

Stability of Rotating Gravitating Streams

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This paper treats the stability of two superposed gravitating streams rotating about the axis transverse to the horizontal magnetic field. The critical wave number for instability is found to be affected by rotation for propagation perpendicular to the axis about which the system rotates. The critical wave number for instability is not affected by rotation when waves propagate along the axis of rotation. The critical wave number is affected by both the magnetic field and the streaming velocity in both cases. Both the magnetic field and the rotation are stabilizing, while the streaming velocity is destabilizing.

Key words: Stability; Gravitating Streams; Rotation; Magnetic Field.

1. Introduction

Jeans [1] considered the gravitational instability of an infinite homogeneous self-gravitating medium in context of the formation of astronomical bodies by the fragmentation of interstellar matter. He has derived a criterion that the medium becomes unstable and breaks up for perturbations of the wave number k less than Jeans [1] wave number $k_j = \frac{\sqrt{G\rho}}{c}$, where ρ is the density, c the velocity of sound in the gas and G the gravitational constant. Chandrasekhar [2] examined the effects of a uniform magnetic field and uniform rotation on the gravitational instability of the static medium and found that both rotations and the magnetic field inhibit the contraction and fragmentation of the interstellar clouds. Since then several researchers have studied this problem under varying assumptions. Tasoul [3] has investigated the gravitational instability of a thermally conducting fluid, while Gliddon [4] has studied this problem for an anisotropic plasma. Mouschovias [5, 6] and Mestel and Paris [7] have pointed out the importance of the gravitational instability of a self-gravitating static homogeneous plasma in the context of fragmentation and collapse in magnetic molecular clouds. Sengar [8, 9] and Radwan and Elazab [10] examined the effect of variable streams on the gravitational instability. Sorker and Sarazin [11] have demonstrated the relevance of this problem in gravitational plasma filaments in cooling flows in clusters and galaxies. Vranjes and Cadez [12] studied the

effect of radiative processes on the gravitational instability in a static medium.

Singh and Khare [13] investigated the instability of superposed gravitating streams in a uniform horizontal magnetic field and rotation about the vertical. Shrivastava and Vaghela [14] have studied the magnetogravitational instability of an interstellar medium with variable streams and radiation. In recent years, several researchers have studied the velocity shear instability in hydrodynamics and plasmas under different assumptions. Benjamin and Bridges [15] have shown that the velocity shear instability problem in hydrodynamics admits a canonical Hamiltonian formulation. Allah [16] has examined the effects of heat and mass transfer on the instability of streams. Luo et al. [17] have investigated the effect of negatively charged dust on the parallel velocity shear instability in a magnetized plasma. More recently Bhatia and Sharma [18] have studied the effects of surface tension and permeability of a porous medium on the stability of superposed viscous conducting streams. In all these studies the streams are not gravitating and are under the action of gravity.

In astrophysical situations the instability of the gravitating rotating streams in a horizontal magnetic field would be interesting when the system rotates about an axis perpendicular to the direction of the magnetic field, in the horizontal plane. This aspect forms the subject matter of this paper where we study the instability of inviscid infinitely conducting gravitating streams.

We study the cases of propagation along and perpendicular to the direction of the magnetic field.

2. Perturbation Equations

We consider two semi-infinitely ideally conducting homogeneous gravitating streaming fluids occupying the regions $z > 0$ and $z < 0$ and separated by a plane interface at $z = 0$. The streams possess uniform densities, ρ_1 and ρ_2 , and move with uniform speeds, V_1 and V_2 . A uniform magnetic field is applied to the system in the direction of the x -axis. The whole system rotates about the y -axis with a small uniform angular velocity Ω .

