A ReMeDI for Microstructure Noise *

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Abstract

We introduce the Realized moMents of Disjoint Increments (ReMeDI) paradigm to measure microstructure noise (the deviation of the observed asset prices from the fundamental values caused by market imperfections). We propose consistent estimators of arbitrary finite moments of a microstructure noise process, which could be serially dependent and nonstationary, based on high-frequency data. We characterize the limit distributions of the proposed estimators and construct robust confidence intervals under infill asymptotics. We further demonstrate that the ReMeDI approach also works on low-frequency, non-infill data. It thus can be applied to many asset pricing and macroeconomic models, in which the time series have a permanent and a transitory component.

We propose two liquidity measures that gauge the instantaneous and average bid-ask spread with potentially autocorrelated order flows. They can be consistently estimated within our framework. We provide an economic model to justify such measures as an intermediary's inventory risk measure when meeting serially dependent liquidity demand. Empirically we find our new liquidity measures are very effective to identify liquidity drains during the Flash Crash, when the market experienced extreme selling pressures.

KEYWORDS: Microstructure noise, liquidity measures, inventory models, order flows, infill asymptotics, permanent and transitory components, Flash Crash

1 Introduction

Economic time series are often modelled as the sum of a latent process obtained from an underlying economic model and another additive term that is either an error reflecting a variety

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of adjustments to the frictionless theoretical model, or a term that represents departures from the economic model, thus

$$\underbrace{Y}_{\text{observed series}} = \underbrace{X}_{\text{underlying process}} + \underbrace{\varepsilon}_{\text{deviation}}. \tag{1}$$

The two processes X and ε are generated by different mechanisms, and can have quite distinct statistical properties and economic interpretations. Both quantities may be of interest as they give interpretation to the underlying economic theory and its relevance for the observed data, or distinguish classes of economic models. However, since only the sum process Y is observable, this makes estimation and inference challenging.

We are concerned with applications of this framework in financial markets where the observed asset price[1] (Y) contains *market microstructure noise* (ε) that blurs the *efficient price* (or *fundamental value)* (X). The fundamental theorem of asset pricing says that X should be a semimartingale process (Delbaen and Schachermayer (1994)). In practice however, there are many market frictions that may cause observed prices to deviate from this ideal price, such as: transaction costs, price discreteness, inventory holdings, information asymmetry, measurement errors. One may also allow for temporary misspricing (French and Roll (1986)) or fads effects (Lehmann (1990)); see also O'Hara (1995), Hasbrouck (2007) and Foucault et al. (2013) for insightful reviews. A lot of early work proceeded on the basis that the microstructure noise process was i.i.d., but recently this assumption has been shown to be too strong; both theoretically and empirically the microstructure noise may exhibit rich dynamics depending on its sources. If the microstructure effects are negligible, the observed price should be close to the efficient price and be unpredictable. Therefore, the dispersion and persistence of the microstructure noise serve as natural measures of market quality. Market quality is of concern to regulators and practitioners as well as academics; proxies for market quality are widely used in empirical analysis, see Linton and Mahmoodzadeh (2018).

We next outline our contribution. We introduce a general econometric approach to measuring microstructure noise in a nonparametric setting. First, we propose a new estimator of the moments of a general dependent noise process called the *Realized moMents of Disjoint Increments* (ReMeDI) based on observed noisy high-frequency transaction prices. We assume that the underlying efficient price follows a semimartingale, which may accommodate stochastic volatility, jumps, etc. We allow the microstructure noise to be weakly dependent and to have serially correlation (of unknown form) that may decay algebraically; this may capture, for instance, the effects of clustered (or hidden) order flows (Gerig (2008)), or herding (Park and Sabourian (2011)). The microstructure noise is allowed to have time-varying heteroskedasticity, which allows for intraday variaiton in the scale of the noise. We develop the joint estimators of arbitrary moments of microstructure noise and derive the associated limit distributions. We provide a consistent estimator of the asymptotic variance that is carefully designed to improve its finite sample performance. Our setting also allows for certain degrees of irregularities in

[1]By *price* it always means the *logarithmic price* in this paper unless stated otherwise.

the observation schemes. *Statistically*, we provide a general method to separate a nonstationary and serially dependent sequence from a semimartingale; from an *economic* point of view, we identify the components of asset prices arising from market frictions.

A second contribution we make is to propose two new liquidity measures based on the autocovariance function of the microstructure noise. The first measure can be considered as a generalization of the Roll effective spread measure that is robust to autocorrelated order flow patterns. It measures the spread associated with a single trade, whence we think of it as an *instantaneous* measure. The second measure is designed to capture the *average* spread of multiple trades, where patterns of the order flow play a role. We develop a simple inventory model to provide economic insights about the average measure: it can be interpreted as an intermediary's inventory risk when he meets a sequence of possibly autocorrelated high-frequency liquidity demands that could reflect order splitting behavior or herding behavior, see Toth et al. (2015) and references therein. We also show how to estimate and conduct inference about these liquidity measures using our ReMeDI procedures.

Our third contribution is to develop several hypotheses tests about the microstructure noise. In particular, we test for the intraday patterns by comparing the magnitudes of noise in two intervals. We also provide tests that the microstructure noise is not autocorrelated. All tests are model-free, and can detect alternatives with rich dynamics and model specifications.

Our fourth contribution is that the ReMeDI approach is robust to data frequencies in a sense that it remains valid when applied to *low-frequency* data under *long-span* asymptotics. We show that the ReMeDI estimators are able to effectively separate a process with independent increments (X) and a stationary mixing sequence (ε). It is a surprising yet intuitive property of the ReMeDI design: the increments of X over disjoint intervals (the efficient returns) are uncorrelated, and what remains is attributed to ε. This property not only distinguishes the ReMeDI approach from alternative high-frequency estimators that rely structurally on the *infill asymptotics*, but also broadens the range of potential applications in other fields of economics and finance that utilize lower frequency data. In this setting the large sample variance is dominated by the variation of the efficient price process; we obtain a CLT for the estimator but at a slower rate than in the infill case. Finally, we provide a novel implementation rule to calculate the asymptotic variance of the ReMeDI estimators if one feels uncertain about the underlying asymptotic framework of his dataset.

We apply our methods to data from the E-mini stock index futures contract during the month of May 2010. There are several interesting empirical findings. First, we find that our liquidity measures are lower during the "European" segment of the trading day, and that these measures are fairly stable except during the Flash Crash day, May 6th, when the liquidity measures during the "American" segment rose dramatically. We find positive and significant autocorrelations only for the first few lags on the non Flash Crash days, much less dependence than for the individual stock case we discuss below. However, during the Flash Crash day this autocorrelation became much stronger and significant for many lags. This is consistent with for example the positive feedback loop hypothesis (Beddington et al. (2012)).

We also apply our method to individual stock price data from 2016 (presented in the supplementary materials Li and Linton (2019)). We find that the microstructure noise exhibits strong and positive autocorrelations in the first several lags, and has minor negative autocorrelations after many ticks. The strong and positive autocorrelations induce significant and persistent differences between the three liquidity measures: the average measure tends to be larger than the instantaneous one, and the classic Roll measure is further downward biased relative to our instantaneous measure. The magnitude of noise in the first five minutes is significantly larger than during the regular trading hours (9:35 — 15:55), but the magnitude of noise in the last five minutes is not so statistically different.

The rest of the paper is organized as follows. Section 1.1 reviews the related literature. Section 2 sets the continuous-time framework for the efficient price and microstructure noise. Section 3 introduces the ReMeDI estimators and outlines the intuition of the design. Section 4 presents the asymptotic theory. Section 5 and Section 6 demonstrate the economic and statistical applications. Section 7 extends the ReMeDI approach to low-frequency settings. The empirical studies are presented in Section 8, and Section 9 concludes. All mathematical proofs, simulation studies, the procedures to select the tuning parameters and additional empirical studies on an individual stock are collected in the supplementary materials Li and Linton (2019).

1.1 Related literature

1.1.1 High-frequency econometrics

There are a number of methods for estimation of the moments of noise and the parameters of the efficient price. Specifically: the two-scale/multi-scale realized volatility by Zhang et al. (2005), Zhang (2006), Aït-Sahalia et al. (2011); the optimal-sampling realized variance by Bandi and Russell (2006, 2008); the maximum likelihood estimators by Aït-Sahalia et al. (2005), Xiu (2010); the pre-averaging method developed in Podolskij and Vetter (2009), Jacod et al. (2009); and the realized kernel by Hansen and Lunde (2006), Barndorff-Nielsen et al. (2008). Most of this literature only considers i.i.d. microstructure noise.

Several recent papers explore richer microstructure models by considering autocorrelated noise. The estimators of the second moments of noise in Da and Xiu (2019) and Li et al. (2019) are by-products of the integrated volatility estimators in the presence of autocorrelated noise. In a recent seminal work, Jacod et al. (2017) introduce the first estimator, called *local averaging* (LA) method to measure arbitrary moments of microstructure noise using high-frequency data. They also introduce a general framework of stochastic observation scheme and microstructure noise with a semimartingale "size process". Our main results are derived under slightly less general assumptions than theirs, although we generalize the consistency result to the precise setting of Jacod et al. (2017). We differentiate our paper from Jacod et al. (2017) as follows. First, the ReMeDI method is based on *differencing*, while the LA is based on deviations from *local averages*, both ideas are widely used in other contexts such as panel data to eliminate nuisance parameters. Second, the ReMeDI approach works beyond the infill framework. It can be applied on low-frequency data and it has a limit distribution under the long-span asymptotics. This statistical property has an economically intuitive interpretation. Thus, the ReMeDI estimators can be used by asset pricers with daily or coarser returns, or researchers in market microstructure working on millisecond prices. The LA method, however, is inconsistent when applied to low-frequency data. Third, we propose new measures of market liquidity and provide economic interpretations. Next, even within the high-frequency framework, the ReMeDI estimators are very robust. The LA estimators are more sensitive to data frequencies. As showed by Jacod et al. (2017) that the LA estimators have a finite sample bias, which is a fraction of the *a priori* unknown integrated volatility of the efficient price. The bias could dominate the noise parameters in practical situations,[2] and this might cause many issues in the implementations with real data.[3] The bias of the ReMeDI estimators by contrast only depends on the slope of the microstructure autocovariance function. Last, the ReMeDI approach has another two advantages in real implementations: it is computationally very efficient,[4] and it has a clear rule to select the tuning parameters.[5]

Chen and Mykland (2017) propose several tests of the intraday pattern, or stationarity of microstructure noise. The tests are formed by comparing the TSRV developed by Zhang et al. (2005) and a modified TSRV introduced by Kalnina and Linton (2008). We distinguish our test of the intraday pattern from two aspects. First, our test is designed to improve the finite sample performance. It is well established (see Hansen and Lunde (2006)) that the realized volatility type estimators of noise variance have a finite sample bias component that is a fraction of the integrated volatility of the underlying efficient price process. Such bias term, as analysed by Li et al. (2019), could wipe out the moments of noise in practical circumstances. Thus to investigate the intraday patterns of microstructure noise, it is essential to effectively separate microstructure noise and volatility in a finite sample, as the latter also exhibits prominent intraday patterns, see Andersen and Bollerslev (1997). Second, we explicitly incorporate autocorrelated microstructure noise. The TSRV approach of volatility estimation allows for certain forms of weakly dependent noise, see Aït-Sahalia et al. (2011). But the statistical tests developed in Chen and Mykland (2017) may not be applicable then, as the asymptotic variances of the test statistics depend on other higher moments of noise.

[2]The detailed analysis by Li et al. (2019) reveals that the bias is determined by both the data frequency and the noise-to-signal ratio (the ratio of the variance of noise and the integrated volatility). Empirically, they show that even using tick by tick data without any filtration, the bias remains significant.

[3]In the empirical analysis by Jacod et al. (2017), the authors are puzzled about the strong dependence in the microstructure noise. As we will show in the simulation studies that the strong and positive dependence in noise after many lags is largely due to the finite sample bias of the LA estimators. However, correcting their bias in practice is not trivial as one needs a proxy of the integrated volatility. Li et al. (2019) encountered a similar situation and they propose a two/multi-step approach to make the bias correction.

[4]For example, the LA (ReMeDI) takes 99.77% (0.23%) of the CPU time to estimate the variance of noise using noisy price from a random walk plus AR(1) noise model, base on 1,000 simulated samples of size 23,400.

[5]Jacod et al. (2017) also propose a heuristic rule to select the tuning parameters for the LA method by comparing the LA estimates to a variant of the realized variances. However, the variant of the realized variances needs an estimate of the integrated volatility to make a bias correction. The estimation of the integrated volatility in the presence of serially dependent microstructure noise is not trivial (Li et al. (2019)). Moreover, the LA estimates have a finite sample bias as well. Thus comparing two sequences of statistics both coupled with significant biases may lack insights on the choice of the tuning parameters.

1.1.2 Market microstructure

Our paper is also related to the market microstructure literature that seek to measure market liquidity and study the impacts of order flow patterns on market liquidity, see Roll (1984), Hasbrouck (1993), Aït-Sahalia and Yu (2009), Corwin and Schultz (2012) and Abdi and Ranaldo (2017). The patterns of order flows affect the measurement of market liquidity. Specifically, Roll (1984) assumes i.i.d. order flows; Hasbrouck and Ho (1987), Choi et al. (1988) consider an AR(1) model with positive coefficient to model the continuation of transactions; AR(1) models with negative coefficient are studied to model the inventory costs, see Ho and Stoll (1981), Hendershott and Menkveld (2014). Some recent studies on infrequent rebalancing (Bogousslavsky (2016)), limit attention (Hendershott et al. (2018)), or the interaction of high-frequency traders and low-frequency traders (Aït-Sahalia and Sağlam (2017a,b)) reveal that the microstructure noise has more complicated patterns. Our setting is more general, and our empirical studies suggest the presence of more complex dynamics that fall outside the above standard frameworks .

Our paper introduces an econometric approach to richer microstructure models. It aims to integrate the market microstructure and financial econometrics literature. It is, however, not the first attempt to push towards the integration of the two fields. Diebold and Strasser (2013) focus on the correlation of efficient price and noise in several leading microstructure models, and study the implications for integrated volatility estimation. Li et al. (2016) model the microstructure noise as a parametric function of the trading information and develop an efficient volatility estimator, see also Chaker (2017) and Clinet and Potiron (2017) for similar approaches. Bandi et al. (2017) develop a novel measure of the staleness of stock returns under the infill asymptotic framework. Bollerslev et al. (2018) study the relationship between trading volume and return volatility around important public news announcements using intraday high-frequency data. The study relies critically on high-frequency econometric techniques to identify jumps. Da and Xiu (2019) advocate the quasi-maximum likelihood approach to estimate both the volatility and the autocovariances of moving-average microstructure noise.

1.1.3 Macroeconomics and asset pricing

The random walk plus stationary component decompositions are popular in macroeconomics and finance. The random walk part carries the *permanent* variations of the observed time series while the stationary component carries the *temporary* or *transitory* fluctuation. Cochrane (1988) measures the random walk component of GNP growth, see also Cochrane (1994), Campbell and Mankiw (1987). Beveridge and Nelson (1981) introduce a general method to decompose time series into permanent and transitory components, and the latter is used to date the business cycles, see recent studies by Oh et al. (2008) and Sinclair (2009). Summers (1986) specifies a random walk plus AR(1) process as an alternative hypothesis to *market efficiency*, see also Poterba and Summers (1988) and Fama and French (1988).

The time series models in the above papers are essentially discrete, and estimations are based on low-frequency data. The discrete-time model we considered in this paper is very general (see Section 7), nesting the popular random walk plus autoregressive processes, and the ReMeDI approach remains effective to separate the two components in such settings.

2 Continuous-Time Framework and Assumptions

In this section we state precisely our assumptions regarding the continuous-time efficient price process, the observation scheme, and the microstructure noise.

