for $\ell \in \mathbb{C}$

$$W(x,\ell)JW(x,\bar{\ell})^* = J, \quad W(x,\bar{\ell})^*JW(x,\ell) = J, \quad x > 0. \tag{2.6}$$

The matrix function $W(\cdot,\ell)$ is entire in $\ell \in \mathbb{C}$ and real, i.e. $\overline{W(\cdot,\bar{\ell})} = W(\cdot,\ell)$. Moreover, $\det W(\cdot,\ell) = 1$ which follows from (2.6) and the normalization in (2.3). In the following theorem the class of Nevanlinna functions is denoted by $\mathbf{N}$. This class consists of all functions $t(\ell)$, holomorphic on $\mathbb{C} \setminus \mathbb{R}$, which satisfy $t(\ell)^* = t(\bar{\ell})$, and $\operatorname{Im} t(\ell)/\operatorname{Im}\ell \geq 0$ for $\ell \in \mathbb{C}\setminus\mathbb{R}$.

Theorem 2.1. *Let $W(\cdot,\ell)$ be the solution of* (2.3). *Then for each $t(\ell) \in \mathbf{N} \cup \{\infty\}$ the limit*

$$Q^+(\ell) = \lim_{x\to\infty} \frac{w_{11}(x,\ell)t(\ell) + w_{12}(x,\ell)}{w_{21}(x,\ell)t(\ell) + w_{22}(x,\ell)}, \quad \ell \in \mathbb{C}\setminus\mathbb{R}, \tag{2.7}$$

exists, is independent of $t(\ell)$, and belongs to $\mathbf{N} \cup \{\infty\}$. Moreover, for each $\ell \in \mathbb{C}\setminus\mathbb{R}$

$$\chi^+(\ell) = \chi^+(\cdot,\ell) = W(\cdot,\bar{\ell})^* \begin{pmatrix} 1 \\ -Q^+(\ell) \end{pmatrix} \in L^2(H,\mathbb{R}^+). \tag{2.8}$$

If $Q^+(\ell)$ is a real constant or ∞, the only solution of (2.1) *which belongs to $L^2(H,\mathbb{R}^+)$ is equivalent to the trivial solution. If $Q^+(\ell)$ is not a real constant, the function in* (2.8) *is the only nontrivial solution of* (2.1) *which belongs to $L^2(H,\mathbb{R}^+)$.*

The function $Q^+(\ell)$ is called the Titchmarsh-Weyl coefficient corresponding to the canonical system (2.1). A proof of this theorem is given in the last section.

Example 2.2. Let the Hamiltonian $H(x)$ be given by

$$H(x) = \xi_\varphi \xi_\varphi^\top, \quad x \in \mathbb{R}^+, \tag{2.9}$$

where ξ_φ denotes

$$\xi_\varphi = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}, \quad 0 \leq \varphi < \pi. \tag{2.10}$$

Then $H(x)$ is trace-normed and has rank 1 on $\mathbb{R}^+$. The solution $W(\cdot,\ell)$ of the corresponding equation (2.3) is given by

$$W(x,\ell) = \begin{pmatrix} 1 - \ell x\cos\varphi\sin\varphi & \ell x\cos^2\varphi \\ -\ell x \sin^2\varphi & 1 + \ell x\cos\varphi\sin\varphi \end{pmatrix}.$$

Hence the Titchmarsh-Weyl coefficient $Q^+(\ell)$ is given by

$$Q^+(\ell) = \cot\varphi.$$

The only solution of (2.1) which belongs to $L^2(H,\mathbb{R}^+)$ is given by

$$\chi^+(\ell) = \chi^+(\cdot,\ell) = W(\cdot,\bar{\ell})^* \begin{pmatrix} 1 \\ -\cot\varphi \end{pmatrix} = \begin{pmatrix} 1 \\ -\cot\varphi \end{pmatrix}.$$

Clearly, $H(x)\chi^+(\ell) = 0$, i.e. the solution $\chi^+(\ell)$ is equivalent to the trivial one.

Define the 2×2 matrix N_ν by

$$N_\nu = \begin{pmatrix} \sin\nu & -\cos\nu \\ \cos\nu & \sin\nu \end{pmatrix}, \quad -\pi/2 < \nu \leq \pi/2, \tag{2.11}$$

which is both unitary and J-unitary, and define the function $Q^{+,\nu}(\ell)$ by

$$Q^{+,\nu}(\ell) = \frac{\tan\nu\, Q^+(\ell) - 1}{Q^+(\ell) + \tan\nu}, \quad \ell \in \mathbb{C} \setminus \mathbb{R}. \tag{2.12}$$

For $\nu = \pi/2$ it coincides with $Q^+(\ell)$. The 2×2 matrix function $W^\nu(\cdot,\ell) = N_\nu W(\cdot,\ell)$ is the unique solution of the initial value problem

$$\frac{dW^\nu(x,\ell)}{dx} J = \ell W^\nu(x,\ell) H(x), \qquad \text{for a.a. } x > 0, \quad W^\nu(0+,\ell) = N_\nu. \tag{2.13}$$

Corollary 2.3. *The function $Q^{+,\nu}(\ell)$ is a Nevanlinna function. For any Nevanlinna function $t(\ell)$*

$$Q^{+,\nu}(\ell) = \lim_{x\to\infty} \frac{w_{11}^\nu(x,\ell)t(\ell) + w_{12}^\nu(x,\ell)}{w_{21}^\nu(x,\ell)t(\ell) + w_{22}^\nu(x,\ell)}, \quad \ell \in \mathbb{C} \setminus \mathbb{R}. \tag{2.14}$$

Moreover, for each $\ell \in \mathbb{C} \setminus \mathbb{R}$

$$\begin{aligned} \chi^{+,\nu}(\ell) = \chi^{+,\nu}(\cdot,\ell) &= W^\nu(\cdot,\bar{\ell})^* \begin{pmatrix} 1 \\ -Q^{+,\nu}(\ell) \end{pmatrix} \\ &= \frac{1}{\cos\nu Q(\ell) + \sin\nu} \chi^+(\ell) \in L^2(H,\mathbb{R}^+). \end{aligned} \tag{2.15}$$

Proof. Rewrite the right side of (2.14) as

$$\frac{w_{11}^\nu(x,\ell)t(\ell) + w_{12}^\nu(x,\ell)}{w_{21}^\nu(x,\ell)t(\ell) + w_{22}^\nu(x,\ell)} = \frac{\dfrac{w_{11}(x,\ell)t(\ell) + w_{12}(x,\ell)}{w_{21}(x,\ell)t(\ell) + w_{22}(x,\ell)} \sin\nu - \cos\nu}{\dfrac{w_{11}(x,\ell)t(\ell) + w_{12}(x,\ell)}{w_{21}(x,\ell)t(\ell) + w_{22}(x,\ell)} \cos\nu + \sin\nu}.$$

Let $x \to \infty$ and apply Theorem 2.1, then (2.14) follows. The definition of $W^\nu(x,\ell)$ and (2.8) imply (2.15). $\square$

Hence, the function $Q^{+,\nu}(\ell)$ is obtained in a similar manner as the function $Q^+(\ell)$, but now relative to the initial condition $W^\nu(0+,\ell) = N_\nu$. From (2.12) it follows that the Titchmarsh-Weyl coefficients $Q^{+,\nu}(\ell)$ and $Q^{+,\mu}(\ell)$ are related by the linear fractional transformation

$$Q^{+,\nu}(\ell) = \frac{Q^{+,\mu}(\ell) + \tan(\nu-\mu)}{1 - \tan(\nu-\mu)\, Q^{+,\mu}(\ell)}, \tag{2.16}$$

see also [9]. The Titchmarsh-Weyl coefficient $Q^{+,\nu}(\ell)$ can be interpreted in terms of an initial value problem of the form (2.3), when the Hamiltonian $H(x)$ is replaced by $N_\nu H(x) N_\nu^*$; cf. [41]. The following observation is sometimes useful; see [37], [46, Lemma 1.16], [47, Lemma 2.2].

Lemma 2.4. *Let two canonical systems have the Hamiltonians $H(x)$ and $\tilde{H}(x)$, and assume that $\tilde{H}(x) = H(x+l)$, $x > 0$. If $W(\cdot, \ell)$ is the solution of the equation* (2.3) *corresponding to $H(x)$, then the Titchmarsh-Weyl coefficients $Q^+(\ell)$ and $\tilde{Q}^+(\ell)$ are related by*

$$Q^+(\ell) = \frac{w_{11}(l,\ell)\tilde{Q}^+(\ell) + w_{12}(l,\ell)}{w_{21}(l,\ell)\tilde{Q}^+(\ell) + w_{22}(l,\ell)}, \quad \ell \in \mathbb{C} \setminus \mathbb{R}. \tag{2.17}$$

3. Maximal and minimal relations

In order to associate with the canonical system (2.1) maximal and minimal linear relations in the Hilbert space $L^2(H, \mathbb{R}^+)$ a closer look at the behaviour of the matrix function $H(x)$ on $\mathbb{R}^+$ is needed. An open subinterval I of $\mathbb{R}^+$ is said to be of positive type, see [40, p.81], if

$$\textstyle\int_I (H(x)e, e)\, dx = 0, \;\; e \in \mathbb{C}^2 \Rightarrow e = 0,$$

and it is called H-indivisible of type φ, $0 \leq \varphi < \pi$, if

$$H(x) = \xi_\varphi \xi_\varphi^{\mathsf{T}}, \qquad \text{for almost all } x \in I,$$

where ξ_φ is given by (2.10). The equation (2.1) is called definite if the whole interval $\mathbb{R}^+$ is of positive type, or equivalently if

$$H(x)e = 0, \qquad \text{for a.a. } x \in \mathbb{R}^+, \; e \in \mathbb{C}^2 \;\; \Rightarrow e = 0. \tag{3.1}$$

The normalization (2.2) implies that either $\operatorname{rank} H(x) = 2$ or $\operatorname{rank} H(x) = 1$, in which case $H(x)$ is an orthogonal projection of the form $H(x) = \xi_{\varphi(x)} \xi_{\varphi(x)}^{\mathsf{T}}$ with

$$\xi_{\varphi(x)} = \begin{pmatrix} \cos\varphi(x) \\ \sin\varphi(x) \end{pmatrix}, \quad 0 \leq \varphi(x) < \pi.$$

The definition (3.1) gives rise to an alternative for the equation (2.1). To prove this alternative the following local result from [37] will be used.

Lemma 3.1. *Let I be an open subinterval of $\mathbb{R}^+$. Then either I is of positive type or I is H-indivisible of type φ for some $0 \leq \varphi < \pi$.*

Proof. Assume that I is not of positive type. Then there exists a nontrivial $e \in \mathbb{C}^2$ for which $\int_I (H(x)e, e)\, dx = 0$ and hence $H(x)e = 0$ almost everywhere on I. This means that $\ker H(x)$ and $\operatorname{ran} H(x)$ and thus also $H(x)$ as a projection onto $\operatorname{ran} H(x)$ are constant a.e. on I, i.e., I is H-indivisible of type φ, $\varphi^{\mathsf{T}} e = 0$. □

Proposition 3.2. *The equation* (2.1) *is either definite or it is of the form* (2.9) *for some $0 \leq \varphi < \pi$.*

Proof. If the equation (2.1) is not definite, then clearly no open subinterval I of $\mathbb{R}^+$ can be of positive type. By Lemma 3.1 this means that each open subinterval I of $\mathbb{R}^+$ must be H-indivisible of type φ_I. However, if two open H-indivisible intervals have a nonempty intersection, their types must coincide. □

The linear relation T^+_{max} in the Hilbert space $L^2(H, \mathbb{R}^+)$ is defined by

$$T^+_{max} = \{\, \{f, g\} \in (L^2(H, \mathbb{R}^+))^2 : \, f \in AC, \; Jf' = -Hg \,\}.$$

This linear relation is made up of pairs of equivalence classes $\{f,g\}$, such that there exist a locally absolutely continuous representative of f, again denoted by f, and a representative of g, again denoted by g, such that $Jf' = -Hg$ a.e. on $\mathbb{R}^+$. Note that $\chi^+(\ell)$ as defined in (2.8) belongs to $\ker(T^+_{max} - \ell)$, i.e. $\{\chi^+(\ell), \ell\chi^+(\ell)\} \in T^+_{max}$. Define the linear relation T^+_{min} as the adjoint of the linear relation T^+_{max} in the graph sense, $T^+_{min} = (T^+_{max})^*$, i.e.,

$$T^+_{min} = \{\, \{f,g\} \in (L^2(H,\mathbb{R}^+))^2 : \langle \{f,g\}, \{h,k\} \rangle = 0 \text{ for all } \{h,k\} \in T^+_{max} \,\},$$

where $\langle \{f,g\}, \{h,k\} \rangle$ denotes $[g,h] - [f,k]$. The relation between T^+_{min} and T^+_{max} depends on the alternative in Proposition 3.2. To show this the following lemma is needed, cf. [33].

Lemma 3.3. *Let $I \subset \mathbb{R}^+$ be an H-indivisible interval of type φ. Assume that f is absolutely continuous and $Jf' = -Hg$ a.e. on I. Then $\xi_\varphi^\top f$ is constant on I.*

Proof. The identity $Jf' = -Hg$ is equivalent to $f' = JHg$. Hence, if the interval I is of type φ it follows from $\xi_\varphi^\top J \xi_\varphi = 0$, that

$$\xi_\varphi^\top f'(x) = \xi_\varphi^\top J \xi_\varphi \xi_\varphi^\top g(x) = 0, \qquad \text{a.e. } x \in I.$$

Therefore $\xi_\varphi^\top f$ is constant on I. □

Proposition 3.4. *Let the equation* (2.1) *be of the form* (2.9). *Then*

$$T^+_{min} = T^+_{max} = \{0\} \oplus L^2(H,\mathbb{R}^+),$$

i.e. T^+_{min} and T^+_{max} are selfadjoint purely multivalued relations.

