Viscosity influence on the stability of a swirling jet with nonrotating core

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The effect of the viscosity on the linear stability of a swirling top hat jet with nonrotating core is studied analytically. Numerical values of the growth rate are obtained. It is shown that the viscosity induces instabilities in the region of wave numbers where no inviscid instability occurs. In the short wave limit, the asymptotic solution reveals that the situation is quite different for the strictly inviscid and the viscous cases. While in the inviscid fluid the growth rate is proportional to the wave number, in the viscous case the growth rate approaches a constant value for large wave numbers. © *2000 American Institute of Physics.* [S1070-6631(00)00405-0]

We study the influence of the viscosity on the stability to axisymmetric disturbances of a swirling top hat jet with nonrotating core of radius R. The velocity is assumed to have a constant circulation Γ outside of the core, and zero circulation in the core of the jet. The axial component of the velocity has a constant value $\tilde{W}$ in the inner region and zero value outside. The components for the basic flow $\mathbf{U}$ are explicitly given by

$$\{U,V,W\}=\begin{cases}\{0,\Gamma/2\pi r,0\} & \text{for } r>R\\ \{0,0\tilde{W}\} & \text{for } r<R\end{cases},$$

in (r,θ,z) cylindrical components. Note that there is a velocity jump in both the azimuthal and the axial velocity.

A similar model, considering the fluid as inviscid, has been investigated by Martin and Meiburg,[1] and previously by Rotunno.[2] In the former, the capacity of the increasing circulation to stabilize the Kelvin–Helmholtz instability[3] is investigated.

In our model, it is not possible to take the viscosity to be nonzero everywhere in the fluid. If the viscosity is taken to be different to zero everywhere, the stress tensor is infinite at $r=R$ by the presence of the velocity jump. In the present model we consider that the viscosity is small but different to zero in the inner region, while the fluid is inviscid in the outer region.

The linearized equations governing the perturbations $\mathbf{v}$ in the inner region are

$$\frac{\partial \mathbf{v}}{\partial t}+\mathbf{U}\cdot\nabla\mathbf{v}=-\frac{1}{\rho}\nabla p+\nu\Delta^2\mathbf{v}, \tag{1}$$

with $\nabla\cdot\mathbf{v}=0$, where p is the perturbation on the pressure, ρ the density, and ν the viscosity of the fluid. In order to solve these equations, we write the disturbance as the sum $\mathbf{v}=\mathbf{v}_i+\mathbf{v}_v$,[4] where the first term is irrotational $\nabla\times\mathbf{v}_i=0$. By using these properties, we obtain that (1) is satisfied if $\mathbf{v}$ verifies

$$\left(\frac{\partial}{\partial t}+\mathbf{U}\cdot\nabla\right)\mathbf{v}_v=\nu\nabla^2\mathbf{v}_v,\quad \frac{\partial \mathbf{v}_i}{\partial t}+\mathbf{U}\cdot\nabla_{\mathbf{v}i}=-\frac{1}{\rho}\nabla p. \tag{2}$$

We assume that the perturbations are of the form

$$\{u_j,v_j,w_j,p\}=\{f_j(r),g_j(r),h_j(r),\pi(r)\}\times\exp[i\alpha(z-ct)]$$

(where $j=i,v$). Substituting these expressions into Eqs. (2), we obtain the modified Bessel equations

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dh_v}{dr}\right)-\mu^2h_v=0,\quad \frac{1}{r}\frac{d}{dr}\left(r\frac{dh_i}{dr}\right)-\alpha^2h_i=0,$$

with $\mu^2=\alpha^2-i\alpha(c-\tilde{W})/\nu$. By requiring the solution to be bounded at the origin, we obtain the solutions $h_v=AI_0(\mu r)$ and $h_i=BI_0(\alpha r)$, and therefore the complete expression of the disturbances in the viscous region are

$$f=-B\frac{i}{\alpha}\frac{d}{dr}I_0(\alpha r)-iA\frac{\alpha}{\mu^2}\frac{d}{dr}I_0(\mu r),$$

$$h=BI_0(\alpha r)+AI_0(\mu r),$$

$$\pi=B(c-\tilde{W})I_0(\alpha r).$$

In the outer region, the flow can be described by the potential theory.[1] Requiring the solution to be bounded at infinity, we find the expressions for the perturbations in the outer region,

$$h=CK_0(\alpha r),\quad f=-C\frac{i}{\alpha}\frac{d}{dr}K_0(\alpha r),\quad \pi=CcK_0(\alpha r).$$