2.1 Efficient price process

We assume that the efficient price process X is an Itô semimartingale defined on a filtered probability space $\left(\Omega^{(0)}, \mathcal{F}^{(0)}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}^{(0)}\right)$ with the Grigelionis representation

$$X_t = X_0 + \int_0^t b_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W_s + \left(\vartheta \mathbf{1}_{\{|\vartheta|\leq 1\}}\right) \star (\mathfrak{p} - \mathfrak{q})_t + \left(\vartheta \mathbf{1}_{\{|\vartheta|>1\}}\right) \star \mathfrak{p}_t, \tag{2}$$

where $W, \mathfrak{p}$ are a Wiener process and a Poisson random measure on $\mathbb{R}_+ \times E$ respectively. Here, $(E, \mathcal{E})$ is a measurable Polish space on $\left(\Omega^{(0)}, \mathcal{F}^{(0)}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}^{(0)}\right)$ and the predictable compensator of $\mathfrak{p}$ is $\mathfrak{q}(\mathrm{d}s, \mathrm{d}z) = \mathrm{d}s \otimes \lambda(\mathrm{d}z)$ for some given σ-finite measure on $(E, \mathcal{E})$, see also Aït-Sahalia and Jacod (2014) and Jacod and Shiryaev (2003) for detailed introduction of the last two integrals. We assume that X satisfies the following assumption:

Assumption (H). *The process b is locally bounded, the process σ is càdlàg , there is a localizing sequence $\{\tau_n\}_n$ of stopping times and for each n a deterministic nonnegative function Γ_n on E satisfying $\int \Gamma_n^2(z)\lambda(\mathrm{d}z) < \infty$ such that $|\vartheta(\omega, t, z)| \wedge 1 \leq \Gamma_n(z)$ for all (ω, t, z) satisfying $t \leq \tau_n(\omega)$.*

2.2 Observation scheme

For each n, let $\{t_i^n : i \in \mathbb{N}_+\}$ be a sequence of strictly increasing finite and deterministic observed transaction times with $0 = t_0^n < t_1^n < \ldots$, where $\mathbb{N}_+$ is the set of nonnegative integers. We denote

$$n_t = \sum_{i\geq 1} \mathbf{1}_{\{t_i^n \leq t\}}, \quad \delta(n, i) = t_i^n - t_{i-1}^n. \tag{3}$$

Here, n_t is the number of observations recorded on the interval $[0, t]$ for $t \in \mathbb{R}_+$, while $\delta(n, i)$ is the i^{th} spacing of the transaction times.

Definition 2.1 (The notation V_i^n)**.** *Since we consider both the* infill *and* long-span *asymptotics, we introduce a unified notation that works in both settings for any process V. First, we use n_t (infill setting) and n (long-span setting) to denote the sample size. If V is a continuous-time function or process, we use V_i^n to denote its discretized observation upon time t, i.e., $V_i^n = V_{t_i^n}, i = 0, \ldots, n_t$; if $\{V_i\}_{i\in\mathbb{Z}}$ is a discrete process, where $\mathbb{Z}$ is the set of integers, we denote $V_i^n = V_i, 0 \leq i \leq n$ or $V_i^n = V_i, 0 \leq i \leq n_t$.*

Let $\{\delta_n\}_n$ be a positive sequence of real numbers satisfying $\delta_n \to 0$ as $n \to \infty$. Then, δ_n can be considered as the time lag between successive observations when the transaction times are equally spaced. It serves as a benchmark to compare with the real observations schemes that will be specified below. More general schemes are relegated to Section 4.5.

Assumption (O). *Let α be a nonnegative deterministic càdlàg process satisfying $0 < c_\alpha \le \alpha_t \le C_\alpha < \infty\ \forall t$ for some positive constants c_α, C_α. Further assume*

$$\left|\alpha_{i-1}^n \delta(n,i) - \delta_n\right| \le C\delta_n^{\frac{3}{2}+\kappa} \tag{4}$$

for some $\kappa > 0$.

Under the above assumptions, it follows that as $n \to \infty$,

$$\delta_n n_t \to A_t := \int_0^t \alpha_s \mathrm{d}s. \tag{5}$$

Thus, α captures the "density" of observations. If the observation scheme is regular, i.e., $\delta(n,i) = \delta_n$ for all i, thus $n_t = [t/\delta_n]$, then $\alpha_t \equiv 1$ for all t satisfies (4) and $A_t = t$.

2.3 Microstructure noise

We allow the microstructure noise to be dependent according to the ρ-mixing property.[6]

Definition 2.2. *Let $\{\chi_i\}_{i\in\mathbb{Z}}$ be a sequence of stationary random variables defined on a probability space $(\Omega^{(1)}, \mathcal{G}, \mathbb{P}^{(1)})$. The probability space has discrete filtrations $\mathcal{G}_p := \sigma\{\chi_k : p \ge k\}, : \mathcal{G}^q := \sigma\{\chi_k : q \le k\}$ satisfying $\mathcal{G}^{-\infty} = \mathcal{G}_\infty = \mathcal{G}$. For any $k \in \mathbb{N}_+$, we define the following* mixing coefficients *for $k \in \mathbb{N}_+$:*

$$\rho_k := \sup\left\{|\mathbb{E}(V_h V_{k+h})| : \mathbb{E}(V_k) = \mathbb{E}(V_{k+h}) = 0, \|V_h\|_2 \le 1, \|V_{k+h}\|_2 \le 1, V_h \in \mathcal{G}_h, V_{k+h} \in \mathcal{G}^{k+h}\right\}. \tag{6}$$

The sequence $\{\chi_i\}_{i\in\mathbb{Z}}$ is ρ mixing *if $\rho_k \to 0$ as $k \to \infty$.*

The following assumption characterises the degree of serial dependence of $\{\chi_i\}_{i\in\mathbb{Z}}$, which is necessary to obtain limit results.

Assumption 2.1 (Polynomially mixing coefficients)**.** *There is some $C > 0, v > 0$ such that*

$$\rho_k \le \frac{C}{k^v} \quad \forall\, k \in \mathbb{N}_+. \tag{7}$$

An immediate consequence of this assumption is that the autocovariance function of $\{\chi_i\}_i$ is decaying at a polynomial rate, i.e.,

$$|\mathbf{Cov}(\chi_i, \chi_{i+k})| \le \frac{C}{k^v}, \tag{8}$$

[6] The key results in this paper also hold under a strong mixing condition, which is weaker than ρ-mixing, see Bradley (2005). However, the moment conditions and the restrictions on the mixing coefficients will be more involved. On the other hand, Jacod et al. (2017) employ ρ-mixing sequence to model microstructure noise. Thus it will facilitate the comparison with their result if we stick to the same setting.

where $C > 0$ is some positive constant. We shall suppose that $v > 1$ for consistency and that $v > 3$ for the CLT, which allows for quite strong dependence close to the long memory boundary.

There is a large literature seeking to characterize the economic mechanisms that govern the dynamic properties of microstructure noise, for example, the continuation of order flows modeled by Hasbrouck and Ho (1987), Choi et al. (1988), reversal order flows due to market maker's risk aversion by Grossman and Miller (1988) and Campbell et al. (1993) or inventory controls by Ho and Stoll (1981), Hendershott and Menkveld (2014), and the presence of inattentive (or infrequent) traders by Bogousslavsky (2016) and Hendershott et al. (2018). Econometric models of microstructure noise focus usually on the i.i.d. case, although there are some works on the MA(q) process, see Hansen and Lunde (2006), Hansen et al. (2008), Hautsch and Podolskij (2013) and the ARMA(p, q) case, see Barndorff-Nielsen et al. (2008), Hendershott et al. (2013). Note that our setting incorporates all the models.

Empirical studies have documented overwhelming evidence of various intraday patterns of microstructure noise, see, e.g., Chan and Lakonishok (1995), Hasbrouck (1993), Madhavan et al. (1997), McInish and Wood (1992) and Wood et al. (1985). The magnitudes of microstructure noise typically exhibit a U-shape or reverse J-shape. To capture such patterns, we introduce a "size function" γ to allow for time-varying microstructure noise.[7]

Assumption (N). *Let $\{\chi_i\}_{i\in\mathbb{Z}}$ be a stationary ρ-mixing random sequence with mixing coefficients $\{\rho_k\}_{k\in\mathbb{N}_+}$ on some probability space $(\Omega^{(1)}, \mathcal{G}, \mathbb{P}^{(1)})$. At stage n, the noise at time t_i^n is given by*

$$\varepsilon_i^n = \gamma_{t_i^n} \cdot \chi_i, \tag{9}$$

where γ_u is a (deterministic) positive Lipschitz continuous function of u. We further assume that $\{\chi_i\}_{i\in\mathbb{Z}}$ is centred at 0 with variance 1 and finite moments of all orders, and $\mathcal{G}$ is independent of $\mathcal{F}^{(0)}$.

Finally, the noisy observed price Y_i^n is given by (for $i = 1, \dots, n_t$)

$$Y_i^n = X_i^n + \varepsilon_i^n. \tag{10}$$

The process ε_i^n is consistent with the model of Engle and Rangel (2008), which allows for stochastic volatility of the GARCH type as well as slow variation coming from γ. We may also allow ε_i^n to have a slowly changing non-zero mean, which may be motivated by the "Drift burst hypothesis" (Christensen et al. (2018)), because our method will difference this out. The assumption of independence between the efficient price and the microstructure noise, however, excludes many interesting microstructure models, in which the efficient returns are usually

[7] Jacod et al. (2017) model γ as a general nonnegative semimartingale process such that γ could depend on X. Our setting is simpler and is motivated by the empirical facts we find that the magnitude of microstructure noise is close to a constant during the trading day and some irregularity arises only in the beginning of the trading session. If, however, the measurement of microstructure noise has a finite sample bias that depends on X, e.g., the integrated volatility of X, one could conclude that the size of microstructure noise is dependent on X and the intraday pattern he discovers may be the manifestation of the intraday pattern of X instead. Or equivalently, any estimators designed to recover the patterns of noise should be able to distinguish the efficient price and microstructure noise effectively.

correlated with the microstructure noise to reflect informational effects, see, e.g., Hendershott et al. (2013). To the best of our knowledge, we are not aware of any nonparametric method that can separate X and ε without assuming uncorrelatedness (Kalnina and Linton (2008) allow for a specific type of small dependence between the noise and the efficient price, Jacod et al. (2017) and Da and Xiu (2019) allow for high-order dependence, but maintain the assumption of uncorrelatedness). However, it seems that the ReMeDI estimators are quite robust to such misspecification, as we show in our extensive simulation studies in the supplementary material Li and Linton (2019). We also present a consistency result for our estimator under the weaker conditions of Jacod et al. (2017).

3 The Design and the Intuition of the ReMeDI Estimators

3.1 The estimator of the autocovariance function

The intuition of the ReMeDI design can be best explained by considering the estimation of the autocovariances of a stationary time series. Let $\{\varepsilon_i\}_{i\in\mathbb{Z}}$ be a mixing sequence with mean zero; we would like to estimate $r_\ell := \mathbf{Cov}(\varepsilon_i, \varepsilon_{i+\ell})$. The natural estimator is the sample analogue

$$\widehat{r}^n_\ell := \frac{1}{n}\sum_{i=0}^{n-\ell} \varepsilon^n_i \varepsilon^n_{i+\ell},$$

which is consistent and asymptotically normal under very mild conditions.

We consider instead an estimator that replaces the "observations" $\varepsilon^n_i, \varepsilon^n_{i+\ell}$ by the "differences", i.e.,

$$\widetilde{r}^n_\ell := \frac{1}{n}\sum_{i=k'_n}^{n-\ell-k_n} \left(\varepsilon^n_i - \varepsilon^n_{i-k'_n}\right)\left(\varepsilon^n_{i+\ell} - \varepsilon^n_{i+\ell+k_n}\right),$$

where k_n, k'_n are integers that will grow at certain rates as the sample size increases. The estimator $\widetilde{r}^n_\ell$ follows the ReMeDI design and it provides another consistent estimator of r_ℓ, provided: $k_n \to \infty$, $k'_n \to \infty$, $k_n/n \to 0$, and $k'_n/n \to 0$. The intuition of the consistency becomes immediate if one rewrites $\widetilde{r}^n_\ell$ as

$$\widetilde{r}^n_\ell = \frac{1}{n}\sum_{i=k'_n}^{n-\ell-k_n} \varepsilon^n_i\varepsilon^n_{i+\ell} - \frac{1}{n}\sum_{i=k'_n}^{n-\ell-k_n} \varepsilon^n_i\varepsilon^n_{i+\ell+k_n} - \frac{1}{n}\sum_{i=k'_n}^{n-\ell-k_n} \varepsilon^n_{i-k'_n}\varepsilon^n_{i+\ell} + \frac{1}{n}\sum_{i=k'_n}^{n-\ell-k_n} \varepsilon^n_{i-k'_n}\varepsilon^n_{i+\ell+k_n}. \quad (11)$$

The first average is (asymptotically) equivalent to the sample analogue, thus it converges in probability to r_ℓ; the remaining three averages are centred at $r_{\ell+k_n}$, $r_{\ell+k'_n}$, and $r_{\ell+k_n+k'_n}$, which themselves converge to zero as $n \to \infty$ at a rate depending on (8).

Taking differences seems redundant if the time series $\{\varepsilon_i\}_i$ is observable. However, in our framework ε is masked by the efficient price X, and what is observable is $Y = X + \varepsilon$. Taking differences removes the effect of the efficient price. The intuition of such removal under infill asymptotics is that the differences of the efficient prices, say, $X^n_i - X^n_{i-k'_n}$, are much smaller than the difference of the noise as n increases. In the absence of infill asymptotics, the method works because of the zero autocorrelations of the efficient returns. We provide more details on this case in Section 7.

3.2 The general ReMeDI design

We next generalize the above approach to formally define our ReMeDI (Realized moMents of Disjoint Increments) estimator. First, we provide some notations that will be used below. Let $\mathfrak{J}$ be the set of all finite sequences of integers satisfying

$$\mathfrak{J} = \left\{ \mathbf{j} = (j_1, j_2, \ldots, j_q) : j_p \in \mathbb{Z}, p = 1, 2, \ldots, q; q \geq 2 \right\}.$$

In the sequel, we will assume without loss of generality that $j_1 = \max\{j_p : j_p \in \mathbf{j}\}$ for any $\mathbf{j} \in \mathfrak{J}$. The $\mathbf{j}$-moments of χ, the stationary component of microstructure noise, are given by

$$\boldsymbol{r}(\mathbf{j}) = \mathbb{E}\left(\prod_{p=1}^{q} \chi_{j_p} \right). \tag{12}$$

We are interested in such moments as target values, both in their own right, but also as they provide information about parameters of a model for χ. For example, Hua et al. (2019) consider an AR(1) process for χ, in which case moments $\mathbb{E}(\chi_i \chi_{i-j})$ can be used to estimate the process' parameters.[8] We may also be interested in functionals of the nonparametric scale function γ_u.

Let $\boldsymbol{k} = (k_1, \ldots, k_q)$ be a q-tuple of integers. For any $\mathbf{j} \in \mathfrak{J}$ and any process V, let $\mathbb{I}(\boldsymbol{k}, \mathbf{j})_t^n$ be the set of observation indices on $[0, t]$ for which the following *multi-difference operator* $\Delta_{\mathbf{j}}^{\boldsymbol{k}}(\cdot)_i^n$ is well defined:

$$\Delta_{\mathbf{j}}^{\boldsymbol{k}}(V)_i^n := \prod_{p=1}^{q} \left(V_{i+j_p}^n - V_{i+j_p-k_p}^n \right). \tag{13}$$

Then the ReMeDI estimator associated with $(V, \mathbf{j}, \boldsymbol{k})$ is defined by

$$\text{ReMeDI}(V; \mathbf{j}, \boldsymbol{k})_t^n := \sum_{i \in \mathbb{I}(\boldsymbol{k}, \mathbf{j})_t^n} \Delta_{\mathbf{j}}^{\boldsymbol{k}}(V)_i^n. \tag{14}$$

Remark 3.1. *Using the above notations, we rewrite the estimator $\widetilde{r}_\ell^n$ as follows*

$$\widetilde{r}_\ell^n = \frac{1}{n} \sum_{i=k_n'}^{n-\ell-k_n} \Delta_{0,\ell}^{-k_n, k_n'}(\varepsilon)_i^n.$$

The general ReMeDI *approach inherits two salient features of this estimator determined by the choices of $\boldsymbol{k}$: 1) the first entry of $\boldsymbol{k}$ will be negative whereas the remaining ones are positive, i.e., the first difference is a* forward difference *and the remaining ones are* backward differences; *2) $\forall 1 \leq p \leq q$, $|k_p| \to \infty$ as $n \to \infty$, and we will often write $\boldsymbol{k}_n = (k_{1,n}, \ldots, k_{q,n})$ in the sequel to reflect such dependence.*

We discuss a little more the intuition of the general ReMeDI design under infill asymp-

[8] A slightly more complicated case arises if χ_i satisfied a GARCH process with parameters θ; in that case, Kristensen and Linton (2006) show how to use the moments $\mathbb{E}\left(\chi_i^2 \chi_{i-j}^2\right)$ to estimate θ.