Proof. Let $\{f,g\} \in T^+_{max}$. Then Lemma 3.3 shows that $\xi_\varphi^\top f$ is constant on $\mathbb{R}^+$. Now $f \in L^2(H,\mathbb{R}^+)$ implies $\xi_\varphi^\top f = 0$, so that $\|f\|_{L^2(H,\mathbb{R}^+)} = 0$. Hence, $T^+_{max} \subset \{0\} \oplus L^2(H,\mathbb{R}^+)$. For the reverse inclusion assume that $g \in L^2(H,\mathbb{R}^+)$, and define

$$f(t) = \int_0^t JH(s)g(s)\,ds, \quad t \in \mathbb{R}^+.$$

Then f is locally absolutely continuous, $\xi_\varphi^\top f(t) = 0$, and $\{f,g\} \in T^+_{max}$. Hence the reverse inclusion is proved. Clearly, $T^+_{max} = \{0\} \oplus L^2(H,\mathbb{R}^+)$ is a selfadjoint relation, so that $T^+_{min} = (T^+_{max})^* = T^+_{max}$. □

In the rest of this section it is assumed that the equation (2.1) is definite. The following lemma goes back to [40].

Lemma 3.5. *Let $I \subset \mathbb{R}^+$ be an interval of positive type. Assume that f is locally absolutely continuous on $\mathbb{R}^+$ and that $Jf' = -Hg$ a.e. on $\mathbb{R}^+$. Then the equation* (2.1) *is definite, and with g fixed, the equivalence class f contains a unique, locally absolutely continuous representative, again denoted by f.*

Proof. The first statement is clear. Suppose that $Jf' = -Hg$, $Jh' = -Hk$, and that f and h, and g and k, respectively, belong to the same equivalence classes. Then

$$J(f' - h') = -H(g-k) = 0, \qquad \text{a.e. on } \mathbb{R}^+,$$

so that $f - h$ is constant on $\mathbb{R}^+$, while $H(f-h) = 0$. Hence, it follows from (3.1) that $f = h$ on $\mathbb{R}^+$. □

This observation shows that it is possible to associate boundary-values with definite equations. The basic result concerning trace-normed definite equations is stated in the next theorem; a proof is given in the last section.

Theorem 3.6. *Let the equation (2.1) be definite. If $\{f,g\},\{h,k\}\in T^+_{max}$, then*

$$\lim_{x\to\infty} h(x)^*Jf(x)=0. \tag{3.2}$$

The relation T^+_{min} is symmetric with defect numbers $(1,1)$. The mapping $\{f,g\}\to f(0+)$ from the graph of T^+_{max} onto $\mathbb{C}$ is a boundary mapping:

$$\langle\{f,g\},\{h,k\}\rangle = h(0+)^*Jf(0+), \tag{3.3}$$

and

$$T^+_{min}=\{\,\{f,g\}\in T^+_{max} : f(0+)=0\,\}. \tag{3.4}$$

Under the assumption that the equation (2.1) be definite, the linear relation T^+_{min} may still be multivalued. A reduction of T^+_{min} by means of its multivalued part $\operatorname{mul} T^+_{min}$ involves the H-indivisible intervals. Define the linear space $L^2_s(H,\mathbb{R}^+)$ of all (equivalence classes of) functions $f\in L^2(H,\mathbb{R}^+)$ such that $H(t)f(t)$ is constant on H-indivisible intervals, cf. [33]. More precisely, if I is an H-indivisible interval of type φ, then $f\in L^2(H,\mathbb{R}^+)$ if and only if $\xi_\varphi^\top f(t)=c_{\varphi,f}$, in which case $H(t)f(t)=c_{\varphi,f}\xi_\varphi$ almost everywhere on I. It follows from Lemma 3.3 that

$$\operatorname{dom} T^+_{max}\subset L^2_s(H,\mathbb{R}^+), \tag{3.5}$$

and in particular

$$\chi^+(\cdot,\ell)=W(\cdot,\bar\ell)^*\begin{pmatrix}1\\ -Q^+(\ell)\end{pmatrix}\in L^2_s(H,\mathbb{R}^+). \tag{3.6}$$

The completeness of $L^2_s(H,\mathbb{R}^+)$ was proved in [33]. It follows also from the next result.

Lemma 3.7. *$L^2_s(H,\mathbb{R}^+)$ is a closed linear subspace of $L^2(H,\mathbb{R}^+)$.*

Proof. Clearly, $L^2_s(H,\mathbb{R}^+)$ is a linear subspace of $L^2(H,\mathbb{R}^+)$. To see that it is closed, let $f_n\in L^2_s(H,\mathbb{R}^+)$, $f\in L^2(H,\mathbb{R}^+)$, and assume that

$$\int_0^\infty (f(t)-f_n(t))^*H(t)(f(t)-f_n(t))\,dt\to 0.$$

Then for each H-indivisible interval I of type φ

$$\int_I |\xi_\varphi^\top f(t)-\xi_\varphi^\top f_n(t)|^2\,dt=\int_I (f(t)-f_n(t))^*H(t)(f(t)-f_n(t))\,dt\to 0.$$

Now $\xi_\varphi^\top f_n(t)=c_n$, $c_n\in\mathbb{C}$, implies that (c_n) is a Cauchy sequence. Hence

$$c_n\to c,\ c\in\mathbb{C},\qquad\text{and}\qquad \int_I |\xi_\varphi^\top f(t)-c|^2\,dt=0.$$

This shows that $f\in L^2_s(H,\mathbb{R}^+)$, and the lemma is proved. □

To describe the (orthogonal) operator part of T^+_{min}, further results are necessary. Some of the technical results below have been greatly influenced by the work [33] of Kac. The next lemma concerns the appearance of H-indivisible intervals.

Lemma 3.8. *Let $f \in \operatorname{dom} T^+_{max}$ and assume that $Hf = 0$ almost everywhere on $\mathbb{R}^+$. If $f(x_0) \neq 0$ for some $x_0 > 0$ or if $f(0+) \neq 0$, then there exists $\varepsilon > 0$ such that*

$$(x_0 - \varepsilon, x_0 + \varepsilon) \quad or \quad (0, \varepsilon), \tag{3.7}$$

respectively, is H-indivisible.

Proof. Since $f \in \operatorname{dom} T^+_{max}$ is continuous there exists $\varepsilon > 0$ such that $f(t) \neq 0$ if $t \in (x_0 - \varepsilon, x_0 + \varepsilon)$. The assumption $H(t)f(t) = 0$ (a.e.) implies

$$H(t) = \xi_{\varphi(t)}\xi^{\top}_{\varphi(t)} \text{ for a.e. } t \in (x_0 - \varepsilon, x_0 + \varepsilon), \tag{3.8}$$

where $0 \leq \varphi(t) < \pi$, so that $H(t)f(t) = 0$ (a.e.) is equivalent to

$$\cos\varphi(t) f_1(t) + \sin\varphi(t) f_2(t) = 0 \text{ for a.a. } t \in (x_0 - \varepsilon, x_0 + \varepsilon). \tag{3.9}$$

It follows from $Jf' = -Hg$ that $f'^{\top} Jf = g^{\top} Hf = 0$, so that

$$f_2'(x) f_1(x) - f_1'(x) f_2(x) = 0 \text{ a.e. on } \mathbb{R}^+. \tag{3.10}$$

If, for instance, $f_1(t) \neq 0$ then (3.10) implies that

$$\frac{f_2(t)}{f_1(t)} = \text{ constant for } t \in (x_0 - \varepsilon, x_0 + \varepsilon). \tag{3.11}$$

Combining (3.9) and (3.11) gives

$$\frac{f_2(t)}{f_1(t)} = -\cot\varphi(t) = \text{ constant for } t \in (x_0 - \varepsilon, x_0 + \varepsilon),$$

so that $\varphi(t)$ is constant, say φ, for $t \in (x_0 - \varepsilon, x_0 + \varepsilon)$. Hence, (3.8) shows that the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ is H-indivisible of type φ. Similarly, one proves that if $f(0+) \neq 0$, then there exists $\varepsilon > 0$ such that the interval $(0, \varepsilon)$ is H-indivisible. □

Introduce the nonnegative number $\kappa \geq 0$ by

$$\kappa = \sup\{\, x > 0 : (0, x) \text{ is } H\text{-indivisible}\,\} \cup \{0\}. \tag{3.12}$$

Then, either $\mathbb{R}^+$ starts with an H-indivisible interval, in which case $\kappa > 0$ is the length of the maximal H-indivisible interval in which it is contained, or $\mathbb{R}^+$ does not start with an H-indivisible interval in which case $\kappa = 0$. Moreover, $\kappa < \infty$ if and only if the equation (2.1) is definite. Note that if $\kappa = \infty$ then $L^2_s(H, \mathbb{R}^+) = \{0\}$.

Corollary 3.9. *Let $f \in \operatorname{dom} T^+_{max}$ and assume that $Hf = 0$ almost everywhere on $\mathbb{R}^+$. Then with κ ($\kappa < \infty$) defined in* (3.12),

$$f(\kappa) = 0. \tag{3.13}$$

Proof. Suppose that $f(\kappa) \neq 0$. Then by Lemma 3.8 there exists $\varepsilon > 0$ such that the interval $(\kappa - \varepsilon, \kappa + \varepsilon)$ when $\kappa > 0$ and the interval $(0, \varepsilon)$ when $\kappa = 0$ is H-indivisible; a contradiction with the definition of κ. □

It follows from (3.5) that $L^2(H, \mathbb{R}^+) \ominus L^2_s(H, \mathbb{R}^+) \subset \operatorname{mul} T^+_{min}$. To show the reverse inclusion a description of $(\operatorname{mul} T^+_{max}) \cap L^2_s(H, \mathbb{R}^+)$ is needed.

Proposition 3.10. *The multivalued part of T^+_{max} satisfies*

$$(\operatorname{mul} T^+_{max}) \cap L^2_s(H, \mathbb{R}^+) = \{\, g \in L^2_s(H, \mathbb{R}^+) : Hg = 0 \ a.e. \ on \ (\kappa, \infty) \,\}. \tag{3.14}$$

Proof. Let $g \in (\operatorname{mul} T^+_{max}) \cap L^2_s(H,\mathbb{R}^+)$, then there exists an element $f \in L^2(H,\mathbb{R}^+)$ such that $\{f,g\} \in T^+_{max}$ and $Hf = 0$ almost everywhere on $\mathbb{R}^+$. The assumption $g \in L^2_s(H,\mathbb{R}^+)$ implies that

$$(3.15) \qquad f(t) = 0, \qquad H(t)g(t) = 0, \qquad \text{a.e. on } (\kappa,\infty).$$

Since $Jf' = -Hg$, the statement (3.15) follows by showing that $f(t) = 0$ for all $t > \kappa$. Assume the converse that $f(x_0) \neq 0$ for some $x_0 > \kappa$. Let (α,β) be the maximal H-indivisible interval with $x_0 \in (\alpha,\beta)$; cf. Lemma 3.8. Since $g \in L^2_s(H,\mathbb{R}^+)$ there exists a constant $c_{\varphi,g} \in \mathbb{C}$ such that $\xi_\varphi^\top g(t) = c_{\varphi,g}$ almost everywhere on (α,β). Furthermore, $f(\alpha) = 0$ and if $\beta < \infty$ then also $f(\beta) = 0$, otherwise (α,β) is not maximal. If $\beta < \infty$, then

$$0 = J(f(\beta) - f(\alpha)) = -\int_\alpha^\beta H(t)g(t)\,dt = -c_{\varphi,g}\xi_\varphi(\beta-\alpha),$$

and hence $c_{\varphi,g} = 0$. If $\beta = \infty$, then $g \in L^2_s(H,\mathbb{R}^+)$ forces $c_{\varphi,g} = 0$. Hence $Hg = 0$ for almost all $t \in (\alpha,\beta)$ and $Jf' = -Hg$ implies that $f' = 0$ for almost all, and $f(t) = 0$ for all, $t \in (\alpha,\beta)$, contradicting $f(x_0) \neq 0$. Hence (3.15) is proved, and this shows that $(\operatorname{mul} T^+_{max}) \cap L^2_s(H,\mathbb{R}^+)$ is contained in the right side of (3.14).

To prove the reverse inclusion, let $g \in L^2_s(H,\mathbb{R}^+)$ be such that $H(t)g(t) = 0$ almost everywhere on (κ,∞). Assume that the H-indivisible interval $(0,\kappa)$ is of type φ, so that $H = \xi_\varphi\xi_\varphi^\top$ on $(0,\kappa)$. Define

$$f(t) = \int_\kappa^t JH(s)g(s)\,ds, \quad t \in \mathbb{R}^+.$$

Then f is absolutely continuous, $Jf' = -Hg$, and $f(t) = f(\kappa) = 0$ for $t \in [\kappa,\infty)$. For $t \in (0,\kappa)$ it follows from the definition that

$$\xi_\varphi^\top f(t) = \int_\kappa^t \xi_\varphi^\top J\xi_\varphi\xi_\varphi^\top g(s)\,ds = 0,$$

so that $Hf = 0$ on $(0,\kappa)$. Thus f is equivalent to the null element in $L^2(H,\mathbb{R}^+)$. Since $\{f,g\} \in T^+_{max}$, this shows that $g \in \operatorname{mul} T^+_{max}$ and completes the proof. □

Corollary 3.11. $L^2(H,\mathbb{R}^+) = L^2_s(H,\mathbb{R}^+) \oplus (\operatorname{mul} T^+_{min})$.

Proof. The statement is equivalent to

$$(3.16) \qquad \operatorname{dom} T^+_{max} \quad \text{is dense in} \quad L^2_s(H,\mathbb{R}^+).$$

To prove this assume that $g \in L^2_s(H,\mathbb{R}^+)$ and that $g \in (\operatorname{dom} T^+_{max})^\perp = \operatorname{mul} T^+_{min}$. Then $Jf' = -Hg$ for some $f \in \operatorname{dom} T^+_{min}$ such that $Hf = 0$ almost everywhere on $\mathbb{R}^+$. Since $T^+_{min} \subset T^+_{max}$ and in particular $\operatorname{mul} T^+_{min} \subset \operatorname{mul} T^+_{max}$, Proposition 3.10 implies that

$$Hg = 0 \text{ a.e. on } (\kappa,\infty).$$

If $\kappa > 0$, then $(0,\kappa)$ is H-indivisible and it follows from $f(0+) = 0$ and (3.13) that

$$0 = J(f(\kappa) - f(0+)) = -c_{\varphi,g}\xi_\varphi\kappa.$$

This gives $c_{\varphi,g} = 0$ so that $Hg = 0$ almost everywhere on $(0,\kappa)$. Therefore $Hg = 0$ almost everywhere on $\mathbb{R}^+$, and (3.16) is proved. □

The multivalued part of T^+_{min} reduces T^+_{min} and T^+_{max}. The corresponding parts in the Hilbert space $L^2_s(H,\mathbb{R}^+)$ are defined by

$$T^+_{min,s} = T^+_{min} \cap (L^2_s(H,\mathbb{R}^+))^2, \quad T^+_{max,s} = T^+_{max} \cap (L^2_s(H,\mathbb{R}^+))^2.$$

The following result is now obtained as a consequence of the above considerations.