The linearized perturbation equations relevant to the problem are

$$\rho_s \left[\frac{\partial}{\partial t} \vec{u}_s + (\vec{V}_s \nabla) \vec{u}_s \right] = -\nabla \delta p_s + \rho_s \nabla \delta \phi_s + 2\rho_s (\vec{u}_s \times \vec{\Omega}) + (\nabla \times \vec{h}_s) \times \vec{H}, \tag{1}$$

$$\frac{\partial}{\partial t} \delta \rho_s + (\vec{V}_s \nabla) \delta \rho_s + \rho_s (\nabla \vec{u}_s) = 0, \tag{2}$$

$$\frac{\partial}{\partial t} \vec{h}_s + (\vec{V}_s \nabla) \vec{h}_s = \text{curl}(\vec{u}_s \times \vec{H}), \tag{3}$$

$$\nabla \vec{h}_s = 0, \tag{4}$$

$$\nabla^2 \delta \phi_s = -G \delta \rho_s, \tag{5}$$

$$\frac{\partial}{\partial t} \delta \rho_s + (\vec{V}_s \nabla) \delta \rho_s = C_s^2 \left[\frac{\partial}{\partial t} \delta \rho_s + (\vec{V}_s \nabla) \delta \rho_s \right]. \tag{6}$$

The equations are same for both streams. The subscript ‘ s ’ distinguishes the two streams, $s = 1$ corresponding to the upper region $z > 0$ and $s = 2$ to the lower region $z < 0$. In the above equations $\vec{h} = (h_x, h_y, h_z)$, $\delta \phi$, δp and $\delta \rho$ are the perturbations, respectively, in the magnetic field \vec{H} , the gravitational potential ϕ , the pressure p and density ρ due to a small disturbance of the system which produces the velocity field $\vec{u} = (u, v, w)$ in the system. Here C is the velocity of sound. As stated above, we take here the horizontal magnetic field along the x -axis i. e. $\vec{H} = (H, 0, 0)$, and streams rotating about an axis in the horizontal plane perpendicular to the direction of the magnetic field, i. e. $\vec{\Omega} = (0, \Omega, 0)$. We investigate the stability problem for the two cases of propagation along and perpendicular to the axis about which the streams rotate.

3. The Governing Differential Equation for Propagation along the Magnetic Field

For this mode of wave propagation along the direction of the magnetic field we assume that all perturbed quantities have the space and time dependence of the form

$$F(z) \exp(ik_x x + nt), \tag{7}$$

where $F(z)$ is some function of z , k_x is the wave number of the perturbation along the x -axis, and n (may be complex) is the rate at which the system departs away from equilibrium. The streaming velocity is also taken along the x -axis in this mode, i. e. $\vec{V} = (V, 0, 0)$. Then for the perturbations of the form (7), (1) to (6) give, on writing $D \equiv \frac{d}{dz}$:

$$\rho_s \sigma_s u_s = -ik_x \delta p_s + \rho_s ik_x \delta \phi_s - 2\rho_s w_s \Omega, \tag{8}$$

$$\rho_s \sigma_s v_s = H ik_x (h_y)_s, \tag{9}$$

$$\rho_s \sigma_s w_s = -D \delta p_s + \rho_s D \delta \phi_s + 2\rho_s u_s \Omega - H [D(h_x)_s - ik_x (h_z)_s], \tag{10}$$

$$\sigma_s \delta \rho_s = -\rho_s (ik_y u_s + D w_s), \tag{11}$$

$$\sigma_s (h_x, h_y, h_z)_s = [D w_s, ik_x v_s, ik_x w_s], \tag{12}$$

$$ik_x (h_x)_s + D(h_z)_s = 0, \tag{13}$$

$$(D^2 - k_x^2) \delta \phi_s = -G \delta \rho_s, \tag{14}$$

$$\sigma_s \delta p_s = C^2 \sigma_s \delta \rho_s, \tag{15}$$

where

$$\sigma_s = n + ik_x V_s. \tag{16}$$

Eliminating the various quantities from the above equations, we finally get the fourth order differential equation in $\delta \phi_s$

$$(D^2 - k_x^2)(D^2 - N_s^2) \delta \phi_s = 0, \tag{17}$$

where

$$N_s^2 = \frac{(C_s^2 k_x^2 + \sigma_s^2 - G \rho_s)(\sigma_s^2 + M_s^2 k_x^2) + 4\Omega^2 \sigma_s^2}{\sigma_s^2 (\sigma_s^2 + M_s^2) + M_s^2 C_s^2 k_x^2}. \tag{18}$$

Here $M_s^2 = \frac{H^2}{\rho_s}$ is the Alfvén velocity. Equation (17) holds for both streams and must be solved subject to the appropriate boundary conditions.