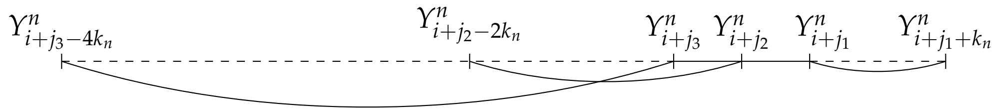

Figure 1: Illustration the ReMeDI estimator of $\mathbf{j}$-moments with $\mathbf{j} = (j_1, j_2, j_3)$ and $\boldsymbol{k}_n = (-k_n, 2k_n, 4k_n)$.

totics. For this purpose, suppose that the noise size process γ is constant and we are estimating $\mathbb{E}\left(\prod_{p=1}^{q} \varepsilon_{i+j_p}^n\right)$. Suppose $\boldsymbol{k}_n$ satisfies the two properties in Remark 3.1. Next, we explain how to connect $\mathbb{E}\left(\prod_{p=1}^{q} \varepsilon_{i+j_p}^n\right)$ and $\Delta_{\mathbf{j}}^{\boldsymbol{k}_n}(Y)_i^n$ by $\Delta_{\mathbf{j}}^{\boldsymbol{k}_n}(\varepsilon)_i^n$.

To see this, we first note $\{i + j_p - k_{p,n}\}_p$ are the "distant" indices of the intervals on which the backward and forward differences are taken. Figure 1 illustrates a simple example with $\mathbf{j} = (j_1, j_2, j_3), \boldsymbol{k}_n = (-k_n, 2k_n, 4k_n)$ for some $k_n \in \mathbb{N}_+$. The forward difference starts at the $(i + j_1)$-th observation and ends at the $(i + j_1 + k_n)$-th observation; for the remaining indices in $\mathbf{j}$, the associated differences start from $i + j_2, i + j_3$ and end at $i + j_2 - 2k_n, i + j_3 - 4k_n$, respectively. The intuition of the ReMeDI approach is that the "distant" noise terms are approximately independent of each other, and are also independent of the "clustered" noise $\{\varepsilon_{i+j_p}^n\}_p$ (recall a special case outlined in (11)), therefore

$$\mathbb{E}\left(\Delta_{\mathbf{j}}^{\boldsymbol{k}_n}(\varepsilon)_i^n\right) \approx \mathbb{E}\left(\prod_{p=1}^{q} \varepsilon_{i+j_p}^n\right).$$

If $k_{p,n}$ is still relatively small such that $\sup_p \delta_n \left|k_{n,p}\right| \to 0$, the differences/increments of the efficient price over the intervals are asymptotically negligible. That is,

$$\Delta_{\mathbf{j}}^{\boldsymbol{k}_n}(Y)_i^n \approx \Delta_{\mathbf{j}}^{\boldsymbol{k}_n}(\varepsilon)_i^n.$$

Thus the averages of $\Delta_{\mathbf{j}}^{\boldsymbol{k}_n}(Y)_i^n$ will converge in probability to $\mathbb{E}\left(\prod_{p=1}^{q} \varepsilon_{i+j_p}^n\right)$ by the law of large numbers. This explains the intuition of the identification.

A salient feature of the ReMeDI design is that the noisy log-return $Y_{i+j_1}^n - Y_{i+j_1+k_{1,n}}^n$, the first factor of $\Delta_{\mathbf{j}}^{\boldsymbol{k}_n}(Y)_i^n$, is taken on an interval disjoint from others. As we will see in Section 7 that such design achieves consistency out of the infill asymptotics framework.

4 The ReMeDI Estimators under Infill Asymptotics

4.1 Consistency

We next give the large sample properties of the ReMeDI (for a given choice of $\boldsymbol{k}_n$) estimator in our general setting. Given $\mathbf{j}$, the noise moments $\mathbb{E}\left(\prod_{p=1}^{q} \varepsilon_{i+j_p}^n\right) = C_\gamma^q \boldsymbol{r}(\mathbf{j})$ if $\gamma_u \equiv C_\gamma \; \forall u$. For a general γ function that satisfies Assumption (N), the "average size" of the noise moments becomes $\int_0^t \gamma_s^q \mathrm{d}A_s / A_t$ instead of C_γ^q, and it appears in the probability limit of the ReMeDI estimators.

Theorem 4.1. *Let Assumptions (H, O, N) hold, assume $v > 1$ and $\boldsymbol{k}_n$ satisfies*

$$\begin{cases} -k_{1,n} \to \infty, k_{p,n} \to \infty, \forall\, p \geq 2, \\ \sup_p \left|\delta_n k_{n,p}\right| \to 0, \forall\, p \geq 1, \\ k_{p+1,n} - k_{p,n} \to \infty, \forall\, p \geq 2, \end{cases} \tag{15}$$

as $n \to \infty$. For $\mathbf{j} \in \mathfrak{J}$, we have the following convergence in probability:

$$\frac{\mathrm{ReMeDI}(Y; \mathbf{j}, \boldsymbol{k}_n)_t^n}{n_t} \xrightarrow{\mathbb{P}} \mathbf{R}(\mathbf{j})_t := \frac{\int_0^t \gamma_s^q \mathrm{d}A_s}{A_t} \boldsymbol{r}(\mathbf{j}), \tag{16}$$

where $\boldsymbol{r}(\mathbf{j})$ is defined in (12) and A_t in (5).

Remark 4.1. *Let $\{k_n\}_n$ be a sequence of integers satisfying $k_n \to \infty$, $k_n\delta_n \to 0$. Let $\boldsymbol{k}_n$ be specified as follows:*

$$k_{p,n} = \begin{cases} -k_n & \text{if } p = 1, \\ (p-1)k_n & \text{if } p \geq 2. \end{cases}$$

Then $\boldsymbol{k}_n$ satisfies the conditions in (15).

The microstructure noise specified in (9) has a multiplicative structure; the following result states that we can separate the components of microstructure noise locally.

Theorem 4.2 (A local estimator of γ)**.** *Let Assumptions (H, O, N) hold and assume $v > 1$. Let $\{k_n\}_n, \{\ell_n\}_n$ be two sequences of integers satisfying*

$$k_n \to \infty, \quad k_n/\ell_n \to 0, \quad \ell_n^2 \delta_n = O(1).$$

Let $\mathbf{j}_0 = (0, 0), \overline{\boldsymbol{k}}_n = (-k_n, k_n)$. Let $\mathbb{I}_{\ell_n}(\mathbf{j}_0, \overline{\boldsymbol{k}}_n)_t^n := \mathbb{I}(\mathbf{j}_0, \overline{\boldsymbol{k}}_n)_t^n \backslash \mathbb{I}(\mathbf{j}_0, \overline{\boldsymbol{k}}_n)_{t - \frac{\delta_n \ell_n}{\alpha_t}}^n$. A ReMeDI *estimator of γ_t is given by*

$$\widehat{\gamma}_t^n := \sum_{i \in \mathbb{I}_{\ell_n}(\mathbf{j}_0, \overline{\boldsymbol{k}}_n)_t^n} \Delta_{\mathbf{j}_0}^{\overline{\boldsymbol{k}}_n}(Y)_i^n / \ell_n \xrightarrow{\mathbb{P}} \gamma_t.$$

One could use this estimator to re-estimate the noise moments using the rescaled data, thereby separating the scaling factor from the noise. We use it below to obtain consistent estimates of the asymptotic variance of the ReMeDI estimator.

4.2 Limit distribution

In the sequel whenever we have two vectors $\mathbf{j} = (j_1, \ldots, j_q), \mathbf{j}' = (j_1', \ldots, j_{q'}') \in \mathfrak{J}$, we suppose without loss of generality that $q \leq q'$. We denote

$$\mathbf{j} \oplus \mathbf{j}' = (j_1, j_2, \ldots, j_q, j_1', j_2', \ldots, j_{q'}'), \quad \mathbf{j}_{-p} = \mathbf{j} \backslash \{j_p\},$$
$$\mathbf{j}(+k) = (j_1 + k, j_2 + k, \ldots, j_q + k), \text{ for } k \in \mathbb{Z},$$
$$\mathbf{j}_{Q_q} = (j_p : p \in Q_q) \text{ for } Q_q \subset \{1, 2, \ldots, q\}.$$

For each $Q_q \subset \{1, 2, \dots, q\}$, there is an associated (unique) pair of subsets:

$$Q_q^c := \{1, 2, \dots, q\} \backslash Q_q, \quad Q_{q'} := Q_q \cup \{q+1, \dots, q'\}. \tag{17}$$

We denote for each $k \in \mathbb{Z}$ the following moments[9]

$$\begin{aligned}
\boldsymbol{s}_0(\mathbf{j}, \mathbf{j}'; k) &:= \boldsymbol{r}\left(\mathbf{j} \oplus \left(\mathbf{j}'(+k)\right)\right) - \boldsymbol{r}\left(\mathbf{j}\right) \boldsymbol{r}\left(\mathbf{j}'\right); \\
\boldsymbol{s}_1(\mathbf{j}, \mathbf{j}'; k) &:= \sum_{Q_q \subsetneq \{1,2,\dots,q\}} \boldsymbol{r}\left(\mathbf{j}_{Q_q} \oplus \left(\mathbf{j}'_{Q_{q'}}(+k)\right)\right) \prod_{p \in Q_q^c} \boldsymbol{r}(j_p, j'_p + k); \\
\boldsymbol{s}_2(\mathbf{j}, \mathbf{j}'; k) &:= \sum_{\substack{j_p \in \mathbf{j}, j'_{p'} \in \mathbf{j}' \\ p \neq p'}} \boldsymbol{r}(j_p, j'_{p'} + k) \boldsymbol{r}(\mathbf{j}_{-p}) \boldsymbol{r}(\mathbf{j}'_{-p'}) - \sum_{j_p \in \mathbf{j}} \boldsymbol{r}(\{j_p\} \oplus \mathbf{j}'(+k)) \boldsymbol{r}(\mathbf{j}_{-p}) \\
&\quad - \sum_{j'_{p'} \in \mathbf{j}'} \boldsymbol{r}(\{j'_{p'} + k\} \oplus \mathbf{j}) \boldsymbol{r}(\mathbf{j}'_{-p'}).
\end{aligned}$$

Now we further specify the choice of $\boldsymbol{k}_n$ to derive the limit distributions of the ReMeDI estimators. $\boldsymbol{k}_n$ will solely be determined by an integer k_n, which is related to v as follows

$$\begin{aligned}
&k_{p,n} = \begin{cases} -k_n & \text{if } p = 1, \\ 2^{p-1} k_n & \text{if } p \geq 2, \end{cases} \\
&v > 3, \quad k_n \asymp \delta_n^{-\varrho}, \varrho \in \left(\frac{1}{2v}, \delta\right), \text{ where } \delta \in \left(\frac{1}{v+2}, \frac{1}{5}\right).
\end{aligned} \tag{18}$$

The lower bound of k_n is to guarantee the estimator converges faster enough to the true parameters than the convergence rate, while the upper bound is to satisfy a Lindeberg's condition.

Remark 4.2. *Note that (18) implies (15). In the sequel, we will omit* $\boldsymbol{k}_n$ *and simply write* $\Delta_{\mathbf{j}}(Y)_i^n$ *and* $\mathrm{ReMeDI}(Y; \mathbf{j})_t^n$ *instead of* $\Delta_{\mathbf{j}}^{\boldsymbol{k}_n}(Y)_i^n$ *and* $\mathrm{ReMeDI}(Y; \mathbf{j}, \boldsymbol{k}_n)_t^n$ *when* $\boldsymbol{k}_n$ *satisfies (18).*

Let

$$\overline{Z}(\mathbf{j})_t^n := \sqrt{n_t} \left(\frac{\mathrm{ReMeDI}(Y; \mathbf{j})_t^n}{n_t} - \mathbf{R}(\mathbf{j})_t \right).$$

The following theorem presents the joint limit distribution of $\overline{Z}(\mathbf{j})_t^n, \overline{Z}(\mathbf{j}')_t^n$.

Theorem 4.3. *Let Assumptions (H), (O) and (N) hold, and* k_n, v *satisfy (18). For any* $t > 0, \mathbf{j}, \mathbf{j}' \in \mathfrak{J}$*, we have the following convergence in law*

$$\left(\overline{Z}(\mathbf{j})_t^n, \overline{Z}(\mathbf{j}')_t^n\right) \xrightarrow{\mathcal{L}} \left(\overline{Z}(\mathbf{j})_t, \overline{Z}(\mathbf{j}')_t\right), \tag{19}$$

where the limit is centred Gaussian with (co)variances:

$$\sigma(\mathbf{j}, \mathbf{j}')_t := \mathbb{E}\left(\overline{Z}(\mathbf{j})_t \overline{Z}(\mathbf{j}')_t\right) = \mathfrak{S}(\mathbf{j}, \mathbf{j}') \frac{\int_0^t \gamma_s^{q+q'} \mathrm{d}A_s}{A_t}, \tag{20}$$

[9] By convention we let $\boldsymbol{r}(\emptyset) = 1$.

and

$$\mathfrak{S}(\mathbf{j}, \mathbf{j}') := \sum_{k\in\mathbb{Z}} \sum_{m=0}^{2} \boldsymbol{s}_m(\mathbf{j}, \mathbf{j}'; k). \tag{21}$$

This limit theory makes use of the assumption that the noise process is bounded away from zero and so relatively large on the small time scale. One can allow a more general setting such as Kalnina and Linton (2008) wherein the noise process shrinks to zero with sample size but at a rate slower than $\sqrt{n_t}$. In that case, one can obtain a self-normalized CLT. We consider this case further in Section 7.

4.3 Estimating the autocovariances of microstructure noise

In this section we consider the special case concerning the estimation of the variance and autocovariances of microstructure noise. Let $\mathbf{j}_\ell = (0, \ell), \ell \in \mathbb{N}_+$.

$$\widehat{R}^n_{t,\ell} := \frac{1}{n_t}\text{ReMeDI}(Y; \mathbf{j}_\ell)^n_t = \frac{1}{n_t}\sum_{i=2k_n}^{n_t-k_n-\ell} \left(Y^n_{i+\ell} - Y^n_{i+\ell+k_n}\right)\left(Y^n_i - Y^n_{i-2k_n}\right). \tag{22}$$

The following corollary provides the limit distribution.

Corollary 4.1. *Under the conditions of Theorem 4.3, we have*

$$\sqrt{n_t}\left(\widehat{R}^n_{t,\ell} - R_{t,\ell}\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \mathfrak{S}_\ell \frac{\int_0^t \gamma_s^4 \mathrm{d}A_s}{A_t}\right),$$

where

$$R_{t,\ell} := r_\ell \frac{\int_0^t \gamma_s^2 \mathrm{d}A_s}{A_t}, \quad \mathfrak{S}_\ell := \sum_{k=-\infty}^{\infty} \left(\mathbb{E}((\chi_0\chi_\ell - r_\ell)(\chi_k\chi_{k+\ell} - r_\ell)) + 3r_k^2\right). \tag{23}$$

In the special case of i.i.d. noise, the asymptotic variance reduces to $\mathbb{E}(\chi_0^4) + 2r_0^2$ up to a constant magnitude of noise. We detail the intuition of asymptotic variances in the next section.

4.4 The asymptotic variance and its estimation

Like the probability limit, the asymptotic variance of the ReMeDI estimators has two components. Specifically, $\mathfrak{S}(\mathbf{j}, \mathbf{j}')$ is the asymptotic variance of the stationary part of the ReMeDI estimators. It has several constituents: $\sum_{k\in\mathbb{Z}} \boldsymbol{s}_0(\mathbf{j}, \mathbf{j}'; k)$ is the asymptotic variance of the sample analogue of $\boldsymbol{r}(\mathbf{j})$, whereas $\sum_{k\in\mathbb{Z}} \boldsymbol{s}_1(\mathbf{j}, \mathbf{j}'; k) + \boldsymbol{s}_2(\mathbf{j}, \mathbf{j}'; k)$ are attributed to noise at the "distant" indices at the ends of each intervals. Consider $\mathfrak{S}_\ell$ defined in (23) as an example. If $\{\chi_i\}_{i\in\mathbb{N}}$ is directly observable and we estimate r_ℓ by the sample analogue $\sum_i \chi_i\chi_{i+\ell}/n_t$, the asymptotic variance will be $\sum_{k=-\infty}^{\infty} \mathbb{E}((\chi_0\chi_\ell - r_\ell)(\chi_k\chi_{k+\ell} - r_\ell))$. The presence of the "distant" noise increases the asymptotic variance by $\sum_{i\in\mathbb{Z}} 3r_i^2$, which is the asymptotic variances of the last three terms in (11) (with $k_n' = 2k_n$).