Theorem 3.12. *Let the equation* (2.1) *be definite. Then the relation $T^+_{min,s}$ is the (orthogonal) operator part of T^+_{min}. It is a closed symmetric operator in $L^2_s(H,\mathbb{R}^+)$ with defect numbers* $(1,1)$ *whose adjoint is $T^+_{max,s}$:*

$$(3.17) \qquad (T^+_{min,s})^* = T^+_{max,s}.$$

It is densely defined if and only if $\kappa = 0$. If $\kappa > 0$ and the interval $(0,\kappa)$ is of type φ, then $\operatorname{mul} T^+_{max,s}$ is one-dimensional and spanned by $\omega \in L^2_s(H,\mathbb{R}^+)$ of the form

$$(3.18) \qquad \omega(t) = \xi_\varphi \text{ a.e. on } (0,\kappa), \quad \omega(t) = 0 \text{ a.e. on } [\kappa,\infty).$$

Proof. The first part of the theorem is obvious by the above considerations. The statements concerning the domain of $T^+_{min,s}$ and $\operatorname{mul} T^+_{max,s}$ can be seen as follows. The identity (3.17) shows that $T^+_{min,s}$ is densely defined in $L^2_s(H,\mathbb{R}^+)$ if and only if $\operatorname{mul} T^+_{max,s} = \{0\}$. Now the definition of $T^+_{max,s}$ shows that

$$\operatorname{mul} T^+_{max,s} = (\operatorname{mul} T^+_{max}) \cap L^2_s(H,\mathbb{R}^+).$$

Hence it remains to use the description (3.14) given in Proposition 3.10. □

If $T^+_{min,s}$ is densely defined, then all selfadjoint extensions are densely defined, and hence they are operators. If $T^+_{min,s}$ is not densely defined, or equivalently, if $\kappa > 0$, then there is precisely one selfadjoint extension of $T^+_{min,s}$, which is not an operator; cf. [18], [23], [25].

4. Selfadjoint realizations on the halfline

Assume that the equation (2.1) is definite so that $T^+_{min,s}$ is a closed symmetric operator with defect numbers $(1,1)$ in $L^2_s(H,\mathbb{R}^+)$. The canonical selfadjoint extensions of $T^+_{min,s}$ will now be characterized. For $\nu \in (-\pi/2,\pi/2]$ define the solution $w^+_\nu(x,\ell)$ of (2.1) by

$$w^+_\nu(x,\ell) = -\frac{1}{\cos\nu Q^+(\ell) + \sin\nu} W(x,\bar{\ell})^* \begin{pmatrix} \cos\nu \\ \sin\nu \end{pmatrix}.$$

It follows from (2.8) that

$$(4.1) \qquad w^+_\nu(x,\ell) - w^+_\mu(x,\ell) = \chi^+(x,\ell)\frac{\sin(\nu-\mu)}{(\cos\nu Q^+(\ell)+\sin\nu)(\cos\mu Q^+(\ell)+\sin\mu)}.$$

Moreover,

$$(4.2) \qquad \chi^+(x,\ell)w^+_\nu(t,\bar{\ell})^* - w^+_\nu(x,\ell)\chi^+(t,\bar{\ell})^* = W(x,\bar{\ell})^* J W(t,\ell).$$

Theorem 4.1. *There is a one-to-one correspondence between the selfadjoint extensions $A^+(\nu)$ of $T^+_{min,s}$ in $L^2_s(H,\mathbb{R}^+)$ and $\nu \in (-\pi/2, \pi/2]$ via*

$$\operatorname{dom} A^+(\nu) = \{ f \in \operatorname{dom} T^+_{max,s} : \sin\nu\, f_1(0+) = \cos\nu\, f_2(0+) \}. \tag{4.3}$$

The corresponding resolvent operator $(A^+(\nu)-\ell)^{-1}$ of $A^+(\nu)$ is given by

$$\begin{aligned}(A^+(\nu)-\ell)^{-1}h(x) \\ = \chi^+(x,\ell)\textstyle\int_0^x w^+_\nu(t,\bar\ell)^* H(t)h(t)\,dt + w^+_\nu(x,\ell)\int_x^\infty \chi^+(t,\bar\ell)^*H(t)h(t)\,dt,\end{aligned} \tag{4.4}$$

where $h \in L^2_s(H,\mathbb{R}^+)$.

Proof. The parametrization follows from Theorem 3.6; see [2]. To prove the last part denote the right side of (4.4) by $y(x,\ell)$. Clearly $y(\cdot,\ell)$ solves the inhomogenous equation

$$Jy'(x,\ell) = -\ell H(x)y(x,\ell) - H(x)h(x), \qquad \text{for a.a. } x > 0, \tag{4.5}$$

see (4.2), (2.4), and (2.6). Moreover,

$$y(0+,\ell) = -\frac{1}{\cos\nu\, Q^+(\ell) + \sin\nu}\begin{pmatrix}\cos\nu\\ \sin\nu\end{pmatrix}[h,\chi^+(\bar\ell)].$$

Denote the left side of (4.4) by $z(x,\ell)$, then $\{z(\cdot,\ell), \ell z(\cdot,\ell)+h\} \in A^+_\nu \subset T^+_{max}$. Hence, $z(\cdot,\ell)$ also satisfies the inhomogeneous equation (4.5). As $z(\cdot,\ell) \in \operatorname{dom} A^+_\nu$, it satisfies the boundary condition in (4.3) and therefore

$$z(0+,\ell) = \begin{pmatrix}\cos\nu\\ \sin\nu\end{pmatrix} c(\ell),$$

for some scalar $c(\ell)$, whose form follows from

$$\begin{aligned}[h,\chi^+(\bar\ell)] &= \langle \{z,\ell z + h\}, \{\chi^+(\bar\ell), \bar\ell\chi^+(\bar\ell)\}\rangle \\ &= \chi^+(0+,\bar\ell)^* Jz(0+,\ell) = -(\cos\nu\, Q^+(\ell) + \sin\nu)c(\ell),\end{aligned}$$

see Theorem 3.6. This shows that $y(0+,\ell) = z(0+,\ell)$. By the uniqueness of the initial value problem for (4.5) it follows that $z(\cdot,\ell) = y(\cdot,\ell)$. □

The boundary condition (4.3) for the selfadjoint extension $A^+(\nu)$ is usually written as $f_2(0+) = \tan\nu\, f_1(0+)$ when $-\pi/2 < \nu < \pi/2$, and as $f_1(0+) = 0$ when $\nu = \pi/2$. For the proof of the following result a device in [16], cf. [13], will be used.

Corollary 4.2. *The operator $T^+_{min,s}$ is completely nonselfadjoint in $L^2_s(H,\mathbb{R}^+)$.*

Proof. Assume that $h \in L^2_s(H,\mathbb{R}^+)$ is orthogonal to all $\chi^+(\ell)$, $\ell \in \mathbb{C}\setminus\mathbb{R}$. Then it follows from (4.4) and (4.2), that

$$(A^+(\nu)-\ell)^{-1}h(x) = W(x,\bar\ell)^* J \textstyle\int_0^x W(t,\ell)H(t)h(t)\,dt.$$

For each x the function on the right side is entire in ℓ. If $E_\nu(t)$ is the spectral family of $A^+(\nu)$ then it follows from the Stieltjes inversion formula that $[E_\nu(t)h,h] = 0$ for all $t \in \mathbb{R}$. Therefore,

$$\text{s-lim}_{t\to\infty} E_\nu(t)h = 0. \tag{4.6}$$

If $T^+_{min,s}$ is densely defined then (4.6) implies $h = 0$. If $T^+_{min,s}$ is not densely defined, select ν such that $A^+(\nu)$ is an operator extension of $T^+_{min,s}$ and reach the same conclusion. By Kreĭn's criterion (see [22]) this means that $T^+_{min,s}$ is completely nonselfadjoint. □

The notion of the Q-function of a (completely nonselfadjoint) symmetric operator and a selfadjoint extension goes back to M.G. Kreĭn (cf. e.g. [22]). In [10], [11] this notion is connected with abstract boundary value spaces and called a Weyl function.

Theorem 4.3. *The Titchmarsh-Weyl coefficient $Q^+(\ell)$ of* (2.1) *is the Q-function (Weyl function) of $A(\pi/2)$ and $T^+_{min,s}$.*

Proof. Clearly $\chi^+(\ell) \in \ker(T^+_{max,s} - \ell)$ for all $\ell \in \mathbb{C} \setminus \mathbb{R}$, so that

$$\{\chi^+(\ell) - \chi^+(\lambda), \ell\chi^+(\ell) - \lambda\chi^+(\lambda)\} \in T^+_{max,s}.$$

It follows from (3.6) and (4.3) that $\chi^+(\ell) - \chi^+(\lambda) \in \operatorname{dom} A^+(\pi/2)$. Hence

$$\frac{\chi^+(\ell) - \chi^+(\lambda)}{\ell - \lambda} = (A^+(\pi/2) - \ell)^{-1}\, \chi^+(\lambda). \tag{4.7}$$

Using (2.5) and (3.6) we observe that

$$\begin{aligned}
&(\ell - \bar\lambda)[\chi^+(\ell), \chi^+(\lambda)] \\
&= \lim_{x\to\infty} \begin{pmatrix} 1 \\ -Q^+(\lambda) \end{pmatrix}^* (\ell - \bar\lambda) \textstyle\int_0^x W(t,\bar\lambda) H(t) W(t,\bar\ell)^*\, dt \begin{pmatrix} 1 \\ -Q^+(\ell) \end{pmatrix} \\
&= -\lim_{x\to\infty} \begin{pmatrix} 1 \\ -Q^+(\lambda) \end{pmatrix}^* (W(x,\bar\lambda) J W(x,\bar\ell)^* - J) \begin{pmatrix} 1 \\ -Q^+(\ell) \end{pmatrix} \\
&= Q^+(\ell) - Q^+(\lambda)^* - \lim_{x\to\infty} \chi^+(x,\lambda)^* J \chi^+(x,\ell).
\end{aligned}$$

According to Theorem 3.6 the last limit is zero, so that

$$\frac{Q^+(\ell) - Q^+(\lambda)^*}{\ell - \bar\lambda} = [\chi^+(\ell), \chi^+(\lambda)]. \tag{4.8}$$

Together (4.7) and (4.8) show that $Q^+(\ell)$ is the Q-function of $T^+_{min,s}$ and $A(\pi/2)$, cf. [22]. □

Since $T^+_{min,s}$ is completely nonselfadjoint, by Corollary 4.2, the Titchmarsh-Weyl coefficient $Q^+(\ell)$, as Q-function determines $T^+_{min,s}$ and its selfadjoint extension $A(\pi/2)$ uniquely, up to isometric isomorphisms. The canonical system (2.1) on $\mathbb{R}^+$ provides the unique model after the reduction described in the previous section. The selfadjoint extensions of $T^+_{min,s}$ can be parametrized by means of Kreĭn's formula. In the following result the parameters in Kreĭn's formula are related to the boundary conditions in Theorem 4.1.

Proposition 4.4. *The resolvents of $A^+(\nu)$ and $A^+(\pi/2)$ are connected by*

$$(A^+(\nu) - \ell)^{-1} = (A^+(\pi/2) - \ell)^{-1} - \chi^+(\ell)\, \frac{1}{Q^+(\ell) + \tan\nu}\, [\cdot, \chi^+(\bar\ell)], \tag{4.9}$$

when $\ell \in \mathbb{C} \setminus \mathbb{R}$.

Proof. Consider (4.4) for ν, $-\pi/2 < \nu < \pi/2$, and for $\nu = \pi/2$. According to (4.1)

$$w^+_\nu(x,\ell) - w^+_{\pi/2}(x,\ell) = -\frac{1}{Q^+(\ell) + \tan\nu}\, \chi^+(x,\ell).$$

This yields (4.9). □

It is a consequence of Theorem 4.3 that the function $Q^{+,\nu}(\ell)$ given in (2.12) is the Q-function of the selfadjoint extension $A^+(\nu)$ in Theorem 4.1 and the minimal operator $T^+_{min,s}$, for each $\nu \in (-\pi/2, \pi/2]$. To see this, note that it follows from (4.7) and (4.9) that $\chi^{+,\nu}(\ell)$ in (2.15) satisfies

$$\frac{\chi^{+,\nu}(\ell)-\chi^{+,\nu}(\lambda)}{\ell-\lambda} = (A^+(\nu)-\ell)^{-1}\,\chi^{+,\nu}(\lambda),$$

while (4.8) implies

$$\frac{Q^{+,\nu}(\ell)-Q^{+,\nu}(\lambda)^*}{\ell-\bar\lambda} = [\chi^{+,\nu}(\ell), \chi^{+,\nu}(\lambda)].$$

If $T^+_{min,s}$ is not densely defined, there is precisely one selfadjoint extension of $T^+_{min,s}$, which is not an operator. This extension will now be characterized in terms of its Titchmarsh-Weyl coefficient.

Theorem 4.5. *Assume that $\kappa > 0$ and that the type of the H-indivisible interval $(0,\kappa)$ is φ. Then for $\nu \neq \varphi + \pi/2$*

$$\lim_{y\to\infty} \frac{Q^{+,\nu}(iy)}{iy} = 0, \tag{4.10}$$

while for $\nu = \varphi + \pi/2$

$$\lim_{y\to\infty} \frac{Q^{+,\nu}(iy)}{iy} = \kappa. \tag{4.11}$$

The only selfadjoint extension $A^+(\nu)$ of $T^+_{min,s}$, which is not an operator corresponds to $\nu = \varphi + \pi/2$.