4. Solution of the Differential Equations

Now we seek the solutions of (17) which remain bounded in the two regions. The appropriate solutions for the two regions are therefore

$$\delta\phi_1 = A_1 e^{-k_x z} + B_1 e^{-N_1 z} \quad (z > 0) \tag{19}$$

and

$$\delta\phi_2 = A_2 e^{k_x z} + B_2 e^{N_2 z} \quad (z < 0), \tag{20}$$

where A_1, A_2, B_1 and B_2 are constants of integration. In writing the solutions (19) and (20) for $\delta\phi$ it is assumed that N_1 and N_2 are so defined that their real parts are positive. The four boundary conditions to be satisfied at the interface $z = 0$ are:

- (i) Continuity of the perturbed gravitational potential, i. e. $\delta\phi_1 = \delta\phi_2$.
 - (ii) Continuity of the normal derivative of the perturbed gravitational potential, i. e. $D(\delta\phi_1) = D(\delta\phi_2)$.
 - (iii) Continuity of the total perturbed pressure, i. e. $\delta p_1 + H(h_x)_1 = \delta p_2 + H(h_x)_2$.
 - (iv) The normal displacement at any point (fluid element) is unique at $z = 0$, i. e. $\frac{w_1}{\sigma_1} = \frac{w_2}{\sigma_2}$.
- These conditions, on applying the solutions (19)–(20), lead to the four equations

$$A_1 + B_1 - A_2 - B_2 = 0, \tag{21}$$

$$k_x A_1 + N_1 B_1 + k_x A_2 + N_2 B_2 = 0, \tag{22}$$

$$Q_1 A_1 + Q_2 B_1 - Q_3 A_2 - Q_4 B_2 = 0, \tag{23}$$

$$T_1 A_1 + T_2 B_1 - T_3 A_2 - T_4 B_2 = 0, \tag{24}$$

where

$$Q_1 = \rho_1 \sigma_1^2 \alpha_1^2 + k_x^2 \rho_1 (M_1^2 k_x^2 + \sigma_1^2) - (M_1^2 k_x^2 + \sigma_1^2 + 4\Omega^2)(C_1^2 k_x^2 + \sigma_1^2)(\alpha_1^2 - k_x^2)/G + 2ik_x \alpha_1 M_1^2 \Omega [\rho_1 k_x^2 + (C_1^2 k_x^2 + \sigma_1^2)(\alpha_1^2 - k_x^2)/G], \tag{25}$$

$$T_1 = \rho_2 \sigma_2 (C_1^2 k_x^2 + \sigma_1^2)(\sigma_2 \alpha_1 + 2ik_x \Omega)(\alpha_1^2 - k_x^2)/G + \rho_1 \rho_2 k_x^2 (\sigma_1^2 \alpha_1 + 2ik_x \Omega). \tag{26}$$

The coefficient Q_2 is obtained from Q_1 by replacing α_1 by N_1 , Q_3 is obtained from Q_1 by replacing α_1 by α_2 , changing i to $-i$ and interchanging the subscripts 1 and 2, and Q_4 is obtained from Q_3 by replacing α_2 by N_2 . Similarly T_2 to T_4 are obtained from T_1 . Here the values of α_1 and α_2 are the same in the two streams and equal to k_x , i. e. $\alpha_1 = \alpha_2 = k_x$ [see (17)].