Compared to the ReMeDI estimators, the local averaging (LA) estimators proposed by Jacod et al. (2017) are asymptotically more efficient in our settings, as only $\sum_{k\in\mathbb{Z}} s_0(\mathbf{j},\mathbf{j}';k)$ appears in the asymptotic variance of the stationary part (although one can improve the efficiency of ReMeDI by taking averages of estimators computed using different $\boldsymbol{k}_n$). However, simulation studies show that the ReMeDI class works better in finite samples with realistic sample sizes or equivalently, data frequency — it has smaller finite sample variance and is almost unbiased under various model specifications. Moreover, the loss of asymptotic efficiency is compensated for by the greater computational efficiency of the ReMeDI approach, which pays off when one is working with massive high-frequency datasets (recall Footnote 4).

We now turn to the inference question. One can just estimate term by term using ReMeDI for diferent combinations ignoring asymptotic independence and using Parzen-Newey-West weighting. However, we find that this approach poorly represents the finite sample variance. We propose instead to first recenter with local ReMeDI estimator; it seems to work better in practice.

For given integers i, ϖ_n, we define a local ReMeDI statistic

$$\text{ReMeDI}_{\varpi_n}(Y;\mathbf{j})_i^n := \sum_{\ell=i+1}^{i+\varpi_n} \Delta_{\mathbf{j}}(Y)_\ell^n. \tag{24}$$

so that $\varpi_n^{-1}\text{ReMeDI}_{\varpi_n}(Y;\mathbf{j})_i^n$ will be a local estimate of the $\mathbf{j}$-moments of noise. We also introduce the Newey-West weights to improve the finite sample performance (see Newey and West (1987)): for any $i, u_n \in \mathbb{N}_+$,

$$\nu_{i,u_n} = 1 - \frac{i}{1+u_n}. \tag{25}$$

Let $\widehat{\boldsymbol{\Sigma}}_t^n$ be a 2×2 matrix with components

$$\widehat{\boldsymbol{\Sigma}}_t^n(1,1) := \widehat{\sigma}(Y;\mathbf{j},\mathbf{j})_t^n, \quad \widehat{\boldsymbol{\Sigma}}_t^n(1,2) = \widehat{\boldsymbol{\Sigma}}_t^n(2,1) := \widehat{\sigma}(Y;\mathbf{j}',\mathbf{j})_t^n, \quad \widehat{\boldsymbol{\Sigma}}_t^n(2,2) := \widehat{\sigma}(Y;\mathbf{j}',\mathbf{j}')_t^n,$$

where

$$\begin{aligned}
&\widehat{\sigma}(Y;\mathbf{j},\mathbf{j}')_t^n := \frac{1}{n_t}\sum_{i=2^{q'-1}k_n}^{n_t-\varpi_n'} \left(\overline{\Delta}_{\mathbf{j}}(Y)_i^n\overline{\Delta}_{\mathbf{j}'}(Y)_i^n + \sum_{k=1}^{i_n}\nu_{k,i_n}\left(\overline{\Delta}_{\mathbf{j}}(Y)_i^n\overline{\Delta}_{\mathbf{j}'}(Y)_{i+k}^n + \overline{\Delta}_{\mathbf{j}'}(Y)_i^n\overline{\Delta}_{\mathbf{j}}(Y)_{i+k}^n\right)\right);\\
&\overline{\Delta}_{\mathbf{j}}(Y)_i^n := \Delta_{\mathbf{j}}(Y)_i^n - \varpi_n^{-1}\text{ReMeDI}_{\varpi_n}(Y;\mathbf{j})_i^n; \quad \overline{\Delta}_{\mathbf{j}'}(Y)_i^n := \Delta_{\mathbf{j}'}(Y)_i^n - \varpi_n^{-1}\text{ReMeDI}_{\varpi_n}(Y;\mathbf{j}')_i^n;\\
&\varpi_n' := \varpi_n + k_n + i_n + j_1 \vee j_1'.
\end{aligned} \tag{26}$$

Theorem 4.4. *Assume all the conditions of Theorem 4.3 hold, let $\{i_n\}_n, \{\varpi_n\}_n$ satisfy*

$$i_n \asymp \delta_n^{-\delta}, \quad \varpi_n \asymp \delta_n^{-\phi}, \quad 2\delta < \phi < 1/2. \tag{27}$$

We have

$$\mathbf{L}_t^n \begin{pmatrix} \overline{Z}(\mathbf{j})_t^n \\ \overline{Z}(\mathbf{j}'^n_t \end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}\left(\mathbf{0}, \mathbf{I}_2\right), \tag{28}$$

where $\mathbf{L}_t^n$ *is a matrix satisfying* $\mathbf{L}_t^n \mathbf{L}_t^{n\top} = \left(\widehat{\boldsymbol{\Sigma}}_t^n\right)^{-1}$.

Corollary 4.2 (Estimation of the autocovariance and autocorrelation function)**.** *Let* $\mathbf{j}_\ell = (0, \ell), \ell \in \mathbb{N}_+$*. Under the conditions of Theorem 4.4, we have*

$$\sqrt{\frac{n_t}{\widehat{\sigma}(Y; \mathbf{j}_\ell, \mathbf{j}_\ell)_t^n}} \left(\widehat{R}_{t,\ell}^n - R_{t,\ell}\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, 1\right), \tag{29}$$

$$\sqrt{\frac{n_t}{\widehat{S}_{t,\ell}^n}} \left(\frac{\widehat{R}_{t,\ell}^n}{\widehat{R}_{t,0}^n} - r_\ell\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, 1\right), \tag{30}$$

where

$$\widehat{S}_{t,\ell}^n := \left(\frac{1}{\widehat{R}_{t,0}^n}, -\frac{\widehat{R}_{t,\ell}^n}{\left(\widehat{R}_{t,0}^n\right)^2}\right) \widehat{\boldsymbol{\Sigma}}_t^n \left(\frac{1}{\widehat{R}_{t,0}^n}, -\frac{\widehat{R}_{t,\ell}^n}{\left(\widehat{R}_{t,0}^n\right)^2}\right)^\top,$$

and $\widehat{\boldsymbol{\Sigma}}_t^n$ *is a* 2×2 *matrix with* $\widehat{\boldsymbol{\Sigma}}_t^n(1,1) = \widehat{\sigma}(Y; \mathbf{j}_\ell, \mathbf{j}_\ell)_t^n$, $\widehat{\boldsymbol{\Sigma}}_t^n(2,2) = \widehat{\sigma}(Y; \mathbf{j}_0, \mathbf{j}_0)_t^n$, $\widehat{\boldsymbol{\Sigma}}_t^n(1,2) = \widehat{\boldsymbol{\Sigma}}_t^n(2,1) = \widehat{\sigma}(Y; \mathbf{j}_\ell, \mathbf{j}_0)_t^n$.

The local ReMeDI statistics (24) are introduced to capture the time-varying noise, and the statistics can be replaced by a global one $\text{ReMeDI}(Y; \mathbf{j})_t^n / n_t$ if stationarity holds; the global estimator requires less tuning parameters and reduces the computational cost.

Corollary 4.3 (Asymptotic (co)variances estimators with stationary noise)**.** *Assume* $\gamma \equiv C_\gamma > 0$. *Let*

$$\widetilde{\sigma}(Y; \mathbf{j}, \mathbf{j}')_t^n := \frac{1}{n_t} \sum_{i=2^{q'-1}k_n}^{n_t - k_n - j_1 \vee j_1' - i_n} \left(\widetilde{\Delta}_{\mathbf{j}}(Y)_i^n \widetilde{\Delta}_{\mathbf{j}'}(Y)_i^n + \sum_{k=1}^{i_n} \nu_{k,i_n} \left(\widetilde{\Delta}_{\mathbf{j}}(Y)_i^n \widetilde{\Delta}_{\mathbf{j}'}(Y)_{i+k}^n + \widetilde{\Delta}_{\mathbf{j}'}(Y)_i^n \widetilde{\Delta}_{\mathbf{j}}(Y)_{i+k}^n\right)\right);$$

$$\widetilde{\Delta}_{\mathbf{j}}(Y)_i^n := \Delta_{\mathbf{j}}(Y)_i^n - \frac{\text{ReMeDI}(Y; \mathbf{j})_t^n}{n_t}; \quad \widetilde{\Delta}_{\mathbf{j}'}(Y)_i^n := \Delta_{\mathbf{j}'}(Y)_i^n - \frac{\text{ReMeDI}(Y; \mathbf{j}')_t^n}{n_t}.$$

Then

$$\widetilde{\sigma}(Y; \mathbf{j}, \mathbf{j}')_t^n \xrightarrow{\mathbb{P}} C_\gamma^{q+q'} \mathfrak{S}(\mathbf{j}, \mathbf{j}').$$

4.5 Extensions to the model setting

In this section, we extend the consistency of the ReMeDI estimators under a broader setting introduced by Jacod et al. (2017) in which the observation times and the size parameters of the noise, γ, are stochastic and possibly related to the efficient price process.

Assumption (E-O) (Assumption (O) in Jacod et al. (2017)). *$\alpha, \overline{\alpha}$ are two Itô semimartingales defined on $\left(\Omega^{(0)}, \mathcal{F}^{(0)}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}^{(0)}\right)$ satisfying Assumption (H). We further assume there is a localizing sequence $\{\tau_m\}_m$ of stopping times and positive constants $\kappa_{m,p}$ and κ such that:*

1. *For $t < \tau_m$ we have $\frac{1}{\kappa_{m,1}} \leq \alpha_{t-} \leq \kappa_{m,1}$ and $\overline{\alpha}_{t-} \leq \kappa_{m,1}$.*
2. *Let $(\mathcal{F}^n_t)_{t\geq 0}$ be the smallest filtration satisfying*
   (a) *$\mathcal{F}_t \subset \mathcal{F}^n_t$,*
   (b) *t^n_i is a $\{\mathcal{F}^n_t\}_{t\geq 0}$ stopping time for $i = 0, 1, 2, \ldots$,*
   (c) *$\delta(n, i)$, conditional $\mathcal{F}^n_t$, is independent of $\mathcal{F}_\infty := \bigvee_{t\geq 0} \mathcal{F}_t$ for $i = 0, 1, 2, \ldots$*
3. *With the restriction $\{t^n_{i-1} < \tau_m\}$, and for all $p > 0$,*

$$\begin{aligned}
&\left| \mathbb{E}\left(\delta(n,i)\,|\mathcal{F}^n_t\right) - \frac{\delta_n}{\alpha^n_{i-1}} \right| \leq \kappa_{m,1}\delta_n^{\frac{3}{2}+\kappa}, \\
&\left| \mathbb{E}\left(\left(\delta(n,i)\alpha^n_{i-1} - \delta_n\right)^2 |\mathcal{F}^n_t \right) - \delta_n^2 \overline{\alpha}^n_{i-1} \right| \leq \kappa_{m,2}\delta_n^{2+\kappa}, \\
&\mathbb{E}\left(\delta(n,i)^p\,|\mathcal{F}^n_t\right) \leq \kappa_{m,p}\delta_n^p.
\end{aligned} \tag{31}$$

Assumption (E-N). *The process χ satisfies Assumption (N). Moreover, it is independent of $\mathcal{F}_\infty$. Assume the process γ is a positive Itô semimartingale on $\left(\Omega^{(0)}, \mathcal{F}^{(0)}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}^{(0)}\right)$ and its coefficients satisfy Assumption (H).*

Theorem 4.5. *Assume Assumptions (H, E-O, E-N) hold, and $\boldsymbol{k}_n$ satisfies (15). For $\mathbf{j} \in \mathfrak{J}$, we have the following convergence in probability:*

$$\frac{\mathrm{ReMeDI}(Y; \mathbf{j}, \boldsymbol{k}_n)^n_t}{n_t} \xrightarrow{\mathbb{P}} \mathbf{R}(\mathbf{j})_t. \tag{32}$$

5 Economic Applications: New Measures of the Bid-Ask Spread

In this section we introduce two liquidity measures and define ReMeDI estimators of them. The measures are based on the rationale that the dispersion of microstructure noise, the deviation of the observed prices from fundamental values, serves as a natural measure of market quality, see Hasbrouck (1993) and Aït-Sahalia and Yu (2009), among others. The first measure, which we call the instantaneous bid-ask spread (IBAS), can be considered as a generalized and robust Roll measure (Roll (1984)). The second liquidity measure gauges the average bid-ask spread (ABAS). It can be interpreted as the inventory risk that an intermediary takes when he meets a liquidity demand that may exhibit certain autocorrelation patterns.

5.1 The Roll measure and the IBAS measure

Roll (1984) proposes a simple measure of the effective bid-ask spread. In Roll's model, the efficient price follows a random walk, and the microstructure noise becomes the half-signed bid-ask spread, i.e., $\varepsilon_i = Sq_i/2$, where $q_i = 1$ or -1 if the trade is initiated by a buyer or seller and S is the spread. He further assumes that the efficient price is independent of the order flow process $\{q_i\}_i$, which is serially uncorrelated with $\mathbb{P}(q_i = 1) = \mathbb{P}(q_i = -1) = 0.5$. The Roll measure is based on the insight that the autocovariance of the adjacent observed returns captures the bid-ask bounce. Using our notations, the Roll measure is

$$\text{Roll}_t^n := 2\sqrt{\frac{1}{n_t}\sum_{i=1}^{n_t-1} \Delta_{0,0}^{-1,1}(Y)_i^n}. \tag{33}$$

In this case, the Roll measure provides a consistent estimator of the bid-ask spread, i.e.,

$$\text{Roll}_t^n \xrightarrow{\mathbb{P}} S = 2\sqrt{\mathbf{Var}(Sq_i/2)}.$$

We consider a general setting introduced in Section 2 for the half-spread, i.e., $\varepsilon_i^n = \gamma_{t_i^n}\chi_i$, where $\{\chi_i\}_i$ is a stationary mixing sequence, which could be serially dependent. The serial dependence in ε may reflect various order flow patterns triggered by economic agents, and γ can capture the time-varying spread. There is considerable evidence that order flow is strongly positively autocorrelated, see, for example, Gerig (2008).

We propose an alternative liquidity measure

$$\text{IBAS}_\text{t} := 2\sqrt{R_{t,0}}, \tag{34}$$

where $R_{t,0}$ is defined in (23). If $\varepsilon_i = Sq_i/2$, and $\{q_i\}_i$ are uncorrelated with $\mathbb{P}(q_i = 1) = \mathbb{P}(q_i = -1) = 0.5$, we are back to the Roll's setting and we have $\text{IBAS}_t = S$. A consistent estimator of IBAS_t is given by

$$\text{IBAS}_t^n := 2\sqrt{\widehat{R}_{t,0}^n} \xrightarrow{\mathbb{P}} \text{IBAS}_t. \tag{35}$$

Compared to the Roll measure, IBAS_t^n is robust to autocorrelations in the half-spread ε_i. In fact, it is well known, see, e.g., Choi et al. (1988), that the Roll measure becomes biased if the order flows are autocorrelated. One can show that $\text{Roll}_t^n \xrightarrow{\mathbb{P}} S\sqrt{1 - 2r_1 + r_2}$, where $r_k = \mathbf{Cov}(q_i, q_{i+k})$. In particular, if $\{q_i\}_i$ is an AR(1) process with coefficient $1 > \varrho_q > 0$, we have $\text{Roll}_t^n \xrightarrow{\mathbb{P}} S\left(1 - \varrho_q\right) < S$. We may apply the ReMeDI estimators to correct the bias of the Roll measure. Let

$$\text{AdjRoll}_t^n := 2\sqrt{(\text{Roll}_t^n/2)^2 + 2\widehat{R}_{t,1}^n - \widehat{R}_{t,2}^n}. \tag{36}$$

The adjusted Roll measure has the same probability limit as the IBAS_t^n: $\text{AdjRoll}_t^n \xrightarrow{\mathbb{P}} \text{IBAS}_t$, which reduces to S if the spread is constant.[10]

IBAS_t is an instantaneous measure as it gauges the spread of a single trade. Next, we propose an average measure that captures the spread of multiple trades.

[10]Empirically, we observe almost perfect matches of AdjRoll_t^n and IBAS_t^n, see Figure C.2 in our supplementary material Li and Linton (2019).