We now impose boundary conditions at the separation surface of the two regions, which can be expressed as $\eta=R+\delta$, where $\delta=E\exp(ik(z-ct))$. The kinematical condition that the surface moves with the fluid gives $D\eta/Dt=d\eta/dt+(\mathbf{U}\cdot\nabla)\eta=u$, that is,

$$\alpha(c-\tilde{W})E=-AI_0'(\alpha R)-\frac{\alpha}{\mu}BI_0'(\mu R), \tag{3}$$

$$\alpha cd=-CK_0'(\alpha R). \tag{4}$$

The dynamical boundary condition is that the stress of the fluid must be continuous across the surface, $\sigma_{ij,2}(R+\delta)=\sigma_{ij,1}(R+\delta)$. Linearizing, these read

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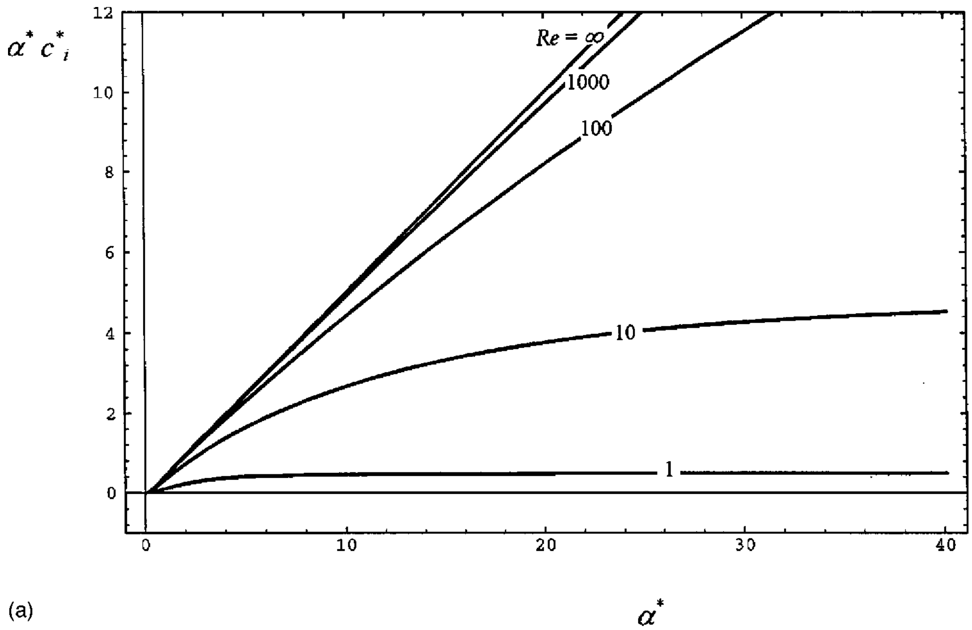