5. The Dispersion Relation

For a non-trivial solution, the determinant of the matrix of the coefficients of $A_1, A_2, B_1,$ and B_2 in (21) to (24) must vanish. This gives the dispersion relation in the general form. Since the expressions for the Q_i 's and T_i 's are complex and quite complicated, an explicit expression for the critical wave number k^* ($= k_x$) cannot be obtained easily analytically. In order to get an insight into the tendencies of the actual situations, we consider now the case of two gravitating streams of the same uniform densities, flowing past each other with the same velocity in opposite directions, and with the same magnetic field and the same velocity of sound in the two streams. The same model has been considered by Singh and Khare [13], we therefore set

$$\rho_1 = \rho_2 = \rho, \quad M_1^2 = M_2^2 = M^2, \tag{27}$$

$$C_1^2 = C_2^2 = C^2, \quad V_1 = V, \quad V_2 = -V.$$

The expressions for Q_1 to Q_4 and T_1 to T_4 are then considerably simplified. Using the values of V_1 and V_2 given by (27) in (16), we find that $\sigma_1^2 = \sigma_2^2 = \sigma^2$ (when $n = 0$) and then $N_1 = N_2 = N$. For the above simple configuration the dispersion relation becomes $N = 0$, i. e.

$$(C^2 k_x^2 - G\rho + \sigma^2)(\sigma^2 + M^2 k_x^2) + 4\Omega^2 \sigma^2 = 0. \tag{28}$$

Now, using the value of $\sigma^2 = -k_x^2 V^2$ (when $n = 0$) in (28), we find that the configuration of rotating gravitating streams is unstable for all wave numbers k_x less than the critical wave number k_B^* , where

$$k_B^* = \sqrt{\frac{G\rho M^2 - G\rho V^2 + 4\Omega^2 V^2}{(C^2 - V^2)(M^2 - V^2)}}. \tag{29}$$

When $V = 0$, i. e. when the streaming velocity vanishes, we obtain Jeans' criterion.

6. Discussion

From (29) we see that in the present case the critical wave number depends on the rotation, magnetic field and streaming velocity. When $\Omega = 0$, i. e. when there is no rotation, the critical wave number below which the configuration is unstable is given by

$$k_s^* = \sqrt{\frac{G\rho}{C^2 - V^2}}, \tag{30}$$

Clearly $k_s^* > k_j$. The streaming velocity has destabilizing influence as it renders the wave number range $k_j < k < k_s^*$ unstable. There is no effect of the magnetic field in this case.

When $M = 0$, i. e. when there is no magnetic field, the critical wave number k_Ω^* is given by

$$k_\Omega^* = \sqrt{\frac{G\rho - 4\Omega^2}{C^2 - V^2}}. \tag{31}$$

Rotation has stabilizing influence on the instability of the configuration as the wave number range $k_\Omega^* < k < k_s^*$ is stabilized by rotation.

Considering now (29) we find that two cases can be distinguished:

(a) $M^2 < V^2$, i. e. when the Alfvén velocity is smaller than the streaming velocity, the effect of rotation is stabilizing as k_s^* decreases on increasing Ω . The magnetic field also has a stabilizing influence in this case as k_B^* increases on increasing M .

(b) $M^2 > V^2$, both rotation and the magnetic field have a destabilizing influence as k_B^* in this case increases on increasing Ω or M , and in this case $k_B^* > k_s^*$.

7. Stability of Streams for Propagation Perpendicular to the Magnetic Field

For propagation perpendicular to the magnetic field, we assume that the perturbed quantities depend on the space coordinates and time as

$$F(z) \exp(ik_y y + nt), \tag{32}$$

where $F(z)$ and n are as explained above and k_y is the wave number of perturbation along the y -axis. Here we take $\vec{V}_s = (0, V_s, 0)$, i. e. the streaming velocity is along the direction of propagation.