5.2 The ABAS measure and its interpretation

We introduce a new liquidity measure that measures the average long run dispersion of the spread,

$$\text{ABAS}_t := 2\sqrt{R_{t,0} + 2\sum_{\ell=1}^{\infty} R_{t,\ell}}. \tag{37}$$

This makes intuitive sense since $\mathbf{Var}\left(\sum_{i=1}^{n_t} \varepsilon_i^n\right)/n_t \to R_{t,0} + 2\sum_{\ell=1}^{\infty} R_{t,\ell}$. The ReMeDI estimator of ABAS is given by (the choice of κ_n will be specified later, also recall the weights ν_{ℓ,κ_n} are defined in (25)):

$$\text{ABAS}_t^n := 2\sqrt{\widehat{R}_{t,0}^n + 2\sum_{\ell=1}^{\kappa_n} \widehat{R}_{t,\ell}^n}. \tag{38}$$

We show below that ABAS_t^n consistently estimates ABAS_t and we derive its limiting distribution.

In the next section we give a specific economic interpretation of the ABAS_t measure.

5.2.1 A simple model of inventory risk when order flows are autocorrelated

We introduce a simple model motivated by the pioneering work of Stoll (1978). The novelty lies in the explicit modelling of the order flows and a simple measure of the imbalances in order flows.

At time τ, a representative intermediary has cash c_τ with a position 0 in a risky stock. There are n trades occurred at time $\{T_i\}_{i=1}^n$, with $\tau < T_1 < \cdots < T_n = \tau + 1$, which could be clustered orders submitted by traders trading at a faster speed (Li (2017)), or orders received at different exchanges (Menkveld (2008)). The intermediary sets a price Y_τ for all trades between τ and $\tau + 1$, and supplies $\{z_i\}_{i=1}^n$; $z_i > 0 (z_i < 0)$ indicates the intermediary is willing to sell (buy) $z_i(-z_i)$ shares. The payoff is realized at time $\tau + 1$ with price equal to the fundamental value $X_{\tau+1}$, which evolves stochastically as

$$X_{\tau+1} = X_\tau + \theta_\tau,$$

where θ_τ has conditional mean 0 and variance $\mathbf{Var}_\tau(\theta\tau) = \sigma_\tau^2$. At time $\tau + 1$, the intermediary has cash

$$c_{\tau+1} = c_\tau + Y_\tau \sum_{i=1}^{n} z_i,$$

and his aggregate wealth becomes

$$w_{\tau+1} = -X_{\tau+1}\sum_{i=1}^{n} z_i + c_{\tau+1} = (Y_\tau - (X_\tau + \theta_\tau))\sum_{i=1}^{n} z_i + c_\tau.$$

Thus, the conditional mean of $w_{\tau+1}$ is $\mathbb{E}_\tau(w_{\tau+1}) = (Y_\tau - X_\tau)\sum_{i=1}^{n} z_i + c_\tau$ and the conditional variance is $\mathbf{Var}_\tau(w_{\tau+1}) = \left(\sum_{i=1}^n z_i\right)^2 \sigma_\tau^2$. The intermediary has mean-variance utility:

$$U_\tau(w) = \mathbb{E}_\tau(w) - \frac{\rho_a}{2}\mathbf{Var}_\tau(w),$$

where ρ_a is the risk-aversion coefficient. To maximize his utility, the intermediary solves

$$\max_{\{z_i\}_{i=1}^n} U_\tau(w_{\tau+1}).$$

At the optimum, the first order condition says that

$$\sum_{i=1}^n z_i = \frac{Y_\tau - X_\tau}{\rho_a \sigma_\tau^2}.$$

In equilibrium, the representative intermediary meets the (stochastic) demand $\{q_i\}_{i=1}^n$, where we assume $q_i = 1$ indicating a buy and $q_i = -1$ a sell. Thus, $\sum_{i=1}^n q_i = \frac{Y_\tau - X_\tau}{\rho_a \sigma_\tau^2}$, which implies that

$$Y_\tau = X_\tau + \rho_a \sigma_\tau^2 \sum_{i=1}^n q_i. \tag{39}$$

The deviation of the transaction price Y_τ from the fundamental value reflects the intermediary's inventory risk to meet an incoming order flow $\{q_i\}_i$. Therefore, the spread compensates the intermediary to provide liquidity.

When there is only one trade, i.e., $n = 1$, then the bid and ask prices are $Y_\tau = X_\tau - \rho_a\sigma_\tau^2$ and $Y_\tau = X_\tau + \rho_a\sigma_\tau^2$; thus the instantaneous spread S_1 is given by $S_1 = 2\rho_a\sigma_\tau^2$. Similar to the Roll's model, $S_1 = 2\sqrt{\mathbf{Var}_\tau(\rho_a\sigma_\tau^2 q_1)}$, twice the dispersion of the deviation $\rho_a\sigma_\tau^2 q_1$, if $\mathbb{P}(q_1 = 1) = \mathbb{P}(q_1 = -1) = 0.5$.

When n is large, we need another measure to reflect the overall inventory risk the intermediary takes. A natural measure of such risk is provided by twice the average dispersion of $\rho_a\sigma_\tau^2\sum_{i=1}^n q_i$, i.e.,

$$S_n := 2\sqrt{\frac{1}{n}\mathbf{Var}_\tau\left(\rho_a \sigma\tau^2 \sum_{i=1}^n q_i\right)}.$$

We denote $S_\infty := \lim_{n\to\infty} S_n$; S_∞ is the long run average measure. Let $r_k = \mathbf{Cov}(q_i, q_{i+k})$, and assume that $\{q_i\}_i$ are independent of σ_τ and $\sum_{k=-\infty}^{\infty}|r_k| < \infty$. We have

$$S_\infty = 2\rho_a\sigma_\tau^2\sqrt{r_0 + 2\sum_{k=1}^{\infty} r_k}.$$

Consider the special case where $\{q_i\}_i$ is an AR(1) process with coefficient ϱ_q. Then, we have $S_\infty = S_1\sqrt{(1+\varrho_q)/(1-\varrho_q)}$. When the order flow is positively autocorrelated, i.e., $\varrho_q \in (0,1)$, the average measure S_∞ is greater than the instantaneous measure S_1, as the intermediary is more likely to accumulate inventories on one side of the market, which entails a larger spread to compensate his risk.

5.3 The limit distribution of IBAS_t^n and ABAS_t^n

Theorem 5.1. *Assume all conditions of Theorem 4.4 hold, and further assume*

$$v > 1 + \max\left\{\frac{2}{\phi - 2\delta}, \frac{1}{1-2\phi}\right\}, \quad \kappa_n \asymp \delta_n^{-\psi}, \quad \frac{1}{2v-2} < \psi < \min\left\{\frac{1}{2}\left(\frac{\phi}{2} - \delta\right), \frac{1}{2} - \phi\right\}. \tag{40}$$

Then we have the following limit distribution

$$\frac{\sqrt{n_t}\left(\mathrm{IBAS}_t^n - \mathrm{IBAS}_t\right)}{\sqrt{\widehat{\mathbf{\Sigma}}_t^{n,\kappa_n}(0,0)/\mathrm{IBAS}_t^n}} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1), \tag{41}$$

$$\frac{\sqrt{n_t}\left(\mathrm{ABAS}_t^n - \mathrm{ABAS}_t\right)}{2\sqrt{\widehat{\Sigma}_t^{n,\kappa_n}/\mathrm{ABAS}_t^n}} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1), \tag{42}$$

where

$$\widehat{\Sigma}_t^{n,\kappa_n} := \widehat{\mathbf{\Sigma}}_t^{n,\kappa_n}(0,0) + 4\sum_{\ell=1}^{\kappa_n} \widehat{\mathbf{\Sigma}}_t^{n,\kappa_n}(\ell,0) + 4\sum_{\ell,\ell'=1}^{\kappa_n} \widehat{\Sigma}_t^{n,\kappa_n}(\ell,\ell'),$$

and

$$\widehat{\mathbf{\Sigma}}_t^{n,\kappa_n}(\ell,\ell') := \widehat{\sigma}(Y;\mathbf{j}_\ell,\mathbf{j}_{\ell'})_t^n. \tag{43}$$

Remark 5.1. *The asymptotic conditions in (40) seem quite complicated. But if the mixing coefficients (recall ρ_k in Definition 2.2) are exponentially decaying, e.g., a stationary invertible ARMA(p,q) model of microstructure noise, the conditions will hold.*

6 Statistical Applications: Testing Statistical Properties of Microstructure Noise

6.1 Testing for intraday patterns

It is well documented that microstructure noise exhibits intraday patterns. This section develops a formal test of the intraday patterns by examining the difference of the average magnitudes of noise over two intervals. Given $s \geq 0$, let $\underline{n}_s = \inf\{i \in \mathbb{N} : t_i^n \geq s\}, \overline{n}_s = \sup\{i \in \mathbb{N} : t_i^n < s\}$. For any $0 \leq T_1 < T_2$, let

$$\mathrm{ReMeDI}(Y;\mathbf{j}_0)_{T_1,T_2}^n := \sum_{i=\underline{n}_{T_1}+2^q k_n}^{\overline{n}_{T_2}-k_n-j_1} \Delta_{\mathbf{j}}(Y)_i^n;$$

$$\widehat{\sigma}(Y;\mathbf{j}_0,\mathbf{j}_0)_{T_1,T_2}^n := \frac{\sum_{i=\underline{n}_{T_1}+2^q k_n}^{\overline{n}_{T_2}-\varpi_n}\left(\overline{\Delta}_{\mathbf{j}}(Y)_i^n \overline{\Delta}_{\mathbf{j}'}(Y)_i^n + \sum_{k=1}^{i_n} \nu_{k,i_n}\left(\overline{\Delta}_{\mathbf{j}}(Y)_i^n \overline{\Delta}_{\mathbf{j}'}(Y)_{i+k}^n + \overline{\Delta}_{\mathbf{j}'}(Y)_i^n \overline{\Delta}_{\mathbf{j}}(Y)_{i+k}^n\right)\right)}{\overline{n}_{T_2} - \underline{n}_{T_1}}.$$

Thus the statistics $\text{ReMeDI}(Y;\mathbf{j}_0)^n_{T_1,T_2}, \widehat{\sigma}(Y;\mathbf{j}_0,\mathbf{j}_0)^n_{T_1,T_2}$ are the ReMeDI estimators based on observations between time T_1 and T_2. The following result provides a device to compare the average variances of microstructure noise on the intervals $[T_1, T_2]$ and $[T_3, T_4]$, corrected for the number of observations.

Let $0 \le T_1 < T_2 \le T_3 < T_4$.

$$\mathbb{H}_0 : \frac{\int_{T_1}^{T_2} \gamma_s^2 \mathrm{d}A_s}{A_{T_2} - A_{T_1}} = \frac{\int_{T_3}^{T_4} \gamma_s^2 \mathrm{d}A_s}{A_{T_4} - A_{T_3}}; \quad \mathbb{H}_A : \frac{\int_{T_1}^{T_2} \gamma_s^2 \mathrm{d}A_s}{A_{T_2} - A_{T_1}} \neq \frac{\int_{T_3}^{T_4} \gamma_s^2 \mathrm{d}A_s}{A_{T_4} - A_{T_3}}. \tag{44}$$

Define the testing statistics

$$T(Y;T_1,T_2,T_3,T_4)_n := \frac{\sqrt{\left(\overline{n}_{T_2} - \underline{n}_{T_1}\right)\left(\overline{n}_{T_4} - \underline{n}_{T_3}\right)} \left(\frac{\text{ReMeDI}(Y;\mathbf{j}_0)^n_{T_1,T_2}}{\overline{n}_{T_2} - \underline{n}_{T_1}} - \frac{\text{ReMeDI}(Y;\mathbf{j}_0)^n_{T_3,T_4}}{\overline{n}_{T_4} - \underline{n}_{T_3}}\right)}{\sqrt{\left(\overline{n}_{T_4} - \underline{n}_{T_3}\right)\widehat{\sigma}(Y;\mathbf{j}_0,\mathbf{j}_0)^n_{T_1,T_2} + \left(\overline{n}_{T_2} - \underline{n}_{T_1}\right)\widehat{\sigma}(Y;\mathbf{j}_0,\mathbf{j}_0)^n_{T_3,T_4}}}. \tag{45}$$

Theorem 6.1. *Under the conditions of Theorem 4.4, we have*

$$\begin{cases} T(Y;T_1,T_2,T_3,T_4)_n \xrightarrow{\mathcal{L}} \mathcal{N}(0,1) & \text{under } \mathbb{H}_0, \\ T(Y;T_1,T_2,T_3,T_4)_n \xrightarrow{\mathbb{P}} \pm\infty & \text{under } \mathbb{H}_A. \end{cases}$$

6.2 Joint tests of zero autocorrelations

This section develops tests of the joint hypothesis of zero autocorrelations based on the distribution theory of the ReMeDI estimators. Recall that for any $\ell \in \mathbb{N}_+$, $\mathbf{j}_\ell = (0,\ell)$. Let $\widehat{\mathbf{\Sigma}}_t^{n,q}$ be a $q \times q$ matrix with the (ℓ,ℓ') component $\widehat{\mathbf{\Sigma}}_t^{n,q}(\ell,\ell') = \widehat{\sigma}(Y;\mathbf{j}_\ell,\mathbf{j}_{\ell'})^n_t$ for all $1 \le \ell,\ell' \le q$. Let $\widehat{\mathbf{R}}_t^{n,q}$ be a q-vector with the ℓ-th element $\widehat{\mathbf{R}}_t^{n,q}(\ell) = \frac{\text{ReMeDI}(Y;\mathbf{j}_\ell)^n_t}{n_t}$, $1 \le \ell \le q$. We first introduce a Box-Pierce (BP) statistic:

$$T^{n,q}_{\text{BP},t} := n_t \left(\widehat{\mathbf{R}}_t^{n,q}\right)^\top \left(\widehat{\mathbf{\Sigma}}_t^{n,q}\right)^{-1} \left(\widehat{\mathbf{R}}_t^{n,q}\right).$$

Next, we will introduce a variance-ratio (VR) type statistic (Lo and MacKinlay (1988), Hong et al. (2017)). Let

$$\widehat{\Sigma}^n_{\text{VR}}(2,2) := \widehat{\sigma}(Y;\mathbf{j}_0,\mathbf{j}_0)^n_t; \quad \widehat{\Sigma}^n_{\text{VR}}(1,2) := 2\sum_{\ell=1}^{q-1} \nu_{\ell,q-1}\widehat{\sigma}(Y;\mathbf{j}_\ell,\mathbf{j}_0)^n_t + \widehat{\sigma}(Y;\mathbf{j}_0,\mathbf{j}_0)^n_t;$$

$$\widehat{\Sigma}^n_{\text{VR}}(1,1) := 4\sum_{\ell=1}^{q-1}\sum_{\ell'=1}^{q-1} \nu_{\ell,q-1}\nu_{\ell',q-1}\widehat{\sigma}(Y;\mathbf{j}_\ell,\mathbf{j}_{\ell'})^n_t + 4\sum_{\ell=1}^{q-1} \nu_{\ell,q-1}\widehat{\sigma}(Y;\mathbf{j}_\ell,\mathbf{j}_0)^n_t + \widehat{\sigma}(Y;\mathbf{j}_0,\mathbf{j}_0)^n_t;$$

$$\widehat{S}^q_t := \widehat{\mathbf{R}}_t^{n,q}(0) + 2\sum_{\ell=1}^{q-1} \nu_{\ell,q}\widehat{\mathbf{R}}_t^{n,q}(\ell); \quad \widehat{\Sigma}^n_{\text{VR}} := \frac{\widehat{\Sigma}^n_{\text{VR}}(1,1)}{\left(\widehat{\mathbf{R}}_t^{n,q}(0)\right)^2} + \frac{\left(\widehat{S}^q_t\right)^2}{\left(\widehat{\mathbf{R}}_t^{n,q}(0)\right)^4}\widehat{\Sigma}^n_{\text{VR}}(2,2) - \frac{2\widehat{S}^q_t\widehat{\Sigma}^n_{\text{VR}}(1,2)}{\left(\widehat{\mathbf{R}}_t^{n,q}(0)\right)^3}.$$

The statistic is

$$T^{n,q}_{\text{VR},t} := \frac{\widehat{S}^q_t}{\widehat{\mathbf{R}}_t^{n,q}(0)},$$

and the studentized version is given by

$$\overline{T}^{n,q}_{\mathrm{VR},t} := \frac{\sqrt{n_t}\left(T^{n,q}_{\mathrm{VR},t} - 1\right)}{\sqrt{\widehat{\Sigma}^n_{\mathrm{VR}}}}.$$

We consider the following pairs of hypotheses:

$$\mathbb{H}^q_0 : r_1 = \cdots = r_q = 0; \qquad \mathbb{H}^q_{\mathrm{A}} : \exists \ell \text{ such that } 1 \le \ell \le q, r_\ell \neq 0. \tag{46}$$

$$\mathbb{H}^\infty_0 : r_\ell = 0 : \forall \ell \ge 1; \qquad \mathbb{H}^\infty_{\mathrm{A}} : \exists \ell \ge 1, r_\ell \neq 0. \tag{47}$$

$$\mathbb{H}'^q_0 : \sum_{\ell=1}^{q} \nu_{\ell,q-1} r_\ell = 0; \qquad \mathbb{H}'^q_{\mathrm{A}} : \sum_{\ell=1}^{q} \nu_{\ell,q-1} r_\ell \neq 0. \tag{48}$$

These null hypotheses are implicit in some stylized models of market microstructure such as the Roll model.