Proof. Let $\{f,g\} \in A^+(\nu)$ so that $\sin\nu f_1(0+) = \cos\nu f_2(0+)$. Assume that $Hf = 0$. Then $\cos\varphi f_1(0+) + \sin\varphi f_2(0+) = 0$. If $\nu \neq \varphi + \pi/2$ then $f(0+) = 0$, so that $\{f,g\} \in T^+_{min,s}$ by Theorem 3.6. Since $T^+_{min,s}$ is an operator, $Hg = 0$. This shows that $A^+(\nu)$ is an operator for $\nu \neq \varphi + \pi/2$ and hence (4.10) holds. It remains to prove (4.11). The Titchmarsh-Weyl coefficient $Q^{+,\nu}(\ell)$ in (2.16) corresponding to $\nu = \varphi + \pi/2$ is given by

$$Q^{+,\varphi+\pi/2}(\ell) = \frac{Q^+(\ell)+\tan\varphi}{1-\tan\varphi\, Q^+(\ell)}.$$

According to Lemma 2.4 and Example 2.2, $Q^+(\ell)$ can be written as

$$Q^+(\ell) = \frac{(1-\kappa\ell\cos\varphi\sin\varphi)\tilde Q^+(\ell)+\kappa\ell\cos^2\varphi}{-\kappa\ell\sin^2\varphi\,\tilde Q^+(\ell)+1+\kappa\ell\cos\varphi\sin\varphi}, \tag{4.12}$$

where $\tilde Q^+(\ell)$ corresponds to the Hamiltonian restricted to $[\kappa,\infty)$. A straightforward calculation shows that

$$Q^{+,\varphi+\pi/2}(\ell) = \kappa\ell + \frac{\tilde Q^+(\ell)+\tan\varphi}{1-\tan\varphi\,\tilde Q^+(\ell)} = \kappa\ell + \tilde Q^{+,\varphi+\pi/2}(\ell). \tag{4.13}$$

The system defined on the interval $[\kappa,\infty)$ is either given by Example 2.2 or it is definite. In the first case $\tilde Q^+(\ell) = \cot\alpha$ with $\alpha \neq \varphi$, since the interval $(0,\kappa)$ is maximal of type φ, and then $\tilde Q^{+,\varphi+\pi/2}(\ell)$ is a real constant. In the second case the corresponding minimal operator is nondensely defined if and only if the system on $[\kappa, \infty)$ starts with an indivisible interval of type $\alpha \neq \varphi$. The corresponding exceptional value is $\alpha + \pi/2 \neq \varphi + \pi/2$. Hence, $\tilde{Q}^{+,\varphi+\pi/2}(\ell)$ correspond to an operator extension. Therefore, in both cases

$$\lim_{y\to\infty} \frac{\tilde{Q}^{+,\varphi+\pi/2}(iy)}{iy} = 0, \tag{4.14}$$

cf. [23], and thus (4.11) follows from (4.13). □

Example 4.6. A simple example of a trace-normed canonical system shows already the existence of a nontrivial multivalued part for its minimal relation, cf. [40]. Assume that $\alpha \neq \varphi$ and let $H(x)$ in (2.1) be given by

$$H(x) = \xi_\varphi \xi_\varphi^\top, \quad x \in (0, \kappa], \text{ and } H(x) = \xi_\alpha \xi_\alpha^\top, \quad x \in (\kappa, \infty).$$

Then $H(x)$ is trace-normed and has rank 1 on $(0, \infty)$. ¿From Example 2.2 and Lemma 2.4 it follows that the Titchmarsh-Weyl coefficient is given by

$$Q^+(\ell) = \frac{\cos\alpha + \ell\kappa \cos\varphi \sin(\alpha - \varphi)}{\sin\alpha + \ell\kappa \sin\varphi \sin(\alpha - \varphi)}.$$

By Proposition 3.2 the equation (2.1) is definite. Let $\{f, g\} \in T^+_{max}$, then $f, g \in L^2(H, \mathbb{R}^+)$ and $Jf' = -Hg$. Since $\operatorname{dom} T^+_{max} \subset L^2_s(H, \mathbb{R}^+)$, $\xi_\varphi^\top f(x)$ is constant for $0 < x \leq \kappa$ and $\xi_\alpha^\top f(x)$ is constant for $x > \kappa$. In fact, $\xi_\alpha^\top f(x) = 0$ for $x > \kappa$, since $f \in L^2(H, \mathbb{R}^+)$. Write

$$f(x) - f(0+) = \textstyle\int_0^x JH(t)g(t)\,dt, \quad x > 0, \tag{4.15}$$

and let

$$\xi_\varphi^\top f(0+) = c_f. \tag{4.16}$$

With $x > \kappa$ (4.15) implies

$$\xi_\alpha^\top (f(x) - f(0+)) = \xi_\alpha^\top \textstyle\int_0^\kappa JH(t)g(t)\,dt = \sin(\alpha - \varphi) \int_0^\kappa \xi_\varphi^\top g(t)\,dt,$$

and hence

$$\xi_\alpha^\top f(0+) = -\sin(\alpha - \varphi) \textstyle\int_0^\kappa \xi_\varphi^\top g(t)\,dt. \tag{4.17}$$

Conversely, for any constant c and any $g \in L^2(H, \mathbb{R}^+)$ there is an element $f \in L^2_s(H, \mathbb{R}^+)$, such that (4.15), (4.16) with $c = c_f$, and (4.17) are satisfied. In particular, $\{f, g\} \in T^+_{max}$ and since the elements $f \in L^2_s(H, \mathbb{R}^+)$ are in one-to-one correspondence with constants $c_f \in \mathbb{C}$ via (4.16), we conclude that

$$T^+_{max} = \{\, \{f, g\} : f \in L^2_s(H, \mathbb{R}^+),\ g \in L^2(H, \mathbb{R}^+) \,\}.$$

Now Theorem 3.6 together with (4.16) and (4.17) shows that

$$T^+_{min} = \{\, \{0, g\} : g \in L^2(H, \mathbb{R}^+),\ \textstyle\int_0^\kappa \xi_\varphi^\top g(t)\,dt = 0 \,\}.$$

It is clear from these representations that $\operatorname{mul} T^+_{min} \oplus L^2_s(H, \mathbb{R}^+) = L^2(H, \mathbb{R}^+)$ and that $\dim(T^+_{max}/T^+_{min}) = \dim(T^+_{max,s}/T^+_{min,s}) = 2$. In fact, $T^+_{min,s} = \{0, 0\}$ and $T^+_{max,s} = \mathbb{C} \oplus \mathbb{C}$. To describe the selfadjoint extensions $A^+(\nu)$ of $T^+_{min,s}$ observe that $f \in \operatorname{dom} A^+(\nu)$ is equivalent to $f(0+) = c\xi_\nu$. It follows from (4.16) and (4.17) that

$$A^+(\nu) = \{\, \{f, g\} \in (L^2_s(H, \mathbb{R}^+))^2 : \cos(\nu - \alpha)c_f = -\cos(\nu - \varphi)\sin(\alpha - \varphi)\kappa c_g \,\}.$$

In particular, this shows that $A^+(\nu)$ with $\nu = \varphi + \pi/2$ is a proper relation. In fact, $Q^{+,\varphi+\pi/2}(\ell) = \cot(\varphi - \alpha) + \ell\kappa$.

5. KAC' INDIVISIBLE INTERVALS AND SMOOTH PERTURBATIONS

If the interval $\mathbb{R}^+$ begins with an H-indivisible interval then all but one of the selfadjoint extensions are operators, and the exceptional selfadjoint extension has a nontrivial multivalued part. This feature can be read off from the corresponding Titchmarsh-Weyl coefficients. Recall that the class $\mathbf{N}$ of Nevanlinna functions has a subdivision which can be formulated in terms of the functions themselves; cf. [25]. Here the subdivision is given in an equivalent form in terms of the corresponding integral representation. If $Q(\ell) \in \mathbf{N}$, then there exist $\alpha \in \mathbb{R}$, $\beta \geq 0$, and a monotone nondecreasing function $\sigma(t)$ for which $\int_{\mathbb{R}} d\sigma(t)/(t^2+1) < \infty$, such that

$$Q(\ell) = \alpha + \beta\ell + \int_{\mathbb{R}} \left(\frac{1}{t-\ell} - \frac{t}{t^2+1} \right) d\sigma(t).$$

The function $Q(\ell)$ belongs to the Kac class $\mathbf{N}_1$ if $\beta = 0$ and $\int_{\mathbb{R}} d\sigma(t)/(|t|+1) < \infty$, so that for $\gamma \in \mathbb{R}$

$$Q(\ell) = \gamma + \int_{\mathbb{R}} \frac{1}{t-\ell}\, d\sigma(t).$$

The function $Q(\ell)$ belongs to $\mathbf{N}_0$ if it belongs to $\mathbf{N}_1$ and $\int_{\mathbb{R}} d\sigma(t) < \infty$. Moreover, $Q(\ell)$ belongs to $\mathbf{N}_{-k}$ for some $k \in \mathbb{N}$ if it belongs to $\mathbf{N}_0$ and $\int_{\mathbb{R}} |t|^k \, d\sigma(t) < \infty$.

Now assume that $\kappa > 0$ and that the H-indivisible interval $(0, \kappa)$ is of type φ. Then the Titchmarsh-Weyl coefficient $Q^+(\ell)$, $\varphi \neq 0$, belongs to the subclass $\mathbf{N}_0$. The only selfadjoint extension $A^+(\nu)$ of $T^+_{min,s}$ which is not an operator corresponds to $\nu = \varphi + \pi/2$. Hence, if $\varphi \neq 0$, then in particular the selfadjoint extension $A^+(\pi/2)$ is an operator. Moreover, all selfadjoint operator extensions of $T^+_{min,s}$ are rank one perturbations of $A^+(\pi/2)$; cf. [18], [25].

Proposition 5.1. *Assume that $\kappa > 0$ and that the H-indivisible interval $(0, \kappa)$ is of type $\varphi \neq 0$. Let the element ω be as in (3.18). Then*

$$A^+(\nu) = A^+(\pi/2) + \frac{1}{\tan\nu + \cot\varphi}[\cdot, \omega]\omega, \quad \nu \neq \varphi + \pi/2, \tag{5.1}$$

and

$$A^+(\nu) = T^+_{min,s} \dot{+} (\{0\} \oplus \operatorname{span}\{\omega\}), \quad \nu = \varphi + \pi/2. \tag{5.2}$$

Proof. Since $\kappa > 0$, $T^*_{min,s}$ is nondensely defined. The Q-functions $Q^{+,\nu}(\ell)$ and $Q^{+,\varphi+\pi/2}(\ell)$ are related via (2.16), and hence (4.11) implies

$$\begin{aligned} \lim_{y\to\infty} Q^{+,\nu}(iy) &= \lim_{y\to\infty} \frac{Q^{+,\varphi+\pi/2}(iy)/iy + \tan(\nu - \varphi - \pi/2)/iy}{1/iy - \tan(\nu-\varphi-\pi/2)Q^{+,\varphi+\pi/2}(iy)/(iy)} \\ &= -\frac{\kappa}{\tan(\nu-\varphi-\pi/2)\kappa} = -\cot(\nu - \varphi - \pi/2), \end{aligned}$$

for $\nu \neq \varphi + \pi/2$. In particular, with $\nu = \pi/2$ this gives

$$\lim_{y\to\infty} Q^+(iy) = \cot\varphi.$$

Now the formulas (5.1) and (5.2) follow from [18, Theorem 1.3] with $1/\tau = \tan\varphi$ and $\gamma = \cot\varphi$. □

Clearly, the symmetric operator $T^+_{min,s}$ is a domain restriction of $A^+(\nu)$:

$$T^+_{min,s} = \{\, \{f,g\} \in A^+(\nu) : [f,\omega] = 0 \,\}, \quad \nu \neq \varphi + \pi/2.$$

This can also be seen directly. Let $\{f,g\} \in T^+_{max,s}$, then

$$[f,\omega] = \int_0^\kappa \omega(t)^* H(t) f(t)\, dt = \kappa c_{\varphi,f} \xi_\varphi^\top f(0+).$$

Note that $\xi_\varphi^\top f(0+) = 0$ if and only if $f \in \operatorname{dom} A^+(\varphi + \pi/2)$. Hence, if $\nu \neq \varphi + \pi/2$ and $f \in \operatorname{dom} A^+(\nu)$ then $[f,\omega] = 0$ implies $f(0+) = 0$.

If $\varphi = 0$ so that $A^+(\pi/2)$ is the only selfadjoint extension of $T^+_{min,s}$ which is not an operator, then there is a similar interpretation of the selfadjoint operator extensions as rank one perturbations of one of them.

Now assume that $\omega \in \operatorname{dom} A^+(\nu) = \operatorname{dom} T^+_{max,s}$, $\nu \neq \varphi + \pi/2$. Let $\kappa_1 = \kappa$ and $\varphi_1 = \varphi$. Then for some $g \in L^2_s(H,\mathbb{R}^+)$, $J\tilde{\omega}' = -Hg$, where $\tilde{\omega}$ denotes the absolutely continuous representative of ω in (3.18). By continuity of $\tilde{\omega}$

$$\tilde{\omega}(\kappa_1) \neq 0, \tag{5.3}$$

while $H(x)\tilde{\omega}(x) = 0$ for a.e. $x \geq \kappa_1$. Applying Lemma 3.8 on the interval (κ_1, ∞) shows that (κ_1,∞) starts with an H-indivisible interval of type $\varphi = \varphi_2 \neq \varphi_1$. If $\kappa_2 = \infty$ the system has the form considered in Example 4.6 and the space $L^2_s(H,\mathbb{R}^+)$ is one-dimensional. If $\kappa_2 < \infty$ then Corollary 3.9 shows that

$$\tilde{\omega}(\kappa_2) = 0. \tag{5.4}$$

Since

$$\tilde{\omega}(\kappa_2) - \tilde{\omega}(\kappa_1) = \int_{\kappa_1}^{\kappa_2} JH(s)g(s)\, ds = (\kappa_2 - \kappa_1) c_{\varphi_2,g} \xi_{\varphi_2},$$

it follows from (5.3) and (5.4) that $c_{\varphi_2,g} \neq 0$, i.e. $Hg \neq 0$ on (κ_1,κ_2). Applying Proposition 3.10 on the interval (κ_1,∞) shows that $\tilde{\omega}(x) = 0$ and $H(x)g(x) = 0$ (a.e.) on (κ_2,∞), cf. (3.15). Hence, if $\omega \in \operatorname{dom} A^+(\nu)^2$, $\nu \neq \varphi + \pi/2$, then $g = A^+(\nu)\omega \in \operatorname{dom} A^+(\nu) = \operatorname{dom} T^+_{max,s}$ and the above arguments when applied to g instead of ω show that (κ_2,∞) starts with an H-indivisible interval. The case $\omega \in \operatorname{dom} A^+(\nu)^n$ can be considered by repeating this process. This leads to the following proposition in which also the more general case $\omega \in \operatorname{dom} |A^+(\nu)|^{k/2}$ for some $k \in \mathbb{N}$ is considered. The corresponding rank one perturbations are called smooth. The smoothness of the perturbation element ω provides certain stability properties of the domains of the operators in (5.1), see [25]. Note that ω in (3.18) cannot belong to the domain of the extension (5.2).