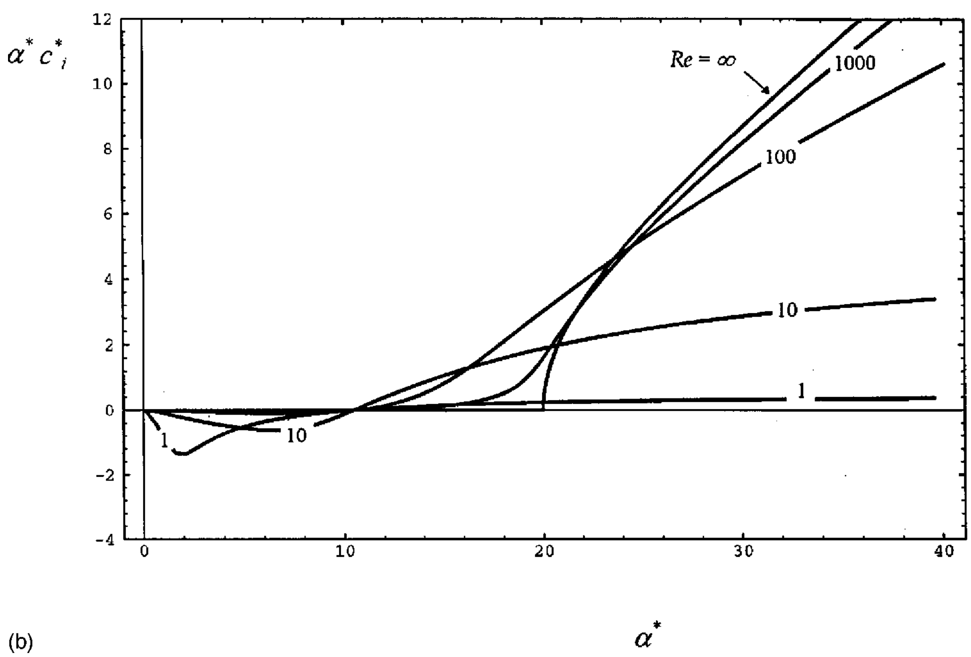

FIG. 1. Growth rate $\sigma=\alpha^* c^*$ for (a) $\Delta=0$. The flow is unstable for all values of Re. The growth rate σ approaches a constant value as α^* tends to infinity. (b) $\Delta=10$. In this case, the circulation existing around the jet stabilizes the flow for small wave numbers. However, the stabilization of the increasing circulation is less effective for the viscous fluid.

$$\tau_{rz,2}(R)=0$$

$$\pi_2(R)+\left.\frac{\partial P_{02}}{\partial \tau}\right|_{r=R}-\tau_{rr,2}(R)=\pi_1(R)+\left.\frac{\partial P_{01}}{\partial r}\right|_{r=R},$$

where τ_{ij} is the deviatoric term of the stress tensor and P_0 is the pressure of the unperturbed flow. Substituting for the expressions of the disturbances we get two equations which together with Eqs. (3) and (4) constitute a linear system for the constants *A, B, C, E*. Elimination of these constants give the following eigenvalue equation:

$$\left(c^*-1+\frac{2i}{\mathrm{Re}}\alpha^*\frac{I_0''(\alpha^*)}{I_0(\alpha^*)}\right)\left(c^*-1+\frac{2i}{\mathrm{Re}}\alpha^*\right)$$
$$-\frac{I_0'(\alpha^*)}{I_0(\alpha^*)}\frac{K(\alpha^*)}{K'(\alpha^*)}c^{*2}-\Delta\frac{I_0'(\alpha^*)}{\alpha^* I_0(\alpha^*)}$$
$$+\frac{4}{\mathrm{Re}^2}\alpha^{*2}\mu^{*2}\frac{I_0''(\mu^*)}{I_0'(\mu^*)}\frac{I_0'(\alpha^*)}{I_0(\alpha^*)}=0, \tag{5}$$

where $\mathrm{Re}=\tilde{W}R/\nu$, $c^*=c/\tilde{W}$, $\alpha^*=\alpha R$, $\mu^*=\sqrt{\alpha^{*2}-i(c^*-1)\alpha^*\mathrm{Re}}$, and $\Delta=(\Gamma/2\pi\tilde{W}R)^2$. The sign of the imaginary part of the eigenvalue $c=c_r+ic_i$ determines