Theorem 6.2. *Assume all the conditions of Theorem 4.4 hold, for a given $q \in \mathbb{N}_+$, and a sequence $\{q_n\}_n$ satisfying*

$$q_n \asymp \delta_n^{-\beta}, \quad 0 < \beta < \min\left\{\frac{2}{3}\left(\frac{\phi}{2} - \delta\right), \frac{2}{3}(1 - 2\phi)\right\}. \tag{49}$$

We have

$$\begin{cases} T^{n,q}_{\mathrm{BP},t} \xrightarrow{\mathcal{L}} \chi^2_q & \text{under the } \mathbb{H}^q_0; \\ T^{n,q}_{\mathrm{BP},t} \xrightarrow{\mathbb{P}} \infty & \text{under the } \mathbb{H}^q_{\mathrm{A}}. \end{cases} \tag{50}$$

$$\begin{cases} \sqrt{\frac{q_n}{2}}\left(\frac{T^{n,q_n}_t}{q_n} - 1\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1) & \text{under the } \mathbb{H}^\infty_0; \\ T^{n,q_n}_t \xrightarrow{\mathbb{P}} \infty & \text{under the } \mathbb{H}^\infty_{\mathrm{A}}. \end{cases} \tag{51}$$

$$\begin{cases} \overline{T}^{n,q}_{\mathrm{VR},t} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1) & \text{under the } \mathbb{H}'^q_0; \\ \overline{T}^{n,q}_{\mathrm{VR},t} \xrightarrow{\mathbb{P}} \pm\infty & \text{under the } \mathbb{H}'^q_{\mathrm{A}}. \end{cases} \tag{52}$$

7 Beyond the Infill Asymptotics and High-frequency Data

All the properties of the ReMeDI estimators introduced in previous sections are developed under infill asymptotics, i.e., the data frequencies increase without bound within a fixed time interval. Other approaches, e.g., the realized volatility estimators of the variance and covariances of noise (see Bandi and Russell (2008), Hansen and Lunde (2006), Zhang et al. (2005), Li et al. (2019)), the local averaging estimators proposed by Jacod et al. (2017), are also developed under the infill asymptotics. This section demonstrates that the ReMeDI approach works well beyond the infill framework, and such robustness to data frequencies distinguishes the ReMeDI estimators from the above alternatives. As a consequence, the two liquidity measures developed in Section 5 can be applied to data at coarser grids, even daily data.

7.1 The Model

Let $\{Y_i\}_{i\in\mathbb{N}_+}$ be an observed time series that can be decomposed into

$$Y_i = X_i + \varepsilon_i, \tag{53}$$

where X and ε satisfy

Assumption 7.1. *1. The process $\{X_i\}_i$ satisfies $X_{i+1} = \mu + X_i + \eta_i$, where $\{\eta_i\}_i$ are centered independent random variables with bounded second moments, i.e., $\sup_i \mathbb{E}(\eta_i^2) \leq C < \infty$.*

2. The process $\{\varepsilon_i\}_i$ is stationary, i.e., $\varepsilon_i = C_\gamma \chi_i$, where the process $\{\chi_i\}_{i=0}^n$ satisfies Assumption 2.1 and is independent of X.

Model (53) and Assumption 7.1 characterize a large class of time series, including any time series with stationary first differences, see Cochrane (1988). The decomposition in (53) are widely applied to model various economic phenomena. In financial economics, X is the efficient price and ε is the *pricing error* (Hasbrouck (1993), Hendershott and Menkveld (2014)), or *transitory component* in stock prices due to, e.g., noise trading (Poterba and Summers (1988)). In macroeconomics, X is identified with the *permanent component* of the observed times series while ε is the *cyclical component* (Cochrane (1994), Beveridge and Nelson (1981) and Campbell and Mankiw (1987)). The random walk component X may also arise from a modification of existing model, say, by adding a random walk in the technology change (King et al. (1991)).

7.2 The ReMeDI estimation

Now we show the ReMeDI approach can effectively separate the two components X and ε under Assumption 7.1. In general, we may estimate the drift term μ root-n-consistently by taking the sample average of the first difference of the observed process. The effect of this initial estimation may be shown to be of smaller order, and so for simplicity here suppose that μ is known and without loss of generality is equal to zero.

Let $\overline{\boldsymbol{k}}_n = (-k_n, k_n), \mathbf{j}_\ell = (0, \ell)$. Let

$$\widehat{R}_\ell^n := \frac{1}{n} \sum_{i=k_n}^{n-k_n-\ell} \Delta_{\mathbf{j}_\ell}^{\overline{\boldsymbol{k}}_n}(Y)_i^n.$$

Theorem 7.1. *Let $v > 0$ and k_n satisfies $k_n \asymp n^u, u \in \left(\frac{1}{3+2v}, \frac{1}{3}\right)$. Under Assumption 7.1, we have*

$$\widehat{R}_\ell^n \xrightarrow{\mathbb{P}} R_\ell := C_\gamma^2 r_\ell. \tag{54}$$

If X has stationary increments with $\mathbb{E}(\eta_i^2) = \sigma_X^2$, we have

$$\widehat{\sigma}_X^{2,n} := \frac{\sum_{i=1}^n \Delta_{0,0}^{1,1}(Y)_i^n}{n} - \widehat{R}_0^n + \widehat{R}_1^n \xrightarrow{\mathbb{P}} \sigma_X^2. \tag{55}$$

Moreover, we have the following limit distribution for $\widehat{R}_\ell$:

$$\frac{1}{\widehat{\sigma}_X^{2,n}}\sqrt{\frac{3n}{2k_n^3}}\left(\widehat{R}_\ell^n - R_\ell\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0,1\right). \tag{56}$$

The relative size of the differences of X and ε varies in empirical studies. In particular, Campbell and Mankiw (1987) find large random walk component (X) while Clark (1987) find small permanent component (ε). Next, we present a result when the relative size of the two components is asymptotically large or small.[11]

Corollary 7.1 (Small and large ε)**.** *Assume all conditions in Assumption 7.1 hold except $\varepsilon_i = n^{-\varphi}\chi_i$ with $\varphi < 1/4$. Let $k_n \asymp n^u, u \in \left(0, \frac{1-4\varphi}{3} \wedge \frac{1}{2}\right)$. A consistent* ReMeDI *estimator of the autocorrelation function is given by*

$$\widehat{r}_\ell^{n,\varphi} := \frac{\sum_{i=k_n}^{n-k_n-\ell} \Delta_{\mathbf{j}_\ell}^{\bar{k}_n}(Y)_i^n}{\sum_{i=k_n}^{n-k_n} \Delta_{\mathbf{j}_0}^{\bar{k}_n}(Y)_i^n} \xrightarrow{\mathbb{P}} r_\ell. \tag{57}$$

We next discuss why the LA method is inconsistent under long-span asymptotics. Suppose that Assumption 7.1 holds and that $\mathbb{E}(\eta_i^2) = \sigma_X^2$. Then, one can show under some regularity conditions that

$$\frac{1}{n\tilde{k}_n}U((0,\ell))_t^n \xrightarrow{\mathbb{P}} \frac{3\sigma_X^2}{2}, \quad \mathbb{E}\left(\frac{1}{n}U((0,\ell))_t^n\right) = \frac{3\tilde{k}_n\sigma_X^2}{2} + O(\tilde{k}_n^{-1}),$$

where $U((0,\ell))_t^n/n$ is the LA estimator applied to the efficient price and $\tilde{k}_n$ is the tuning parameter of the LA method.[12] Thus, the LA method is inconsistent when applied to low-frequency data under long span asymptotics, and this is in line with our observations in the simulation studies. The ReMeDI estimator by contrast has a bias that only depends on the microstructure noise, and so its magnitude is not affected by the sampling scheme.

7.3 The intuition and consequence of the robustness to data frequency

The results presented in Theorem 7.1 have a very clear economic intuition, which is quite different from the identification under infill asymptotics illustrated in Section 3. The consistency, without any restrictions on the data frequencies, relies on the zero autocorrelations of the efficient returns. Therefore, the ReMeDI estimators effectively "remove" the efficient price as the efficient returns on non-overlapping intervals are uncorrelated. What remains is simply due to the market microstructure, and by tuning the "distance" and "lengths" of the non-overlapping intervals, we can freely estimate the targeted moments of noise.

[11]See also the studies on "small noise" by Da and Xiu (2019) and Aït-Sahalia and Xiu (2019) on high-frequency analysis.

[12]The result is consistent with Equation (3.35) in Jacod et al. (2017), in which they show, under *infill asymptotics* that the LA applied to the efficient price has a bias (after proper scaling) equal to $\frac{3\mathrm{QV}_t}{2}$, where QV_t is the quadratic variation of the efficient price process.

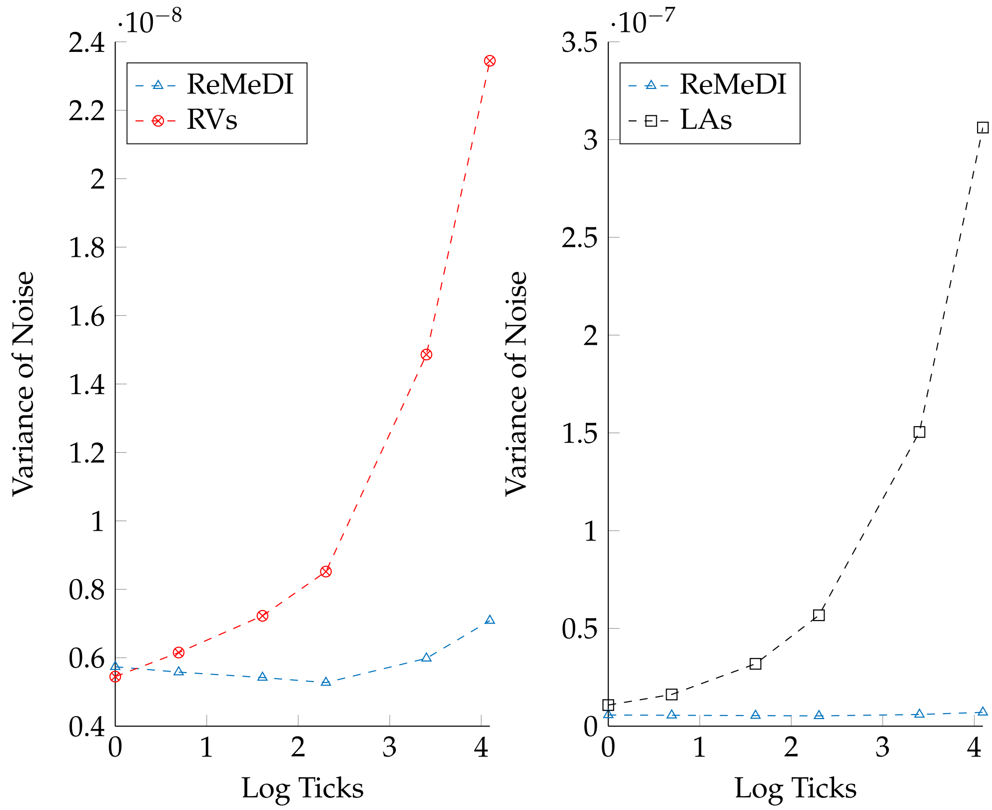

Figure 2: Estimates of the variance of noise using transaction prices of E-mini S&P 500 futures on May 3, 2010 using samples at different frequencies.

Why is the robustness to data frequency crucial? First, such robustness greatly broadens the applications of the ReMeDI estimators. They can be used by microstructure researchers using tick by tick data, or by a macroeconomist using monthly or annual data. Second, the sampling frequency matters in the estimation of the volatility of X, especially if one uses high-frequency data which may be deemed "noisy" whence a de-noise procedure is essential (Jacod et al. (2009), Zhang et al. (2005), Barndorff-Nielsen et al. (2008), Xiu (2010)). Such de-noise methods rely on the estimation of the moments of noise, in particular, the variance of noise. Using real transaction prices, Figure 2 compares the ReMeDI estimates of the variance of noise to two alternatives: the realized volatility (RV) estimator and the local averaging (LA) estimators. The estimation is performed using the transaction prices of the same stock but sampled at different frequencies, from tick by tick data to 1 minute returns. Both the RV and LA estimators suggest that the magnitude of noise is larger when the sampling grid is coarser. However, such monotone correspondence is a consequence of the low-frequency bias.[13] Thus, any de-noise method using the RV or LA estimators to correct the microstructure effects will return very different estimates when the sampling frequency varies. The ReMeDI estimators, however, are quite robust to the sampling frequencies. Third, an economic concern arises when one treats the magnitude of noise as a proxy of liquidity, see Hasbrouck (1993), Aït-Sahalia and Yu (2009) and Chen and Mykland (2017). Trading frequencies have increased enormously in the

[13] See the analysis in Hansen and Lunde (2006), Jacod et al. (2017) and Li et al. (2019).

past decade, in tandem with improvements in liquidity, see Hendershott et al. (2011). Figure 2 provides a warning to use such measures: the RV and LA measures suggest that the liquidity improves as data frequency increases—this is not intuitive as the measures are obtained on subsamples of the transaction prices generated by the same market mechanism.

7.4 Implementation: Which asymptotics?

We have shown that the ReMeDI estimator is robust to the data frequency, however the limiting distribution of the ReMeDI estimator does depend on the sampling scheme. In many practical circumstances, it is hard to decide which asymptotic distribution provides a better approximation to a given sample of transaction prices, e.g., 5-min returns over one trading month. In this section we introduce a practical method to construct estimators of the asymptotic variance for the ReMeDI estimators that are also "robust" to data frequencies.

Let R_0 be the variance of ε and $\sigma_n^2 = \mathbb{E}\big((X_i^n - X_{i-1}^n)^2\big)$. Define a signal-to-noise ratio[14]

$$\pi_n := \frac{R_0}{R_0 + n^{u'}\sigma_n^2} = \frac{1}{1 + n^{u'}\sigma_n^2/R_0}.$$

$R_0 = O_p(1)$ in either asymptotic framework; under infill asymptotics, $\sigma_n^2 = O_p(\delta_n)$, thus for $u' \in (0,1)$, $\pi_n \to 1$; under long-span asymptotics σ_n^2 is a constant, thus $\pi_n \to 0$. Now we define the following asymptotic variance estimator[15]

$$\widehat{\sigma}^2_{\ell,\text{robust}} := \frac{2k_n^3}{3n}\left(\widehat{\sigma}_X^{2,n}\right)^2 + \frac{\pi_n \widehat{\sigma}(Y;\mathbf{j}_\ell,\mathbf{j}_\ell)_t^n}{n_t}.$$

For u' close to 1, we have

$$\widehat{\sigma}^2_{\ell,\text{robust}} = \begin{cases} \frac{2k_n^3}{3n}\left(\widehat{\sigma}_X^{2,n}\right)^2 + o_p(1/n), & \text{long span;} \\ \frac{\widehat{\sigma}(Y;\mathbf{j}_\ell,\mathbf{j}_\ell)_t^n}{n_t} + O_p(k_n^3\delta_n^2), & \text{infill.} \end{cases}$$

The designed estimator $\widehat{\sigma}^2_{\ell,\text{robust}}$ will reduce to infill estimator or long-span estimator when the dataset tells that the noise is relatively large or small. Thus, $\widehat{\sigma}^2_{\ell,\text{robust}}$ will automatically decide which asymptotic framework is proper after collecting some information (signal-to-noise ratio) from the dataset.

8 Empirical Studies

Our empirical study relies on the E-mini S&P 500 futures contracts from Chicago Mercantile Exchange (CME) Globex, which are futures on the S&P 500 stock market index. We use the

[14] To calculate π_n, we need estimates of R_0, σ_n^2. R_0 can be estimated by the ReMeDI estimator, and σ_n^2 can be estimated by the likelihood approach (Aït-Sahalia et al. (2005)). Note that accurate estimates of R_0, σ_n^2 are not required; only the asymptotic order of the ratio is needed.