Theorem 5.2. *Let $k \in \mathbb{N}$. Assume that $\kappa > 0$ and that the interval $(0, \kappa)$ is of type φ. Then $\omega \in L_s^2(H, \mathbb{R}^+)$ given by (3.18) satisfies*

$$\omega \in \operatorname{dom} |A^+(\nu)|^{k/2}, \quad \nu \neq \varphi + \pi/2, \tag{5.5}$$

if and only if either the interval $(0, \infty)$ starts with $k/2 + 1$ H-indivisible intervals of finite total length K when k is even and with $[k/2]+1$ H-indivisible intervals of finite total length K and the canonical system restricted to the interval (K, ∞) has a Q-function which belongs to the Kac class $\mathbf{N}_1$ when k is odd, or the interval $(0, \infty)$ consists of $l \leq [k/2] + 1$ H-indivisible intervals, in which case $L_s^2(H, \mathbb{R}^+)$ is $(l - 1)$-dimensional.

Proof. For the proof we will use the function-theoretic results in [25] concerning the behaviour of Q-functions under fractional linear transforms and the connection to (5.5).

The statement is obvious for $k = 0$. Assume that $k > 0$. Let $\kappa_1 = \kappa$, $\varphi_1 = \varphi$, and let $Q^{+,\varphi_1+\pi/2}(\ell)$ be the Titchmarsh-Weyl coefficient corresponding to $\nu = \varphi + \pi/2$, i.e. the Q-function of $A^+(\varphi + \pi/2)$ in (5.2). If $k = 1$ the statement follows e.g. from [23]. Assume that $k > 1$. Let $\tilde{Q}^{+,\varphi_1+\pi/2}(\ell)$ be the Titchmarsh-Weyl coefficient of the system restricted to the interval (κ_1, ∞) and corresponding to the same value $\nu = \varphi_1 + \pi/2$. Then

$$Q^{+,\varphi_1+\pi/2}(\ell) = \kappa_1 \ell + \tilde{Q}^{+,\varphi_1+\pi/2}(\ell),$$

cf. (4.13), and $\tilde{Q}^{+,\varphi_1+\pi/2}(\ell)$ satisfies the counterpart of (4.14). According to [25, Theorem 5.1], (5.5) is equivalent to $Q^{+,\nu}(\ell) \in \mathbf{N}_{-k}$. Moreover, Theorem 4.2 and Theorem 4.4 in [25] show that

$$Q^{+,\nu}(\ell) \in \mathbf{N}_{-k}, \nu \neq \varphi_1 + \pi/2, \text{ if and only if } \tilde{Q}^{+,\varphi_1+\pi/2}(\ell) \in \mathbf{N}_{-k+2}. \tag{5.6}$$

This means that (κ_1, ∞) starts with an H-indivisible interval (κ_1, κ_2), say of type $\varphi_2 \neq \varphi_1$. If $\kappa_2 = \infty$, $L_s^2(H, \mathbb{R}^+)$ is one-dimensional, and if $\kappa_2 < \infty$ the minimal operator corresponding to the interval (κ_1, ∞) is nondensely defined. Applying the same reasoning now to the system defined on the interval (κ_1, ∞), and starting with the H-indivisible interval (κ_1, κ_2), produces a function $\tilde{Q}^{+,\varphi_2+\pi/2}(\ell) \in \mathbf{N}_{-k+4}$ (here $\mathbf{N}_2$ denotes $\mathbf{N}$), and decreases the dimension of the space $L_s^2(H, \mathbb{R}^+)$ by one. Repeating this process $[k/2] - 1$ times yields the desired result. □

If the interval $(0, \kappa)$ is H-indivisible of type φ, and $\tilde{Q}^+(\ell)$ corresponds to the Hamiltonian $H(x)$ restricted to (κ, ∞), then it follows from (4.12), that $Q^+(\ell)$ can be written as a continued fraction:

$$Q^+(\ell) = \cot \varphi + \cfrac{1}{-\kappa \ell \sin^2 \varphi + \cfrac{1}{\tilde{Q}^+(\ell) - \cot \varphi}}, \quad \varphi \neq 0. \tag{5.7}$$

Here

$$\lim_{y \to \infty} \frac{1}{iy(\tilde{Q}^+(iy) - \cot \varphi)} = 0,$$

which implies

$$\lim_{y \to \infty} iy(Q^+(iy) - \cot \varphi) = -\frac{1}{\kappa \sin^2 \varphi}.$$

This means that the zero order moment of $Q^+(\ell)$ is $m_0(\pi/2) = \frac{1}{\kappa \sin^2 \varphi}$. Note that the function ω is normalized by $\|\omega\|^2 = \kappa$. This is different from [25] due to the present normalization of the Q-function and the linear fractional transforms.

This process can be repeated for $\tilde{Q}^+(\ell)$ if $\omega \in \operatorname{dom} A^+(\nu)$, $\nu \neq \varphi + \pi/2$, i.e. if $\tilde{Q}^{+,\varphi+\pi/2}(\ell) \in \mathbf{N}_0$. Then either $\tilde{Q}^+(\ell)$ has an expression as above with $\lim_{y\to\infty} \tilde{Q}^+(iy) = \cot\varphi_2$, $\varphi_2 \neq 0$, where $\varphi_2 \neq \varphi_1$ is the type of the second H-indivisible interval (κ_1, κ_2) or it has an expression of the form (4.13) if $\varphi_2 = 0$. For connections to moments, see [1], [25].

6. Selfadjoint realizations on the real line with interface conditions at zero

In the present section the trace-normed canonical system (1.1) is studied on $\mathbb{R}$, with an interface condition at 0. This requires first a treatment of the system on the halfline $\mathbb{R}^-$:

$$Jy' = -\ell H(x)y, \quad \text{on } \mathbb{R}^-. \tag{6.1}$$

Let $L^2(H, \mathbb{R}^-)$ be the Hilbert space associated with $H(x)$ on $\mathbb{R}^-$ with inner product

$$\int_{-\infty}^0 g(x)^* H(x) f(x)\, dx.$$

Define the 2×2 matrix function $W(\cdot, \ell)$ as the solution of the initial value problem

$$\frac{dW(x,\ell)}{dx} J = \ell W(x,\ell)H(x), \quad \text{for a.a. } x<0, \quad W(0-,\ell) = I, \tag{6.2}$$

so that the 2×2 matrix function $W(\cdot, \bar{\ell})^*$ is the solution of the initial value problem

$$J\frac{dW(x,\bar{\ell})^*}{dx} = -\ell H(x)W(x,\bar{\ell})^*, \quad \text{for a.a. } x<0, \quad W(0-,\bar{\ell})^* = I. \tag{6.3}$$

It follows that for $\ell, \lambda \in \mathbb{C}$

$$W(x,\ell)JW(x,\lambda)^* - J = (\ell - \bar{\lambda})\int_0^x W(t,\ell)H(t)W(t,\lambda)^*\, dt, \quad x<0, \tag{6.4}$$

and that the counterpart of (2.6) holds. For any $-t(\ell) \in \mathbf{N} \cup \{\infty\}$ the limit

$$Q^-(\ell) = -\lim_{x\to-\infty} \frac{w_{11}(x,\ell)t(\ell) + w_{12}(x,\ell)}{w_{21}(x,\ell)t(\ell) + w_{22}(x,\ell)}, \quad \ell \in \mathbb{C}\setminus\mathbb{R}, \tag{6.5}$$

exists, is independent of $t(\ell)$, and belongs to $\mathbf{N}\cup\{\infty\}$. Moreover, for each $\ell \in \mathbb{C}\setminus\mathbb{R}$

$$\chi^-(\ell) = \chi^-(\cdot,\ell) = W(\cdot,\bar{\ell})^* \begin{pmatrix} 1 \\ Q^-(\ell) \end{pmatrix} \in L^2(H,\mathbb{R}^-). \tag{6.6}$$

In the Hilbert space $L^2(H,\mathbb{R}^-)$ define the linear relation T^-_{max} by

$$T^-_{max} = \{\, \{f,g\} \in (L^2(H,\mathbb{R}^-))^2 : f \in AC,\ Jf' = -Hg \,\},$$

and the linear relation T^-_{min} by $T^-_{min} = (T^-_{max})^*$, i.e.

$$T^-_{min} = \{\, \{f,g\} \in (L^2(H,\mathbb{R}^-))^2 : [g,h] - [f,k] = 0 \text{ for all } \{h,k\} \in T^-_{max} \,\}.$$

Clearly, $\chi^-(\ell) \in \ker(T^-_{max} - \ell)$, i.e. $\{\chi^-(\ell), \ell\chi^-(\ell)\} \in T^-_{max}$. The equation (6.1) is called definite if the whole interval $(-\infty, 0)$ is of positive type, i.e. if the implication (3.1) holds when $x \in \mathbb{R}^-$. If the equation (6.1) is not definite, then $T^-_{min} = T^-_{max}$ is a purely multivalued selfadjoint relation. If the equation (6.1) is definite, then the relation T^-_{min} is closed, symmetric, and has defect numbers $(1,1)$. It is given by

$$T^-_{min} = \{\,\{f,g\} \in T^-_{max} : \, f(0-) = 0\,\}.$$

As on the halfline $\mathbb{R}^-$ there is an orthogonal decomposition

$$L^2(H,\mathbb{R}^-) = (\operatorname{mul} T^-_{min}) \oplus L^2_s(H,\mathbb{R}^-).$$

Define in the Hilbert space $L^2_s(H,\mathbb{R}^-)$ the corresponding parts of T^-_{min} and T^-_{max}:

$$T^-_{min,s} = T^-_{min} \cap (L^2_s(H,\mathbb{R}^-))^2, \quad T^-_{max,s} = T^-_{max} \cap (L^2_s(H,\mathbb{R}^-))^2.$$

Then $T^-_{min,s}$ is a closed symmetric operator with defect numbers $(1,1)$ and its adjoint is given by

$$(T^-_{min,s})^* = T^-_{max,s}.$$

Define

$$w^-_\nu(x,\ell) = -\frac{1}{\cos\nu Q^-(\ell) + \sin\nu}\, W(x,\bar{\ell})^* \begin{pmatrix} \cos\nu \\ -\sin\nu \end{pmatrix}.$$

Then $w^-_\nu(x,\ell)$ satisfies the counterparts of (4.1) and (4.2). The selfadjoint extensions $A^-(\nu)$ of $T^-_{min,s}$ in $L^2_s(H,\mathbb{R}^-)$ are in one-to-one correspondence with $\nu \in (-\pi/2, \pi/2]$ via

$$\operatorname{dom} A^-(\nu) = \{\, f \in \operatorname{dom} T^-_{max} : \, -\sin\nu f_1(0-) = \cos\nu f_2(0-)\,\}.$$

The corresponding resolvent operator $(A^-(\nu)-\ell)^{-1}$ of $A^-(\nu)$ is given by

$$\begin{aligned} (6.7) \qquad & (A^-(\nu)-\ell)^{-1}h(x) \\ & = w^-_\nu(x,\ell)\textstyle\int_{-\infty}^x \chi^-(t,\bar{\ell})^* H(t)h(t)\,dt + \chi^-(x,\ell)\int_x^0 w^-_\nu(t,\bar{\ell})^* H(t)h(t)\,dt, \end{aligned}$$

where $h \in L^2_s(H,\mathbb{R}^-)$. It is connected to the resolvent operator $(A^-(\pi/2)-\ell)^{-1}$ of the selfadjoint extension $A^-(\pi/2)$ by Kreĭn's formula

$$(6.8) \qquad (A^-(\nu)-\ell)^{-1} = (A^-(\pi/2)-\ell)^{-1} - \chi^-(\ell)\,\frac{1}{Q^-(\ell)+\tan\nu}\,[\cdot, \chi^-(\bar{\ell})],$$

when $\ell \in \mathbb{C}\setminus\mathbb{R}$.

Now the equation (1.1) will be considered on $\mathbb{R}$ with an interface condition at 0. It is assumed that the restrictions of (1.1) to $\mathbb{R}^+$ and $\mathbb{R}^-$ are definite. Define the orthogonal sums

$$T^+_{min,s} \oplus T^-_{min,s}, \quad A(\pi/2) = A^+(\pi/2) \oplus A^-(\pi/2).$$

Interpret $\chi^+(\cdot,\ell)$ and $\chi^-(\cdot,\ell)$ as functions on $\mathbb{R}$, by a trivial extension to $\mathbb{R}^-$ and to $\mathbb{R}^+$, respectively, and form $\chi(\cdot,\ell) = (\chi^+(\cdot,\ell), \chi^-(\cdot,\ell))$. Then $\chi(\cdot,\ell)$ is a basis for the eigenspaces of $(T^+_{min,s})^* \oplus (T^-_{min,s})^*$, and the function $Q(\ell) = \operatorname{diag}(Q^+(\ell), Q^-(\ell))$ is the corresponding Q-function for the above orthogonal sums. For $h \in L^2(H,\mathbb{R}) = L^2(H,\mathbb{R}^+) \oplus L^2(H,\mathbb{R}^-)$ the inner product $[h, \chi(\cdot,\bar{\ell})]$ is defined by

$$[h, \chi(\cdot,\bar{\ell})] = \begin{pmatrix} [h, \chi^+(\cdot,\bar{\ell})] \\ [h, \chi^-(\cdot,\bar{\ell})] \end{pmatrix}.$$

A pair of 2×2 matrices (P,T) is called a Nevanlinna pair, when P is an orthogonal projection, T is selfadjoint, and

$$PT = TP, \quad T(I-P) = I-P.$$

Note that for any pair of 2×2 matrices (U,V) which satisfies

$$\operatorname{rank}\begin{pmatrix} U \\ V \end{pmatrix} = 2, \quad U^*V = V^*U,$$

there exists an invertible 2×2 matrix X and a unique Nevanlinna pair (P,T) such that $U = PX$, $V = TX$; see [22].