For the perturbations of the form (32), (1) to (6) give

$$\rho_s \sigma_s u_s = -2\rho_s w_s \Omega, \tag{33}$$

$$\rho_s \sigma_s v_s = -ik_y \delta p_s + \rho_s ik_y \delta \phi_s - H ik_y (h_x)_s, \tag{34}$$

$$\rho_s \sigma_s w_s = -D\delta p_s + \rho_s D\delta \phi_s + 2\rho_s u_s \Omega - HD(h_x)_s, \tag{35}$$

$$\sigma_s \delta \rho_s = -\rho_s (ik_y v_s + Dw_s), \tag{36}$$

$$\sigma_s (h_x, h_y, h_z)_s = [-H(ik_y v_s + Dw_s), 0, 0], \tag{37}$$

$$ik_y (h_y)_s + D(h_z)_s = 0, \tag{38}$$

$$(D^2 - k_y^2) \delta \phi_s = -G\delta \rho_s, \tag{39}$$

$$\sigma_s \delta p_s = C_s^2 \sigma_s \delta \rho_s, \tag{40}$$

where in this case

$$\sigma_s = n + ik_y V_s. \tag{41}$$

Elimination of the variables leads to the differential equation

$$(D^2 - k_y^2)(D^2 - J_s^2) \delta \phi_s = 0, \tag{42}$$

where

$$J_s^2 = \frac{(4\Omega^2 + \sigma_s^2)(\sigma_s^2 + M_s^2 k_y^2 + C_s^2 k_y^2 - G\rho_s)}{\sigma_s^2 (C_s^2 + M_s^2)}. \tag{43}$$

The solutions of the differential equation (42) for the two regions are therefore

$$\delta \phi_1 = A_1 e^{-k_y z} + B_1 e^{-J_1 z} \quad (z > 0) \tag{44}$$

and

$$\delta \phi_2 = A_2 e^{k_y z} + B_2 e^{-J_2 z} \quad (z < 0). \tag{45}$$

In this mode it is also assumed that J_1 and J_2 are so defined that their real parts are positive. The boundary conditions are the same and lead to the relations (21) to (24), where in this case k_y replaces k_x in (22) and

$$Q_1 = G\rho_1 \alpha_1^2 - [G\rho_1 k_y^2 (4\Omega^2 + \sigma_1^2) / \sigma_1^2] - (\alpha_1^2 - k_y^2) (4\Omega^2 + \sigma_1^2) \cdot [\sigma_1^2 + (M_1^2 + C_1^2) k_y^2] / \sigma_1^2, \tag{46}$$

$$T_1 = (G\alpha_1 k_y^2 / \sigma_1^2) + \alpha_1 (\alpha_1^2 - k_y^2) [\sigma_1^2 + (M_1^2 + C_1^2) k_y^2] / \rho_1 \sigma_1^2. \tag{47}$$

Again the coefficients Q_2 to Q_4 and T_2 to T_4 follow from Q_1 and T_1 , respectively, in exactly the same way as for the other mode.

Now in this mode also, we consider the same model for the streams. Using therefore (27) and proceeding as for the other mode, we find that the dispersion relation for the considered mode is $J = 0$, i. e.

$$(4\Omega^2 + \sigma^2)(\sigma^2 + M^2 k_y^2 + C^2 k_y^2 - G\rho) = 0. \tag{48}$$

Using $\sigma^2 = -k_y^2 V^2$ (when $n = 0$), in (48) we find that the critical wave number k^* , below which the system is unstable, is given by

$$k^* = k_y = \sqrt{\frac{G\rho}{M^2 + C^2 - V^2}}. \tag{49}$$

We observe that k^* is independent of the Coriolis force. The magnetic field has stabilizing influence, as k^* decreases on increasing the magnetic field. The streaming velocity is destabilizing, as k^* increases on increasing V . The results obtained for this mode are the same as for the mode of propagation along the magnetic field.

8. Conclusion

We thus conclude, that when the streams rotate about an axis perpendicular to the magnetic field in

the horizontal plane, the critical wave number, below which the system is unstable, is affected by rotation only for the mode of wave propagation perpendicular to the axis about which the system rotates. Rotation and the magnetic field suppress the instability while the streaming velocity has destabilizing influence.

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