[15] Note that we stick to the notations introduced in previous sections. It should be clear that both n_t, n denote the number of observations of a given sample. Similarly, $1/n$ and δ_n are of the same order.

transaction data for all 20 trading days in May, 2010, including May 6, 2010 — the day of the "Flash Crash" — when U.S. equity indices collapsed and rebounded abruptly after a rapid execution of a large selling program. Thus, it is of particular interest to measure market liquidity during period of market stress using our proposed method.

Throughout the empirical studies, we select various tuning parameters via the algorithm we developed in the supplementary material, see Li and Linton (2019).[16]

8.1 Liquidities in three trading zones

We focus on the trading session that spans from 18:00 of the prior day to 16:15 Eastern Standard Time (ET). This session covers three trading zones: 18:00 to 3:00 ET, 3:00 to 9:30 ET and 9:30 to 16:15 ET are the periods of regular trading in Asia, Europe and the U.S., respectively. Table 1 reports some descriptive statistics of the trading activities for the three segments.[17] It is evident that trading in the U.S. segment is more active.

Table 1 also reports our spread measures[18], the IBAS and ABAS as measures of liquidity costs of the E-mini futures market. Among the three trading zones, the European trading segment has relatively smaller liquidity cost, as measured by both the IBAS and ABAS. The relatively magnitudes of liquidity costs in the three trading zones are further compared in the top panel of Figure 3 on a daily level. Figure 6 depicts pairwise comparisons of the variances of microstructure noise in three trading sessions. We label the paired estimates that are *statistically* different using our testing devices developed in Section 6. The top and the bottom panels illustrate that the variance of noise in the European session is significantly smaller than the Asia (14 days out of 20) and the U.S. sessions (17 days out of 20). We also investigate other statistics of the U.S. session that reflect the turbulent market conditions on 6 May. For example, we find $\mathbf{R}(\mathbf{0}_3)_t / R_{t,0}^{3/2}$ equal to -0.5793 and 1.8531 for the non Flash Crash days and the Flash Crash day, respectively; and the estimates of $\mathbf{R}(\mathbf{0}_4)_t / R_{t,0}^2$ are 31.2912 and 159.5389, respectively.

	All days in May, 2010 (exclude 6, May)			May 6, 2010		
	Asia	Europe	The U.S.	Asia	Europe	The U.S.
Volume	1.00×10^5	3.72×10^5	2.43×10^6	7.82×10^4	4.89×10^5	5.09×10^6
Transactions	1.36×10^4	4.20×10^4	1.90×10^5	7.97×10^3	4.77×10^4	3.83×10^5
IBAS	1.55 (0.013)	1.38 (0.0026)	1.57 (0.0070)	1.38 (0.013)	1.29 (0.0073)	2.78 (0.041)
ABAS	1.61 (0.18)	1.53 (0.026)	1.69 (0.053)	1.51 (0.043)	1.33 (0.021)	4.78 (0.49)

Table 1: Statistics of trading activities and liquidity costs of E-mini S&P 500 futures in May, 2010. In the left panel, the trading volume and the number of transactions are averages over the 20 trading days. The two liquidity measures IBAS and ABAS are estimates based on all observations in the 20 trading days. The standard deviations are reported in parentheses. The right panel reports the statistics using the transactions on May 6 only.

[16] Typical choices of k_n on the Flash Crash day are around 8 or 9, depending on the parameters to execute the algorithm; for the remaining days, k_n would be 3 or 4.

[17] The statistics are computed after some minor data filtration using the procedures proposed by Andersen et al. (2018).

[18] The results are stated in basis points. One basis point is equivalent to 0.01% or 0.0001 in decimal form.

8.2 Order flow patterns and autocovariances of microstructure noise

As we observe from Table 1 that the ABAS measure is statistically quite large compared with the IBAS in all segments. We have similar observations on a daily basis — as revealed in the bottom panel of Figure 3 — the ABAS measures are persistently larger than the IBAS measures, which are persistently larger than the Roll measures. The discrepancy, as suggested by earlier analysis, reflects the order imbalances induced by the (positively) autocorrelated order flows.

The autocovariances of microstructure noise provide an angle to look at the dynamics of order flow. We estimate the autocovariances of microstructure noise on each trading day for the U.S. trading segment. The results are presented in Figure 4. The microstructure noise exhibits positive albeit weak autocorrelation patterns on most trading days. The autocovariances usually die out after three lags. However, the $(1, 4)$-subplot in Figure 4 reveals that the autocovariance function of noise becomes extremely strong on May 6, 2010 — when the U.S. market experienced chaotic market conditions — an episode dubbed the Flash Crash.

8.3 The intraday pattern of spread measures

To study the intraday pattern of the spread measures, we estimate IBAS and ABAS in each 15 minutes window. The top and bottom panels of Figure 5 report the estimates for the non Flash Crash days and the Flash Crash day. In line with the statistics reported in Table 1, the European session has smaller spreads. An interesting observation, as is evident in the top panel of Figure 5 is that when switched from the Asia trading session to the European trading session, both spreads exhibit significant drops, and gradually revert to a level that is close to the spreads in the U.S. trading zone. Moreover, the trading volumes have approximately a U-pattern in each trading session. On the Flash Crash day, we observe the simultaneous abrupt increases in spreads measures and trading volume.

8.4 The Flash Crash: Liquidity evaporation under large selling pressure

The right panel of Table 1 suggests that May 6 is "an unusually turbulent day for the markets" CFTC-SEC (2010): trading activities and liquidity costs almost double. This is an unique phenomenon only when the U.S. market is active, which "opened to unsettling political and economic news from overseas concerning the European debt crisis" CFTC-SEC (2010). As the negative market sentiment continued to grow in the afternoon, a large fundamental trader initiated a sell program to sell rapidly a total of 75,000 E-Mini contracts at 2:32 p.m. (ET). It triggered larger selling pressure and a positive feedback loop that drove prices down. The positive feedback loop appears to be consistent with positively serially correlated market microstructure noise as we demonstrated in Figure 4. As a consequence, the E-mini price dropped sharply in a short time period, and liquidity drained quickly.

Now we demonstrate how to apply our new liquidity measures to identify the liquidity evaporation. The left bottom panel of Figure 3 shows that the three liquidity measures yield very different estimates on the day of the Flash Crash. All three measures have peaks in impact during May 6, but such liquidity crisis is less likely to be picked up by the Roll measure, whereas the IBAS and ABAS both pick it up with the ABAS giving a much higher prominence.

The effective identification of the liquidity drains by ABAS still applies when we focus on the intraday price of May 6, 2010. Figure 7 plots ABAS on each 1-minute window.[19] The ABAS measure detects extreme liquidity drains from 14:45 (ET) till 14:53 (ET).[20] This period has the most intensive price pressure as it coincides with the segment when the large seller sold most of his position, and it also coincides with the period where there existed large and persistent price differentials between E-mini and SPY, thus a period when the price pressure can not be transmitted from E-mini to SPY, see Menkveld and Yueshen (2018). Economic theory rationalizes price pressures as the premium charged by risk-averse intermediaries to provide immediacy, see, e.g., Grossman and Miller (1988) and Hendershott and Menkveld (2014), while empirical studies (Kirilenko et al. (2017)) characterize large net position, or order imbalances as a period of large and temporary price pressure. This is exactly the economic intuition behind the design of our liquidity measure conveyed by our inventory model: ABAS captures the price pressure caused by the imbalanced order flow, which manifests statistically as the autocovariances of microstructure noise.

Some empirical studies have found earlier warning signs of the Flash Crash, see Easley et al. (2012) and Menkveld and Yueshen (2018), although others including Andersen and Bondarenko (2014) dispute some of these findings. We propose a normalized ABAS measure to detect imbalanced order flows, which is given by $\mathrm{ABAS}_t^n/\mathrm{IBAS}_t^n$. It reduces to 1 if the order flow is uncorrelated; a value larger than 1 indicates a positively autocorrelated, thus a potentially imbalanced order flow. Figure 8 plots the normalized liquidity measure. We observe a cluttering of liquidity measures above 1, depicted in green, lasting from 14:15 to 15:10. Thus, our normalize liquidity measures provide, at least some mild early warnings of the imbalanced order flows before the Flash Crash.

9 Concluding Remarks

We introduced a nonparametric method to separate the microstructure noise from the underlying semimartingale efficient prices. We have concentrated on the infill setting primarily and the univariate case. The method naturally extends to the multivariate case, although in that case several issues arise. First, the nonsynchronous trading issue has to be faced. Second, even when the assets trade on a common clock there are some remaining theoretical results that need to be established for the infill case. In the long span case (with common discrete time clock), it is possible to establish a CLT for the multivariate estimated error autocovariance matrix under some additional assumptions. These methods permit a better understanding of the permanent-transitory decompositions without making strong assumptions. Our methods

[19]We truncated four negative estimates of ABAS at zero.

[20]This period is identified as follows. We first calculate the mean and the standard deviation of the 405 estimates obtained on each 1-minute window. We then pick up the ABAS measures that are 2 standard deviations larger than the mean. In this way, we obtain 10 extreme estimates that identify the trading time from 14:45 — 14:50, 14:52, 14:53, 15:00 and 15:06. Then we consider the "liquidity evaporation period" as 14:45 — 14:53.

also allow us to develop best linear predictors (Kalman filter) of the unobserved efficient price given the history of observed transactions, under the assumption of stationarity, and we plan to investigate this in the sequel. We have not discussed efficiency in great deal, but one can improve efficiency in two ways: first, by combining the estimators associated with different choices of $\boldsymbol{k}_n$ by minimum distance, and second by doing a kind of GLS procedure using our local estimator of γ_u.

In our Flash Crash application we found some evidence of the predictability of the price declines during that episode, although that was based on ex post estimation. We also found that many features of the microstructure noise process changed on the Flash Crash day. Clearly, the lack of stationarity of the persistence of microstructure noise poses both theoretical and practical questions. Why does the microstructure noise process apparently change so radically during flash episodes? How can one best model the process to provide accurate real time forecasts?

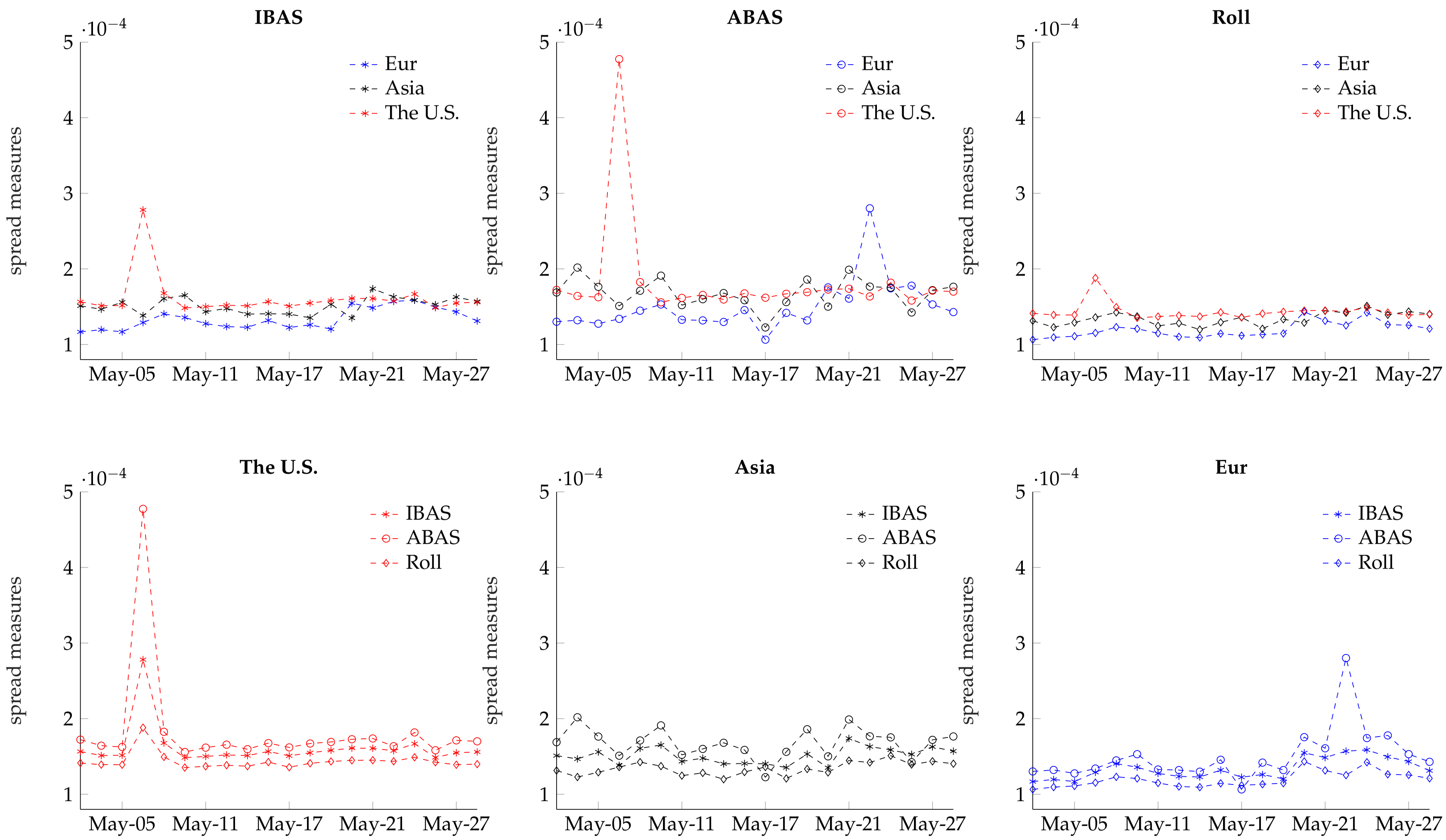

Figure 3: Liquidity measures for the Asia, Europe and the U.S. trading zones using E-mini transaction prices over May, 2010. The three liquidity measures are defined in (33), (35) and (38). In each plot of the top panel, we fix the spread measure and compare the estimates obtained in three trading sessions; in each plot of the bottom panel, we fix the trading session and compare the three measures.

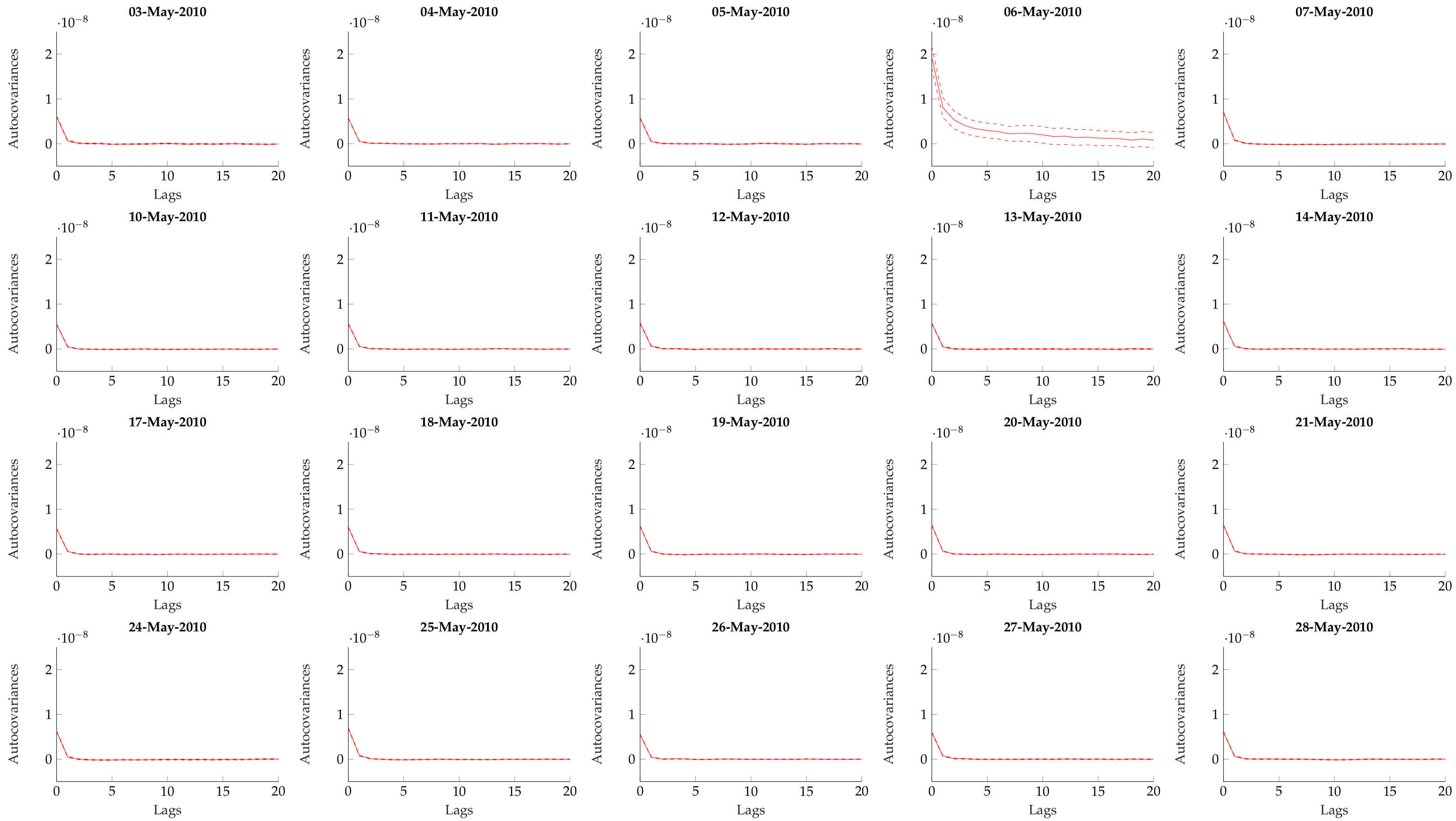

Figure 4: ReMeDI estimates of the autocovariances of microstructure noise using E-mini transaction prices in the U.S. trading zone. The dash lines are the 95% confidence intervals.