Theorem 6.1. *There is a one-to-one correspondence between the selfadjoint extensions H of $T^+_{min,s}\oplus T^-_{min,s}$ in $L^2_s(H,\mathbb{R}^+)\oplus L^2_s(H,\mathbb{R}^-)$ and the Nevanlinna pairs (P,T) of 2×2 matrices via*

$$(H-\ell)^{-1} = (A(\pi/2)-\ell)^{-1} - \chi(\ell)P(Q(\ell)P+T)^{-1}[\cdot,\chi(\bar\ell)]. \tag{6.9}$$

Moreover,

$$\operatorname{dom} H = \left\{ f \in \operatorname{dom} T^+_{max,s}\oplus \operatorname{dom} T^-_{max,s} : P\begin{pmatrix} f_2(0+) \\ -f_2(0-)\end{pmatrix} = T\begin{pmatrix} f_1(0+) \\ f_1(0-)\end{pmatrix} \right\}. \tag{6.10}$$

Proof. The present form (6.9) of Kreĭn's formula can be found in [22]. It is straightforward to check that for any Nevanlinna pair (P,T), the right side of (6.10) defines a selfadjoint extension. Now let H be a selfadjoint extension corresponding to the Nevanlinna pair (P,T) as in (6.9). Let $h\in L^2(H,\mathbb{R})$ and let $y(\cdot,\ell) = (H-\ell)^{-1}h$. Evaluate the right and left limits of $y(x,\ell)$ at $x=0$ via formula (6.9) with the above interpretation of $\chi(\cdot,\ell)$. It follows from (4.4) and $\nu=\pi/2$, that

$$y(0+,\ell) = \begin{pmatrix} 0 \\ -1\end{pmatrix}[h,\chi^+(\bar\ell)] - \begin{pmatrix} 1 & 0 \\ -Q^+(\ell) & 0\end{pmatrix} P(Q(\ell)P+T)^{-1}[h,\chi(\bar\ell)],$$

which gives

$$y(0+,\ell) = \left\{ \begin{pmatrix} -1 & 0 \\ 0 & 0\end{pmatrix} P + \begin{pmatrix} 0 & 0 \\ -1 & 0\end{pmatrix} T \right\} (Q(\ell)P+T)^{-1}[h,\chi(\bar\ell)]. \tag{6.11}$$

Similarly, it follows from (6.7) and $\nu=\pi/2$, that

$$y(0-,\ell) = \begin{pmatrix} 0 \\ 1\end{pmatrix}[h,\chi^-(\bar\ell)] - \begin{pmatrix} 0 & 1 \\ 0 & Q^-(\ell)\end{pmatrix} P(Q(\ell)P+T)^{-1}[h,\chi(\bar\ell)],$$

and this gives

$$y(0-,\ell) = \left\{ \begin{pmatrix} 0 & -1 \\ 0 & 0\end{pmatrix} P + \begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix} T \right\} (Q(\ell)P+T)^{-1}[h,\chi(\bar\ell)]. \tag{6.12}$$

The equations (6.11) and (6.12) can be rewritten as

$$\begin{pmatrix} y_1(0+,\ell) \\ y_1(0-,\ell)\end{pmatrix} = -P(Q(\ell)P+T)^{-1}[h,\chi(\bar\ell)],$$

and

$$\begin{pmatrix} y_2(0+,\ell) \\ -y_2(0-,\ell)\end{pmatrix} = -T(Q(\ell)P+T)^{-1}[h,\chi(\bar\ell)].$$

Now $PT = TP$ yields the desired boundary condition in (6.10). □

The Nevanlinna pairs of 2×2 matrices (P, T) can be classified according to the rank of P, cf. [19], [20], [21]. This leads to a classification of the corresponding selfadjoint extensions H, relative to the selfadjoint extension $A(\pi/2)$. If $\operatorname{rank} P = 2$, then (P, T) is given by

$$(6.13) \qquad (P,T) = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \right).$$

The selfadjoint extensions H which correspond to the Nevanlinna pairs of the form (6.13) are characterized by the property

$$(6.14) \qquad H \cap A(\pi/2) = T^+_{min,s} \oplus T^-_{min,s}.$$

If $\operatorname{rank} P = 1$, then there exist $r_1, r_2 \in \mathbb{C}$, $|r_1|^2 + |r_2|^2 = 1$, and $\tau \in \mathbb{R}$, such that (P, T) is given by

$$(6.15) \qquad (P,T) = \left(\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \begin{pmatrix} \bar{r}_1 & \bar{r}_2 \end{pmatrix}, \tau P + (I - P) \right).$$

In this case Kreĭn's formula (6.9) takes the form

$$(6.16) \qquad \begin{aligned} (H - \ell)^{-1} &= (A(\pi/2) - \ell)^{-1} \\ &\quad - \chi_{r_1,r_2}(\ell) \frac{1}{|r_1|^2 Q^+(\ell) + |r_2|^2 Q^-(\ell) + \tau} [\cdot, \chi_{r_1,r_2}(\bar{\ell})], \end{aligned}$$

where $\chi_{r_1,r_2}(\cdot, \ell) = r_1 \chi^+(\cdot, \ell) + r_2 \chi^-(\cdot, \ell)$. Define the symmetric extension S_{r_1,r_2} of $T^+_{min,s} \oplus T^-_{min,s}$ by

$$(6.17) \qquad S_{r_1,r_2} = \{ \{f, g\} \in A(\pi/2) : [g - \ell f, \chi_{r_1,r_2}(\bar{\ell})] = 0 \}.$$

The selfadjoint extensions H which correspond to the Nevanlinna pairs of the form (6.15) are characterized by the property

$$(6.18) \qquad H \cap A(\pi/2) = S_{r_1,r_2}.$$

Clearly, the cases $r_1 = 0$ and $r_2 = 0$ correspond to selfadjoint extensions of $A^+(\pi/2) \oplus T^-_{min,s}$, and of $T^+_{min,s} \oplus A^-(\pi/2)$, essentially taking place in $L^2(H, \mathbb{R}^-)$ and in $L^2(H, \mathbb{R}^+)$, respectively. If $\operatorname{rank} P = 0$, then $(P, T) = (0, I)$ and H in (6.9) corresponds to $A(\pi/2)$.

Now consider the selfadjoint extension H of $T^+_{min,s} \oplus T^-_{min,s}$ in $L^2_s(H, \mathbb{R}^+) \oplus L^2_s(H, \mathbb{R}^-)$ as an extension of $T^+_{min,s}$ by means of the exit space $L^2_s(H, \mathbb{R}^-)$. For each $\ell \in \mathbb{C} \setminus \mathbb{R}$ there is a corresponding Štrauss extension of $T^+_{min,s}$ given by

$$T(\ell) = \{ \{R^+(H - \ell)^{-1} h, (I + \ell R^+(H - \ell)^{-1}) h\} : h \in L^2_s(H, \mathbb{R}^+) \},$$

where R^+ is the orthogonal projection from $L^2_s(H, \mathbb{R})$ onto $L^2_s(H, \mathbb{R}^+)$. In terms of boundary conditions, the Štrauss extension $T(\ell)$ of $T^+_{min,s}$ is given by

$$(6.19) \qquad T(\ell) = \{ \{f, g\} \in T^+_{max,s} : f_2(0+) = S(\ell) f_1(0+) \}.$$

An equivalent form in terms of Kreĭn's formula for compressed resolvents is

$$(6.20) \qquad R^+(H - \ell)^{-1} |_{L^2_s(H, \mathbb{R}^+)} = (A^+(\pi/2) - \ell)^{-1} - \chi^+(\ell)(Q^+(\ell) + S(\ell))^{-1} [\cdot, \chi^+(\bar{\ell})].$$

In (6.19) and (6.20) $S(\ell)$ is a Nevanlinna function, depending on $Q^-(\ell)$ and the Nevanlinna pair (P, T).

Corollary 6.2. *The selfadjoint extensions H of $T^+_{min,s} \oplus T^-_{min,s}$ with the property* (6.14) *are in one-to-one correspondence with the 2×2 selfadjoint matrices $T = (t_{ij})$ via*

$$\begin{aligned} \operatorname{dom} H = \{\, f \in \operatorname{dom} T^+_{max,s} \oplus \operatorname{dom} T^-_{max,s} : \\ f_2(0+) = t_{11} f_1(0+) + t_{12} f_1(0-),\ f_2(0-) = -t_{21} f_1(0+) - t_{22} f_1(0-) \,\}. \end{aligned}$$

The boundary condition in (6.19) *is given by the Nevanlinna function $S(\ell)$ in* (1.5).

The symmetric extension S_{r_1,r_2} of $T^+_{min,s} \oplus T^-_{min,s}$ defined in (6.17) can be expressed in terms of boundary conditions involving r_1 and r_2:

$$\begin{aligned} S_{r_1,r_2} = \{\, \{f, g\} \in T^+_{max,s} \oplus T^-_{max,s} : \\ \bar{r}_1 f_2(0+) = \bar{r}_2 f_2(0-),\ f_1(0+) = f_1(0-) = 0 \,\}. \end{aligned}$$

Corollary 6.3. *The selfadjoint extensions H of $T^+_{min,s} \oplus T^-_{min,s}$ with the property* (6.18) *are in one-to-one correspondence with $\tau \in \mathbb{R}$ via*

$$\begin{aligned} \operatorname{dom} H = \{\, f \in \operatorname{dom} T^+_{max,s} \oplus \operatorname{dom} T^-_{max,s} : \\ \bar{r}_1 f_2(0+) - \bar{r}_2 f_2(0-) = \tau(\bar{r}_1 f_1(0+) + \bar{r}_2 f_1(0-)),\ r_2 f_1(0+) - r_1 f_1(0-) = 0 \,\}. \end{aligned}$$

If $r_1 \neq 0$, then the boundary condition in (6.19) *is given by the Nevanlinna function $S(\ell)$ in* (1.6).

The result in (6.20) can be rewritten as

$$\begin{aligned} (T(\ell) - \ell)^{-1} h(x) = W(x, \bar{\ell})^* \Omega(\ell) \textstyle\int_0^\infty W(t, \ell) H(t) h(t)\, dt \\ + \tfrac{1}{2} W(x, \bar{\ell})^* J \textstyle\int_0^x W(t, \ell) H(t) h(t)\, dt - \tfrac{1}{2} W(x, \bar{\ell})^* J \textstyle\int_x^\infty W(t, \ell) H(t) h(t)\, dt, \end{aligned} \tag{6.21}$$

when $h \in L^2(H, \mathbb{R}^+)$ has a compact support. Here the 2×2 matrix function $\Omega(\ell)$ is given by

$$\Omega(\ell) = -\frac{1}{2} \begin{pmatrix} Q(\ell) & 1 \\ S(\ell) & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -Q(\ell) \\ 1 & S(\ell) \end{pmatrix}. \tag{6.22}$$

Clearly, $\Omega(\ell)$ is a Nevanlinna function:

$$\frac{\Omega(\ell) - \Omega(\lambda)^*}{\ell - \bar{\lambda}} = \begin{pmatrix} \frac{Q(\ell) - Q(\lambda)^*}{\ell - \bar{\lambda}} & 0 \\ 0 & \frac{S(\ell) - S(\lambda)^*}{\ell - \bar{\lambda}} \end{pmatrix}. \tag{6.23}$$

Boundary-value problems similar to (2.1), (1.3) have been studied by A.V. Štrauss; see for instance [42], [43]. A corresponding exit space is constructed via the Nevanlinna kernel of $S(\ell)$, or rather, via the characteristic function which results after the Cayley transform of $S(\ell)$. The identity (6.23) reflects this construction again, cf. [12], [13], [14]. The work in this section has some connections to the investigations of Kac on the spectral multiplicity in [31] (see also [32]); these connections will be further studied elsewhere.

7. Auxiliary results

This section provides complete proofs for the fundamental Theorems 2.1 and 3.6. If the equation (2.1) is not definite, then $H(x)$ is of the form (2.9). In this case Example 2.2 takes care of Theorem 2.1, and Proposition 3.4 gives a full description of the situation. So it suffices to consider the case where (2.1) is definite. The following corollary of Proposition 3.2 characterizes definite equations, cf. [13], [38].

Lemma 7.1. *The equation* (2.1) *is definite if and only if there exists a compact subinterval I of $(0,\infty)$ such that I is of positive type.*

Proof. The sufficiency is obvious. To prove the necessity, assume that each compact subinterval of $(0,\infty)$ is H-indivisible, cf. Lemma 3.1. Hence for each $n \in \mathbb{N}$, $n > 1$, the interval $[1/n, n]$ is, say, of type φ_n, i.e. $H(x) = \xi_{\varphi_n}\xi_{\varphi_n}^{\top}$ almost everywhere on $[1/n, n]$. The type φ_n of $[1/n, n]$ does not depend on n, i.e. $\varphi_n = \varphi$, since the intersection of two intervals is the smaller one. Since $(0,\infty)$ is a countable union of intervals $[1/n, n]$, $n \in \mathbb{N}$, $n > 1$, it follows that $H(x) = \xi_{\varphi}\xi_{\varphi}^{\top}$ almost everywhere on $(0,\infty)$, so that the equation (2.1) is not definite. This completes the proof. □

In the rest of this section it is assumed that the equation (2.1) is definite. The results in Section 3 up till Theorem 3.6 will be used. In particular, by Lemma 3.5 each element in $\operatorname{dom} T^+_{max}$ has a unique representative which is locally absolutely continuous on $(0,\infty)$. In order to show that T^+_{min} is symmetric, introduce the relation T_0^+ in $L^2(H,\mathbb{R}^+)$ by

$$T_0^+ = \{\, \{f,g\} \in T^+_{max} : \operatorname{supp} f \text{ compact} \,\}.$$

Clearly, the relation T_0^+ is linear.