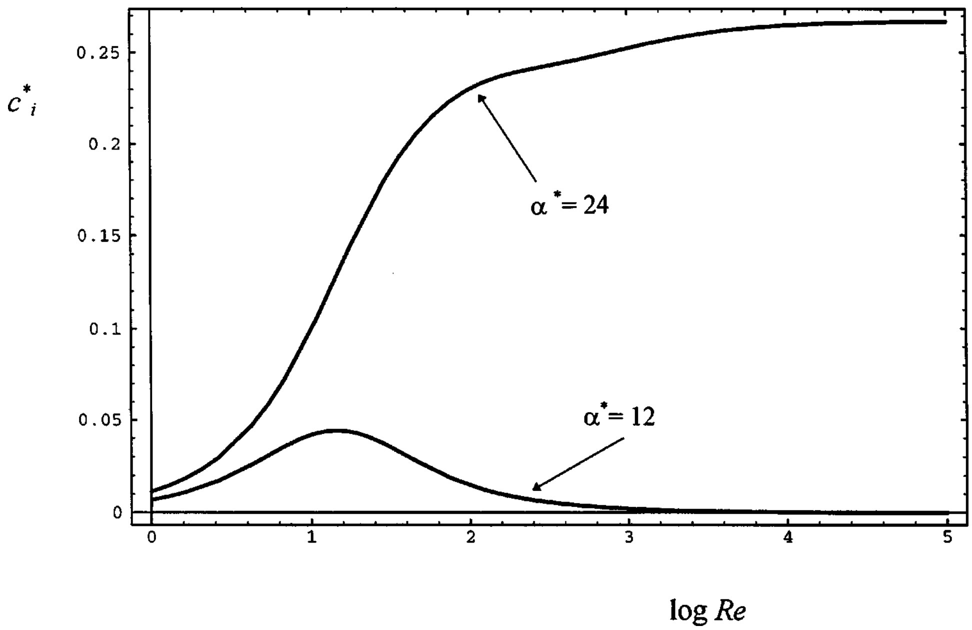

FIG. 2. The variation of the imaginary part of the velocity c_i^* as a function of log Re, for $\alpha^* = 12$ and $\alpha^* = 24$. In the first case ($\alpha^* < 2\Delta$), the growth rate has a maximum at the value Re $= 14.6$, while in the second one ($\alpha^* > 2\Delta$), the mode is most unstable as Re$\rightarrow\infty$.

the stability of the jet. When c_i is positive, the disturbance grows exponentially and the flow is unstable.

Let us consider Re to be the Reynolds number of the basic flow. As Re tends to infinity, the eigenvalue equation for the inviscid case of Martin and Meiburg[1] is recovered from (5) for the particular case of zero circulation in the core.

We solve the eigenvalue equation by use of the Newton method to determine c^* for different given values of Re and the wave number α^*. Figure 1 shows the nondimensional velocity growth $\sigma = \alpha^* c_i^*$ for the most unstable solution, i.e., the eigenvalue with maximum imaginary part, as a function of α^* for different values of Re.

This figure reveals that when Δ is different to zero, the flow is stable for perturbations with small wave numbers, but unstable for sufficient large wave numbers at all values of Re. However, with $\Delta = 0$ the flow is unstable for all wave numbers at any value of Re.

Let us consider the limit of small wave numbers. The asymptotic expression of Eq. (5) (as $\alpha^* \rightarrow 0$) results in the following eigenvalue equation:

$$\left(c^* - 1 + \frac{i}{\mathrm{Re}}\right)\left(c^* - 1 + \frac{i2\alpha}{\mathrm{Re}}\right) - c^{*2}\frac{\alpha^*}{2}\ln\alpha^* - \frac{\Delta}{2} + \frac{2\alpha^{*2}}{\mathrm{Re}^2} = 0.$$

To solve this quadratic equation, we conserve lower order terms in α^*, considering in separate form those terms depending on Re and those independent of it. Hence, the expression of c^* for small wave numbers is

$$c^* = 1 - \frac{i}{2\,\mathrm{Re}}\left(1 \pm \sqrt{1 - 2\Delta\,\mathrm{Re}^2 - 2\mathrm{Re}^2\alpha^{*2}\ln\alpha^*}\right).$$

When $\Delta > 0$, the last term under the square root is negligible and then the flow is stable for all values of Re. With $\Delta = 0$, the term under the square root is greater than 1 and then the basic flow is unstable for long waves.

The stabilization which takes place for $\Delta > 0$ appears to be related to the Rayleigh stability,[3] since this case corresponds to the situation in which the outer circulation is greater than the circulation in the core (which is null in fact). We show here that the Rayleigh stability remains dominant at small wave numbers even in the presence of viscosity. This is, for all values of Re as in the inviscid case.[1]

For arbitrary Re the flow is unstable to sufficiently large wave number. Moreover, Fig. 1 shows that the growth rate of the disturbances with wave number between Δ and 2Δ has a maximum for a finite value of Re. In fact, there are modes which are stable in the inviscid case, but become unstable in the viscous case. To study in detail this property, let us consider that $\Delta \gg 1$ and Re is very high but finite. Employing the asymptotic expression for the modified Bessel functions for large values of the argument, Eq. (5) becomes $(c^* - 1 + i2\alpha^*/\mathrm{Re})^2 + c^{*2} - \Delta/\alpha^* + 4\alpha^*\mu^*/\mathrm{Re}^2 = 0$.