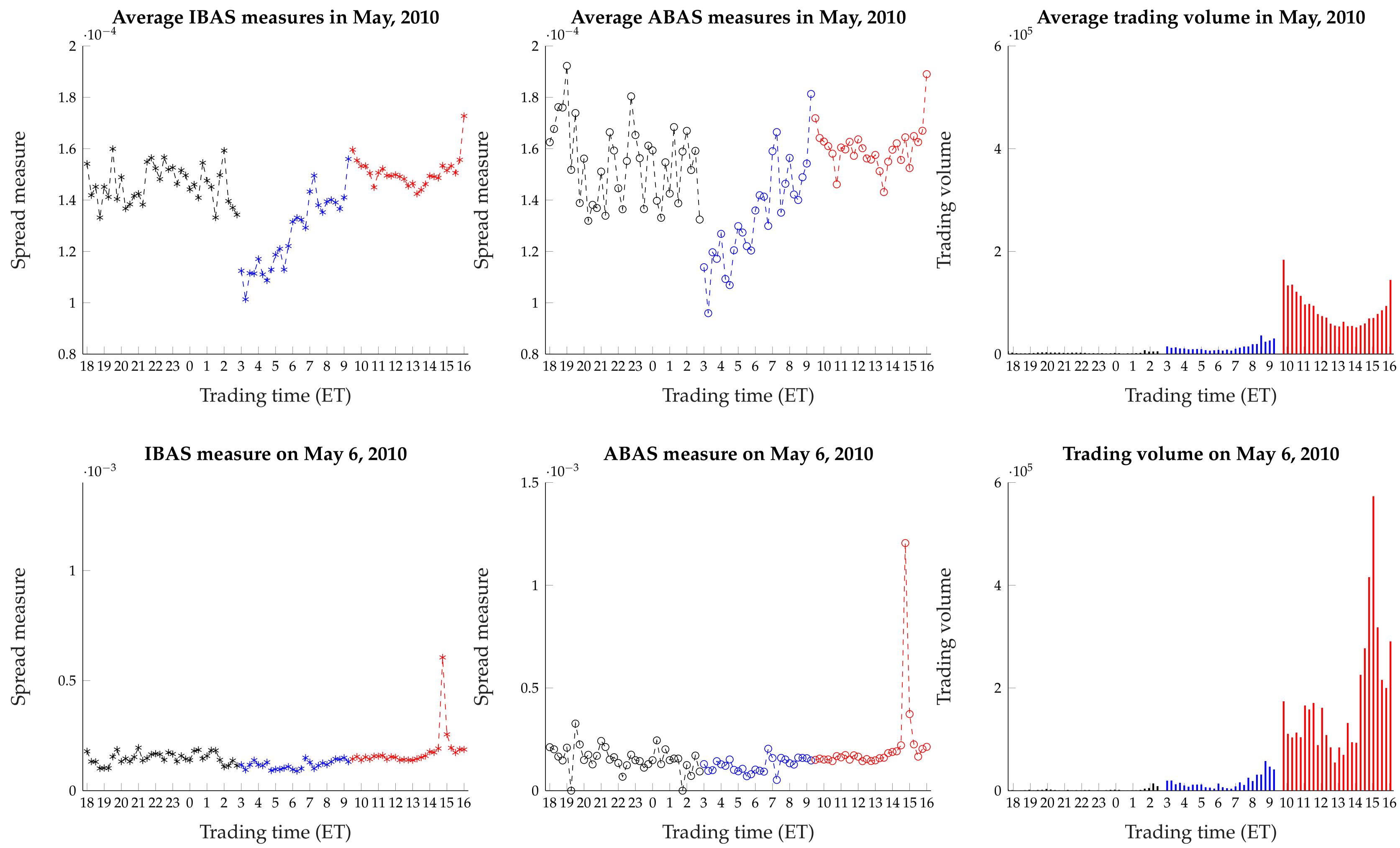

Figure 5: Intraday liquidity measures and trading volume. The liquidity measures are defined in (35) and (38). The estimates are performed in each 15 minutes window. The top panel reports the average across all trading days in May, 2010 except 6, May. The bottom panel reports the estimates on 6 May, 2010. The black, blue and red plots correspond to the Asia, European and the U.S. trading sessions.

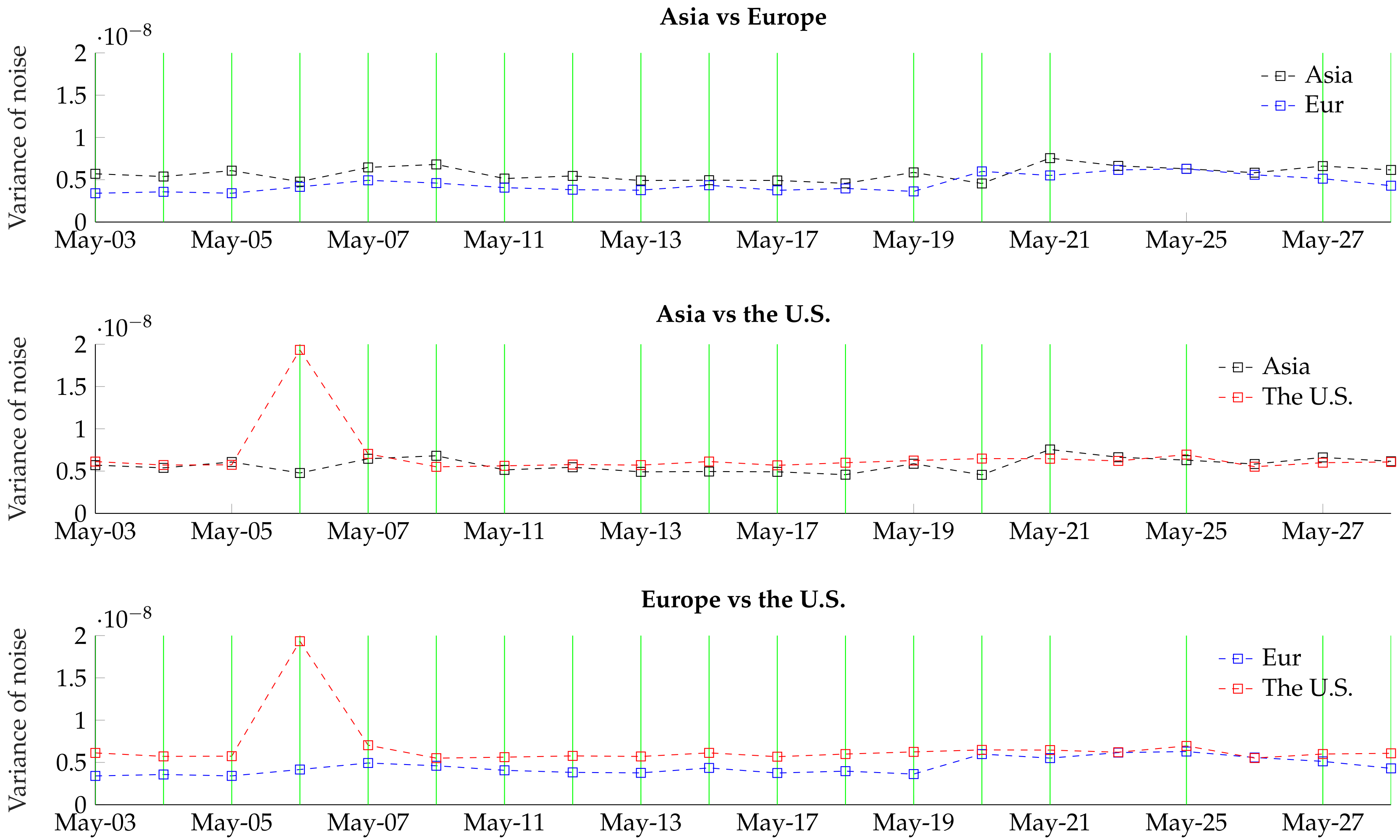

Figure 6: This figure compares the estimated variances of noise in different trading segments. The green vertical bars identify the paired estimates that are statistically different, using the testing devices developed in Theorem 6.1.

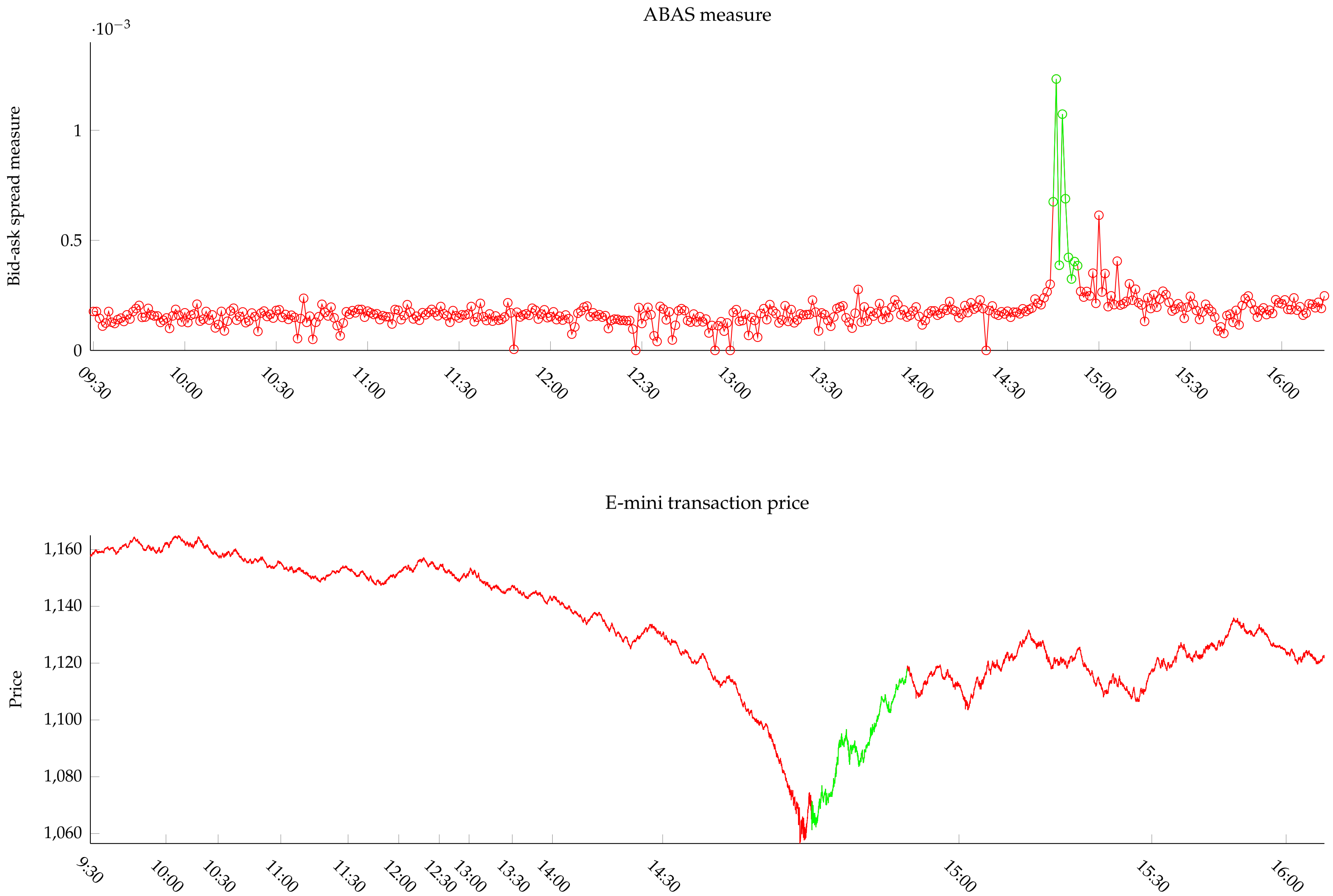

Figure 7: ABAS measures of liquidity and transaction prices of the E-mini futures on May, 6, 2010. The ABAS estimates are obtained in each 1-minute window. The green parts of the ABAS and transaction price identify liquidity-drained period from 14:44 (ET) till 15:07 (ET). This period is identified as follows. We first calculate the mean and the standard deviation of the 405 estimates obtained on each 1-minute window. We then pick up the ABAS measures that are 2 standard deviations larger than the mean. In this way, we obtain 10 extreme estimates that identify the trading time from 14:45 — 14:50, 14:52, 14:53, 15:00 and 15:06. Then we consider the "liquidity evaporation period" as 14:45 — 14:53.

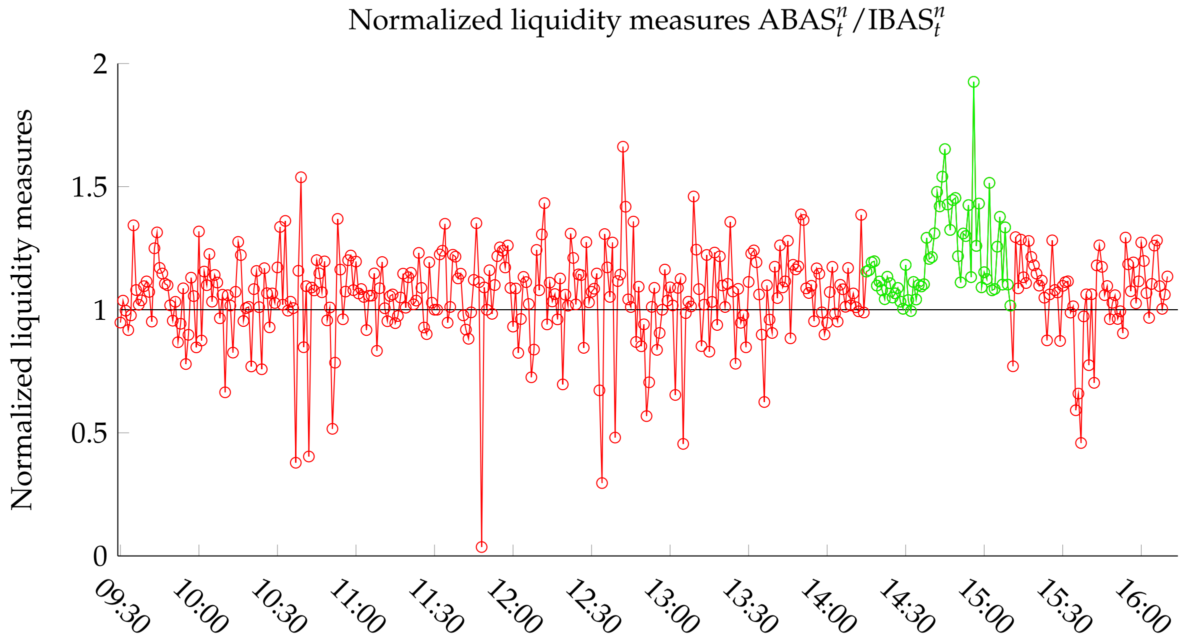

Figure 8: Normalized liquidity measures of order flow imbalances, given by $ABAS_t^n/IBAS_t^n$. The green part highlights a clustering of measures above 1, indicating imbalanced order flows. The corresponding trading period lasts from 14:15 — 15:10.

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