Lemma 7.2. *Let $[\alpha,\beta] \subset (0,\infty)$ be a compact interval. If $\{\varphi,\psi\} \in T_0^+$ and $\operatorname{supp}\varphi \subset [\alpha,\beta]$, then the function ψ satisfies*

$$\operatorname{supp} H\psi \subset [\alpha,\beta], \qquad \int_\alpha^\beta H(t)\psi(t)\,dt = 0. \tag{7.1}$$

Conversely, if the function $\psi \in L^2(H,\mathbb{R}^+)$ satisfies (7.1)*, then there exists an element φ, such that $\{\varphi,\psi\} \in T_0^+$ and $\operatorname{supp}\varphi \subset [\alpha,\beta]$.*

Proof. If $\{\varphi,\psi\} \in T_0^+$, then $\varphi,\psi \in L^2(H,\mathbb{R}^+)$ and $\varphi' = JH\psi$. Hence $\operatorname{supp} H\psi \subset [\alpha,\beta]$ and since $\varphi(\alpha) = 0$,

$$\varphi(x) = \int_\alpha^x JH(t)\psi(t)\,dt.$$

Since $\varphi(\beta) = 0$, also the second fact in (7.1) has been shown. To see the converse, let $\psi \in L^2(H,\mathbb{R}^+)$ and define

$$\varphi(x) = \int_\alpha^x JH(t)\psi(t)\,dt.$$

Then $\operatorname{supp}\varphi \subset [\alpha,\beta]$ and $J\varphi' = -H\psi$. Hence $\{\varphi,\psi\} \in T_0^+$ and $\operatorname{supp}\varphi \subset [\alpha,\beta]$. □

Lemma 7.3. *The linear relation T_0^+ is symmetric and*

$$T_0^+ \subset (T_0^+)^* = T^+_{max}.$$

Proof. Let $\{f,g\} \in T^+_{max}$ and $\{\varphi,\psi\} \in T^+_0$. Then

$$[g,\varphi] - [f,\psi] = -\varphi^* J\psi \,|_\alpha^\beta = 0,$$

which shows that $T^+_{max} \subset (T^+_0)^*$. To show the reverse inclusion assume that $\{h,k\} \in (T^+_0)^*$. Then, by definition, $h,k \in L^2(H,\mathbb{R}^+)$. Let u be a solution of the differential equation $Ju' = -Hk$. Let $[\alpha,\beta]$ be a compact interval. For $\psi \in L^2(H,\mathbb{R}^+)$ satisfying (7.1), let φ be as in Lemma 7.2, so that $\{\varphi,\psi\} \in T^+_0$. Then $[\psi,h] - [\varphi,k] = 0$ implies

$$\begin{aligned}\int_\alpha^\beta h(t)^* H(t)\psi(t)\,dt &= \int_\alpha^\beta k(t)^* H(t)\varphi(t)\,dt \\ &= \int_\alpha^\beta u'(t)^* J\varphi(t)\,dt = -\int_\alpha^\beta u(t)^* J\varphi'(t)\,dt = \int_\alpha^\beta u(t)^* H(t)\psi(t)\,dt,\end{aligned}$$

and hence

$$\int_\alpha^\beta (h(t) - u(t))^* H(t)\psi(t)\,dt = 0.$$

According to Lemma 7.2 the functions $\psi(t)$ span the orthogonal complement of (the equivalence classes of) constants on $[\alpha,\beta]$. Hence, $h(t) - u(t)$ is equivalent to a constant on $[\alpha,\beta]$. Therefore, h has a representative again denoted by h, which is absolutely continuous and satisfies $Jh' = Ju' = -Hk$ a.e. on $[\alpha,\beta]$. Let $I \subset (0,\infty)$ be a compact subinterval of positive type, cf. Lemma 7.1. Lemma 3.5 (more precisely its local analog on $[\alpha,\beta] \supset I$) shows that the absolutely continuous representative h does not depend on the interval $[\alpha,\beta] \supset I$. Since $[\alpha,\beta]$ was arbitrary, it follows that $\{h,k\} \in T^+_{max}$. Hence, $(T^+_0)^* = T^+_{max} \supset T^+_0$ and T^+_0 is symmetric. □

Corollary 7.4. *The linear relation T^+_{min} is closed and symmetric.*

Proof. Since $T^+_{min} = (T^+_{max})^*$, the relation T^+_{min} is closed. Lemma 7.3 implies that $T^+_{min} = \operatorname{clos} T^+_0 \subset T^+_{max} = (T^+_{min})^*$ and hence T^+_{min} is symmetric. □

The following lemma is proved along the lines of [2, p. 396].

Lemma 7.5. *For each $u \in \mathbb{C}^2$ there exists $\{\varphi,\psi\} \in T^+_{max}$ such that φ has a compact support and $\varphi(0+) = u$.*

Proof. Choose $[0,\gamma]$ so that it contains an open subinterval of positive type. Then the 2×2 matrix $\int_0^\gamma H(t)\,dt$ is invertible. Hence, there is a vector $c \in \mathbb{C}^2$ such that

$$\left(\int_0^\gamma H(t)\,dt\right) c = Ju.$$

Define $\psi \in L^2(H,\mathbb{R}^+)$ by

$$\psi(t) = c, \quad 0 \le t \le \gamma, \quad \psi(t) = 0, \quad t > \gamma,$$

and define φ by

$$\varphi(x) = u + \left(\int_0^x JH(t)\,dt\right)\psi(x).$$

Then φ belongs to $L^2(H,\mathbb{R}^+)$, is absolutely continuous and $\operatorname{supp}\varphi \subset [0,\gamma]$. Moreover, $\varphi(0+) = u$ and $J\varphi' = -H\psi$, so that $\{\varphi,\psi\} \in T^+_{max}$. □

Lemma 7.6. *Let $\{f,g\},\{h,k\} \in T^+_{max}$. Then the following limit exists:*

$$\lim_{x\to\infty} h(x)^*Jf(x) = h(0+)^*Jf(0+) - \langle\{f,g\},\{h,k\}\rangle. \tag{7.2}$$

Proof. Let $\{f,g\},\{h,k\} \in T^+_{max}$. Integration by parts gives

$$\begin{aligned}
&\textstyle\int_0^x h(t)^*H(t)g(t)\,dt - \int_0^x k(t)^*H(t)f(t)\,dt \\
&\quad = -\textstyle\int_0^x h(t)^*Jf'(t)\,dt - \int_0^x h'(t)^*Jf(t)\,dt \\
&\quad = -\textstyle\int_0^x \frac{d}{dt}h(t)^*Jf(t)\,dt \\
&\quad = h(0+)^*Jf(0+) - h(x)^*Jf(x).
\end{aligned} \tag{7.3}$$

By definition the left side converges to $[g,h]-[f,k]$ as $x \to \infty$. Hence the limit of $h(x)^*Jf(x)$ exists as $x \to \infty$ and (7.2) is proved. □

The next lemma gives estimates for the defect elements of T^+_{min}. It makes it possible to study the boundary behaviour of any (locally absolutely continuous) element $\{f,g\} \in T^+_{max}$ at 0 and ∞.

Lemma 7.7. *Let $\{f,\ell f\} \in T^+_{max}$, $\ell \in \mathbb{C}\setminus\mathbb{R}$. Then for $i=1,2$*

$$|f_i(x) - f_i(y)| \le \sqrt{6}\,|\ell|\sqrt{|x-y|}\,\|f\|, \quad x,y>0.$$

In particular, for $i=1,2$

$$|f_i(x)| \le \sqrt{x}(|f_i(0+)| + \sqrt{6}\,|\ell|\,\|f\|), \quad x \ge 1.$$

Proof. Since the 2×2 matrix $H(x)$ is assumed to be nonnegative and trace-normed, it has the form

$$H(x) = \begin{pmatrix} \alpha(x) & \beta(x) \\ \beta(x) & 1-\alpha(x) \end{pmatrix},$$

$0 \le \alpha(x) \le 1$, $\beta(x)^2 \le \alpha(x)(1-\alpha(x))$. The eigenvalues of $H(x)$ are real, have the form $\delta(x)$ and $1-\delta(x)$, and satisfy $0 \le \delta(x) \le 1$. Assume without loss of generality that $\delta(x) \le 1-\delta(x)$. Clearly,

$$0 \le \det H(x) = \alpha(x)(1-\alpha(x)) - \beta(x)^2 = \delta(x)(1-\delta(x)) \le \delta(x),$$

$$0 \le \sqrt{\alpha(x)(1-\alpha(x))} - |\beta(x)| \le \sqrt{\delta(x)}. \tag{7.4}$$

Now introduce

$$\gamma(x) = \operatorname{sgn}(\beta(x))\sqrt{\alpha(x)(1-\alpha(x))}, \quad \eta(x) = \beta(x) - \gamma(x).$$

Then the first and the second inequality in (7.4) give

$$(\beta(x)+\eta(x))^2 \le \alpha(x)(1-\alpha(x)), \quad \eta(x)^2 \le \delta(x), \tag{7.5}$$

respectively. Define the matrix functions $H_1(x)$ and $H_2(x)$ by

$$H_1(x) = \begin{pmatrix} \alpha(x) & \gamma(x) \\ \gamma(x) & 1-\alpha(x) \end{pmatrix}, \qquad H_2(x) = \begin{pmatrix} 0 & \eta(x) \\ \eta(x) & 0 \end{pmatrix},$$

so that $H(x) = H_1(x) + H_2(x)$. The first inequality in (7.5) gives $H_1(x) + 2H_2(x) \geq 0$, which leads to

$$\pm H_2(x) \leq H(x) \text{ and } 0 \leq H_1(x) \leq 2H(x). \tag{7.6}$$

Now let $\{f, \ell f\} \in T^+_{max}$ so that (2.1) holds. Then the components of f' can be rewritten as

$$f_1'(x) = -\ell(\sqrt{1-\alpha(x)}\xi_1(x) + \eta(x)f_1(x)), \quad f_2'(x) = \ell(\sqrt{\alpha(x)}\xi_2(x) + \eta(x)f_2(x)),$$

where

$$\xi_1(x) = \sqrt{1-\alpha(x)}f_2(x) + \operatorname{sgn}(\beta(x))\sqrt{\alpha(x)}f_1(x), \tag{7.7}$$

$$\xi_2(x) = \sqrt{\alpha(x)}f_1(x) + \operatorname{sgn}(\beta(x))\sqrt{1-\alpha(x)}f_2(x). \tag{7.8}$$

Observe that

$$|\xi_1(x)|^2 = f(x)^* H_1(x) f(x) = |\xi_2(x)|^2. \tag{7.9}$$

Now assume that e.g. $x > y$. Since $f_1(x) - f_1(y) = \int_y^x f_1'(t)\,dt$, it follows that

$$\begin{aligned}
&|f_1(x) - f_1(y)|^2 \\
&\quad = |\ell|^2 \left| \int_y^x [\sqrt{1-\alpha(t)}\xi_1(t) + \eta(t)f_1(t)]\,dt \right|^2 \\
&\quad \leq |\ell|^2 \left(\int_y^x 1\,dt \right) \left(\int_y^x |\sqrt{1-\alpha(t)}\xi_1(t) + \eta(t)f_1(t)|^2\,dt \right) \\
&\quad \leq 2|\ell|^2|x-y| \left(\int_y^x (1-\alpha(t))|\xi_1(t)|^2\,dt + \int_y^x |\eta(t)|^2|f_1(t)|^2\,dt \right) \\
&\quad \leq 2|\ell|^2|x-y| \left(\int_y^x |\xi_1(t)|^2\,dt + \int_y^x \delta(t)|f_1(t)|^2\,dt \right),
\end{aligned}$$

where the last inequality follows from (7.5). Due to (7.6) and (7.9),

$$\int_y^x |\xi_1(t)|^2\,dt = \int_y^x f(t)^* H_1(t) f(t)\,dt \leq 2\int_0^\infty f(t)^* H(t) f(t)\,dt < \infty. \tag{7.10}$$

Moreover, since $\delta(x)I \leq H(x)$,

$$\int_y^x \delta(t)|f_1(t)|^2\,dt \leq \int_y^x \delta(t)|f(t)|^2\,dt \leq \int_0^\infty f(t)^* H(t) f(t)\,dt < \infty.$$

This shows that $|f_1(x) - f_1(y)| \leq \sqrt{6}\,|\ell|\sqrt{|x-y|}\,\|f\|$. The proof for the second component is similar. □

Lemma 7.8. *Let $\{f, \ell f\}, \{h, \lambda h\} \in T^+_{max}$ for some $\ell, \lambda \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$\lim_{x\to\infty} h(x)^* J f(x) = 0. \tag{7.11}$$

Proof. The statement (7.11) follows by means of the polarization formula as soon as it is shown that

$$\lim_{x\to\infty} u(x)^* J u(x) = 0, \tag{7.12}$$

for $u(x) = af(x) + bh(x)$, $a, b \in \mathbb{C}$. Define the function ξ by

$$\xi(x) = \sqrt{1-\alpha(x)}u_2(x) + \operatorname{sgn}(\beta(x))\sqrt{\alpha(x)}u_1(x),$$

so that

$$2\left|\operatorname{Im}\frac{\xi(x)\overline{u_2(x)}}{\sqrt{\alpha(x)}}\right| = |u(x)^*Ju(x)| = 2\left|\operatorname{Im}\frac{u_1(x)\overline{\xi(x)}}{\sqrt{1-\alpha(x)}}\right|, \tag{7.13}$$

when $\alpha(x) \neq 0$ and $\alpha(x) \neq 1$, respectively. Since $|\xi(x)|^2 = u(x)^*H_1(x)u(x)$, the integral $\int_0^\infty |\xi(x)|^2dx$ converges; cf. (7.6). Hence, for each $\varepsilon > 0$ there exists a set $M_\varepsilon \subset \mathbb{R}^+$, whose Lebesque measure is infinite, such that

$$|\xi(x)| < \frac{\varepsilon}{\sqrt{x}}, \quad x \in M_\varepsilon. \tag{7.14}$$

Define

$$K_{\ell,\lambda} = |a|(|f(0+)| + \sqrt{6}\,|\ell|\,\|f\|) + |b|(|h(0+)| + \sqrt{6}\,|\lambda|\,\|h\|).$$

Then it follows from Lemma 7.7 that for $i = 1, 2$,

$$|u_i(x)| \leq K_{\ell,\lambda}\sqrt{x}, \quad x \geq 1. \tag{7.15}$$

If $\alpha(x) \geq \frac{1}{2}$, then it follows from the first equality in (7.13), (7.14), and (7.15) that for $x \in M_\varepsilon$ and $x \geq 1$

$$|u(x)^*Ju(x)| \leq 2\sqrt{2}|\xi(x)||u_2(x)| \leq 2\sqrt{2}\frac{\varepsilon}{\sqrt{x}}\sqrt{x}K_{\ell,\lambda} = 2\sqrt{2}\varepsilon K_{\ell,\lambda}.$$

Similarly, if $\alpha(x) < \frac{1}{2}$, then it follows from the second equality in (7.13), (7.14), and (7.15) that for $x \in M_\varepsilon$ and $x \geq 1$

$$|u(x)^*Ju(x)| \leq 2\sqrt{2}|u_1(x)||\xi(x)| \leq 2\sqrt{2}\sqrt{x}K_{\ell,\lambda}\frac{\varepsilon}{\sqrt{x}} = 2\sqrt{2}\varepsilon K_{\ell,\lambda}.$$

Hence, for arbitrary $\varepsilon > 0$ there exists a sequence $\{x_n\} \subset M_\varepsilon$, $x_n \geq 1$, such that $x_n \to \infty$ as $n \to \infty$ and

$$|u(x_n)^*Ju(x_n)| \leq 2\sqrt{2}\varepsilon K_{\ell,\lambda}.$$

Together with Lemma 7.6 this implies (7.12). □

Proof of Theorem 3.6. First (3.2) will be proved, i.e. that the limit in (7.2) is zero. Observe that if either $\{f, g\}$ or $\{h, k\}$ belongs to T^+_{min}, then

$$\lim_{x\to\infty} h(x)^*Jf(x) = 0. \tag{7.16}$$

To see this assume that $\{f, g\} \in T^+_{min}$. For each $u \in \mathbb{C}^2$, there exists $\{\varphi, \psi\} \in T^+_{max}$ as in Lemma 7.5, so that

$$0 = [g, \varphi] - [f, \psi] = u^*Jf(0+),$$

which implies that $f(0+) = 0$. Therefore, if $\{f, g\} \in T^+_{min}$, then (7.2) implies (7.16). The statement (3.2) for arbitrary $\{f, g\}, \{h, k\} \in T^+_{max}$ follows now from (7.16) and Lemma 7.8 by means of von Neumann's formula.