Since we assumed Re to be very high, we consistently consider only the least order terms (in 1/Re). Then, the following approximate expression for c^* is obtained:

$$c^* = \frac{1}{2}\left(1 \pm \sqrt{2\Delta/\alpha^* - 1}\right) + \left(-1 \pm 1/(1 - 2\Delta/\alpha^*)\right)\frac{\alpha^*}{\mathrm{Re}}\,i. \tag{6}$$

From the instability condition $c_i^* > 0$, we find that the flow is unstable for all disturbances with $\alpha^* > \Delta$, for finite Re. This last result is unexpected since usually it has been accepted that the viscosity acts purely as a stabilizing agent for swirling flows in the absence of rigid boundaries.[5] Recently, Mayer and Powell[6] and Khorrami[7] reported viscous instabilities in their study of the trailing vortex stability. They found that a stable range of wave numbers in the inviscid case was destabilized by increasing viscosity.

It can be pointed out that this destabilizing influence of viscosity can appear in other flows like boundary layer or parallel flows.[3,8,9]

The dependence of c_i^* as a function of log Re is shown in Fig. 2. For wave numbers less than 2Δ, the growth rate has a maximum for a relative low Re, and tends to zero as Re $\rightarrow\infty$, decreasing with 1/Re.

On the other hand, for a fixed wave number $\alpha^* > 2\Delta$, the disturbance is most unstable as Re tends to infinity.

It is interesting to compare these results with those obtained in recent works by Mayer and Powell[6] and Khorrami,[7] in which the stability of the trailing vortex is studied by numerical methods. Results from these earlier investigations have several similar features to what we have observed. In particular, in Ref. 6 the authors report instabilities with growth rate decreasing as 1/Re at a fixed wave number, in agreement with the expression for c^* in Eq. (6). These coincidences are remarkable, taking into account the differences in the models.

In the limit of large α (finite Re), the approximation $\mu^* \simeq \alpha^* - (1/2)i(c^* - 1)\mathrm{Re}$ holds and hence Eq. (5) becomes a quadratic equation with solutions

$$c^* = 1/2(1 - i\alpha^*/\mathrm{Re}) \pm + 1/2\sqrt{(i\alpha^*/\mathrm{Re}+1)^2 - 2 + 2\Delta/\alpha^*}.$$

Taking the limit $\alpha^* \rightarrow \infty$, we find the two solutions for the complex velocity approach $c^* \rightarrow 1$, $c^* \rightarrow -i\alpha^*/\mathrm{Re}$. The first one is the unstable solution, and its imaginary part tends to zero as $1/\alpha^*$ in the limit $\alpha^* \rightarrow \infty$. Then the growth rate approaches a constant value for large wave numbers, for all values of Re (see Fig. 1). The situation is quite different from the strictly inviscid model case, for which $c^* \rightarrow 1/2(1 \pm i)$ as $\alpha^* \rightarrow \infty$, and as a consequence, the growth rate tends to infinity in the limit of large wave numbers. Viscous analysis thus demonstrates that although large wave number modes are the most unstable in the strictly inviscid case, they are also the most susceptible to the stabilizing action of the viscosity.

In this work, we present analytical evidence of the novel phenomenon that a swirling flow can be destabilized by the presence of the viscosity, without the presence of rigid walls. Rayleigh stability predominates in the region of small wave numbers. However, the stable region is reduced in the presence of viscosity.

These results show that the effects of the viscosity are drastic in the range of large wave numbers. The conclusions drawn from inviscid models require great care in the short wave region.

The present model is restrictive since it takes the fluid to be viscous in a limited zone of the volume. Analytically, we have obtained results which are in agreement with the more computationally intensive investigations performed in Refs. 6 and 7.

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