The first part of the proof shows that T^+_{min} is contained in the right side of (3.4). Conversely, if $\{f, g\} \in T^+_{max}$ and $f(0+) = 0$ then (3.2) shows that the right side of (7.3) is zero for every $\{h,k\} \in T^+_{max}$. Thus, $\{f,g\} \in T^+_{min}$. Finally, the identity (3.3) shows that the mapping $\{f,g\} \to f(0+)$ is a boundary mapping from T^+_{max} onto $\mathbb{C}$. This implies that the defect numbers of T^+_{min} are $(1,1)$. □

Lemma 7.9. *The linear fractional transform $m(z) = \frac{az+b}{cz+d}$, $ad-bc \neq 0$, maps the closed upper halfplane into itself if and only if the matrix $W = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies*

$$\frac{1}{i}(W^*JW - |\det W|J) \geq 0. \tag{7.17}$$

Proof. Observe that

$$\frac{1}{i}(m(z_1) - \overline{m(z_2)})(\overline{cz_2+d})(cz_1+d) = \frac{1}{i}\begin{pmatrix} z_2 \\ 1 \end{pmatrix}^* W^*JW \begin{pmatrix} z_1 \\ 1 \end{pmatrix}. \tag{7.18}$$

For $z = z_1 = z_2 = x+iy$ the right side can be rewritten as $\alpha x^2 + 2\mathrm{Im}\,\beta x + \gamma + \alpha y^2 - 2\mathrm{Re}\,\beta y$, where

$$\frac{1}{i}W^*JW = \begin{pmatrix} \alpha & \beta/i \\ -\bar\beta/i & \gamma \end{pmatrix}.$$

Moreover, $|\beta|^2 - \alpha\gamma = |\det W|^2$. Now $\mathrm{Im}\,m(z) \geq 0$ implies $\alpha \geq 0$, $\gamma \geq 0$, and

$$2\alpha|cz+d|^2\mathrm{Im}\,m(z) \geq |\alpha y - \mathrm{Re}\,\beta|^2 - |\det W|^2 \geq 0,$$

where the lower bound corresponds to $x = -\mathrm{Im}\,\beta/\alpha$, $\alpha > 0$. For $y=0$ this shows that $|\mathrm{Re}\,\beta| \geq |\det W|$ and further for $y > 0$ that $-\mathrm{Re}\,\beta \geq 0$, i.e., $-\mathrm{Re}\,\beta \geq |\det W|$. If $\alpha = 0$ then $\mathrm{Im}\,m(z) \geq 0$ yields $-\mathrm{Re}\,\beta = |\det W|$. On the other hand, (7.17) is equivalent to $\alpha \geq 0$, $\gamma \geq 0$, and

$$|\beta + |\det W||^2 \leq \alpha\gamma, \quad \text{or} \quad 2(|\det W|^2 + \mathrm{Re}\,b|\det W|) \leq 0.$$

This shows that $\mathrm{Im}\,m(z) \geq 0$ for $\mathrm{Im}\,z \geq 0$ implies (7.17). In view of (7.18) the converse statement is obvious. □

Proof of Theorem 2.1. Recall that it suffices to assume that the equation (2.1) is definite. It follows from (2.5) that for $\ell \in \mathbb{C}\setminus\mathbb{R}$

$$\frac{W(x,\ell)JW(x,\ell)^* - J}{\ell - \bar\ell} = \int_0^x W(t,\ell)H(t)W(t,\ell)^*\,dt \geq 0. \tag{7.19}$$

Thus also

$$\frac{W(x,\ell)^*JW(x,\ell) - J}{\ell - \bar\ell} \geq 0. \tag{7.20}$$

Let $a > 0$ be such that $(0,a]$ contains an interval of positive type, see Lemma 7.1. Then for every $x \geq a$ the matrix in (7.19), and hence also the matrix in (7.20), is invertible. To see this assume that for some e

$$\left(\int_0^x W(t,\ell)H(t)W(t,\ell)^*\,dt\right)e = 0.$$

This implies $H(t)W(t,\ell)^*e = 0$ so that by (2.4) $W(t,\ell)^*e$ is constant almost everywhere on $(0,x]$, and hence $e=0$, since the interval $(0,x]$ is of positive type. The invertibility of (7.19) shows that for each $x \geq a$ and $\ell \in \mathbb{C} \setminus \mathbb{R}$ the diagonal elements of (7.19), and therefore all the elements of $W(x,\ell)$, are nonzero. It follows from (2.6) and (7.20) that $w_{12}(x,\ell)/w_{11}(x,\ell)$ and $w_{22}(x,\ell)/w_{21}(x,\ell)$ are Nevanlinna functions. Let $x \geq a$, $\ell \in \mathbb{C}^+$, and consider the mappings

$$f(z) = \frac{w_{11}(x,\ell)z + w_{12}(x,\ell)}{w_{21}(x,\ell)z + w_{22}(x,\ell)}, \quad \hat{f}(z) = -\frac{w_{21}(x,\ell)z + w_{22}(x,\ell)}{w_{11}(x,\ell)z + w_{12}(x,\ell)},$$

$z \in \mathbb{C}^+ \cup \mathbb{R} \cup \{\infty\}$. Observe that $\hat{f}$ corresponds to $\widehat{W}(x,\ell) = JW(x,\ell)$, for which the equality

$$\widehat{W}(x,\ell)^* J \widehat{W}(x,\ell) = W(x,\ell)^* J W(x,\ell)$$

holds, and recall that $\det W(x,\ell) = 1$. By Lemma 7.9, $f(z)$ and $\hat{f}(z)$ map $\mathbb{C}^+ \cup \mathbb{R} \cup \{\infty\}$ into the closed upper halfplane. It follows from (7.18) and (7.20) that if $f(z) = 0$ (or $\hat{f}(z) = 0$) then $z \in \mathbb{R} \cup \{\infty\}$, where the case $z = \infty$ is not possible if $x > a$. This means that $z = -w_{12}(x,\ell)/w_{11}(x,\ell)$ (or $z = -w_{22}(x,\ell)/w_{21}(x,\ell)$) as a function of ℓ is a real constant. Thus, in (7.19) the first (respectively the second) diagonal element is zero, which is not possible if $x > a$, since for $x > a$ (7.19) is a positive invertible matrix. Hence, if $x > a$ the mappings $f(z)$ and $\hat{f}(z)$ do not have any zeros or poles in $\mathbb{C}^+ \cup \mathbb{R} \cup \{\infty\}$. The standard properties of Möbius transforms show that $f(z)$ maps the closed upper halfplane onto a disk $D(x)$ in the upper halfplane with the center at $f(c_0)$, $c_0 = -\overline{w}_{22}(x,\ell)/\overline{w}_{21}(x,\ell)$, and the radius

$$r(x,\ell) = |f(0) - f(c_0)| = \frac{1}{|w_{22}(x,\ell)\overline{w}_{21}(x,\ell) - w_{21}(x,\ell)\overline{w}_{22}(x,\ell)|}.$$

Moreover, $D(x_2) \subset D(x_1)$ for $x_2 > x_1$. To see this, let $\widetilde{H}(x) = H(x + x_1)$, $x > 0$, and let $\widetilde{W}(x,\ell)$ be the solution of (2.1) corresponding to $\widetilde{H}(x)$. Then $W(x + x_1,\ell)$ and $W(x_1,\ell)\widetilde{W}(x,\ell)$ satisfy the same initial value problem, which implies that

$$W(x + x_1,\ell) = W(x_1,\ell)\widetilde{W}(x,\ell).$$

Therefore,

$$f_{x_2,\ell}(z) = W(x_2,\ell) \circ z = W(x_1,\ell)\widetilde{W}(x_2 - x_1,\ell) \circ z = W(x_1,\ell) \circ \tilde{z} \in D(x_1).$$

Similarly $\hat{f}(z)$ maps the closed upper halfplane onto a disk $\hat{D}(x)$ in the upper halfplane with the center at $\hat{f}(d_0)$, $d_0 = -\overline{w}_{12}(x,\ell)/\overline{w}_{11}(x,\ell)$, and the radius

$$\hat{r}(x,\ell) = |\hat{f}(0) - \hat{f}(d_0)| = \frac{1}{|w_{12}(x,\ell)\overline{w}_{11}(x,\ell) - w_{11}(x,\ell)\overline{w}_{12}(x,\ell)|},$$

such that $\hat{D}(x_2) \subset \hat{D}(x_1)$ for $x_2 > x_1$. Observe that for $x > a$, $r(x,\ell) < \infty$ and $\hat{r}(x,\ell) < \infty$. Now consider the following solutions of (2.1):

$$v_j(x,\ell) = \begin{pmatrix} \overline{w}_{j1}(x,\bar{\ell}) \\ \overline{w}_{j2}(x,\bar{\ell}) \end{pmatrix}, \quad j = 1,2.$$

By Theorem 3.6 the defect numbers of T^+_{min} are $(1,1)$, so that one of the solutions cannot belong to $L^2(H,\mathbb{R}^+)$. Hence, it follows from (2.5) and

$$\frac{1}{r(x,\ell)} = |v_2(x,\ell)^* J v_2(x,\ell)|, \quad \frac{1}{\hat{r}(x,\ell)} = |v_1(x,\ell)^* J v_1(x,\ell)|,$$

that either $\lim_{x\to\infty} r(x,\ell) = 0$ or $\lim_{x\to\infty} \hat{r}(x,\ell) = 0$. In fact, these properties are equivalent since $f = -1/\hat{f}$ and f, $\hat{f}$ do not have any zeros or poles in $\mathbb{C}^+ \cup \mathbb{R} \cup \{\infty\}$. Thus, the limit $Q^+(\ell)$ in (2.7) exists and is independent of $t(\ell)$. Clearly, $Q^+(\ell)$ is finite and belongs to $\mathbb{C}^+ \cup \mathbb{R}$. That $Q^+(\ell)$ depends holomorphically on $\ell \in \mathbb{C}^+$ follows easily from the fact that the fraction in the right side of (2.7) is uniformly bounded on $[1,\infty)$ with $\ell \in K$, K a compact set in $\mathbb{C}^+$, cf. e.g. [1] and [28, Lemma 3.1].

To see (2.8) let $c > a$ and define

$$Q_c(\ell) = \frac{\overline{w}_{11}(c,\bar{\ell})}{\overline{w}_{21}(c,\bar{\ell})}, \quad \chi_c(x,\ell) = W(x,\bar{\ell})^* \begin{pmatrix} 1 \\ -Q_c(\ell) \end{pmatrix}.$$

Observe that $Q_c(\ell) \to Q^+(\ell)$ as $c \to \infty$ and that $Q_c(\ell)$ is a Nevanlinna function. In particular, (2.6) shows that $Q_c(\bar{\ell}) = \overline{Q}_c(\ell)$ and hence $\lim_{c\to\infty} Q_c(\bar{\ell}) = \overline{Q}^+(\ell) = Q^+(\bar{\ell})$. Clearly, $\chi_c(c,\ell)^* J \chi_c(c,\ell) = 0$ and $\chi_c(0,\ell)^* J \chi_c(0,\ell) = Q_c(\ell) - \overline{Q}_c(\ell)$, so that (7.19) implies

$$0 \le \int_0^b \chi_c(t,\ell)^* H(t) \chi_c(t,\ell)\, dt \le \int_0^c \chi_c(t,\ell)^* H(t) \chi_c(t,\ell)\, dt = \frac{Q_c(\ell) - \overline{Q}_c(\ell)}{\ell - \bar{\ell}}, \tag{7.21}$$

for $b < c$. Here $\chi_c(t,\ell) \to \chi^+(t,\ell)$ uniformly for $t \in [0,b]$ as $c \to \infty$. Therefore, letting $c \to \infty$ in (7.21) gives

$$0 \le \int_0^b \chi^+(t,\ell)^* H(t) \chi^+(t,\ell)\, dt \le \frac{Q^+(\ell) - \overline{Q}^+(\ell)}{\ell - \bar{\ell}}, \tag{7.22}$$

for every $b > 0$, which implies (2.8).

Finally, observe that if $Q^+(\ell) \in \mathbb{R}$ then (7.22) implies $H(t)\chi^+(\ell) = 0$ for almost every $t \in \mathbb{R}^+$. Hence by (2.4) $\chi^+(\cdot,\ell)$ is constant which means that the equation (2.1) is not definite and $H(x)$ is given by Example 2.2. □

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