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ON NORMAL VERBAL EMBEDDINGS OF GROUPS

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For the case of an arbitrary group H and an arbitrary word set V, we establish a necessary and sufficient condition under which there exists a group G such that H is isomorphic to a normal subgroup $\tilde{H}$ of G such that $\tilde{H}$ lies in $V(G)$. This is a generalization of results of Burnside and Blackburn (concerning the cases of the commutator word and some much more special classes of groups) as well as of the first author (establishing a criterion for the case of one word w and finite p-group H). Some related special cases are considered.

1. Introduction

It is not difficult to show that for each nontrivial set V of words each (finite, finitely generated) group is embeddable in the verbal V-subgroup of a certain (finite, finitely generated) group. Still the following problem of *normal embeddings* of this type is far from being trivial even for such a "simple" word subgroup as a commutator subgroup: *for the given group H and given set of words $V \subseteq F_\infty$ there exists a group G with a subgroup $\tilde{H}$ such that $H \cong \tilde{H}$, $\tilde{H} \triangleleft G$, and $\tilde{H} \subseteq V(G)$.*

In 1912, Burnside proved that a nonabelian group with a cyclic center or a nonabelian group the index of whose derived group is p^2 cannot be the derived group of a p-group [3, Theorems on pp. 241 and 242]. On the other hand, Blackburn has found all the 2-generated p-groups which occur as derived groups of p-groups [2]. Finally, the first author generalized these results and proved that a finite p-group H is invariant in some finite p-group G and lies for the word $w \in F_\infty$ in the verbal subgroup $w(G)$ if and only if $w(L) \supseteq \mathrm{Inn}\,(H)$, where L is a Sylow-p-subgroup of $\mathrm{Aut}\,(H)$ [5].

The main aim of this paper is to study such embeddings in a more general situation.

First, we establish a necessary and sufficient condition for arbitrary V and arbitrary H under which H is invariant embeddable in some group G and lies in $V(G)$ (Theorem 1 in Sec. 3).

Next we consider embeddings with the property mentioned for the case of abelian groups (Sec. 4). It turns out that some additional properties can be provided. Each abelian group H can be normally embedded in some *nilpotent* group G such that its image $\tilde{H}$ is normal in G and $\tilde{H} \subseteq V(G)$. If H is finite (finitely generated), G can be chosen finite (finitely generated) and nilpotent (Theorem 2). Meanwhile, it is known that some finite solvable groups cannot even be *subnormally* embedded in any finite group and simultaneously be contained in the commutator subgroup of the latter [6].

In Sec. 5, we impose a restriction on the word set and obtain "economical" embeddings for certain word subgroups — commutator subgroup, nth degree, terms of the lower central series, etc. Some of them are generalizations of the above-mentioned results on their own.

The general criterion we establish can be used not only to obtain many concrete examples of the groups H that *can* or *cannot* be normally embedded into some group G such that $\tilde{H} \subseteq V(G)$ but also can be used over whole classes of groups or constructions of groups (direct and cartesian products, matrix and symmetric groups, etc.). This will be the subject of another paper [7]; here we restrict ourselves only to an illustration of the use of Theorem 1, namely with a criterion of the possibility of embeddings of the above-mentioned type for the *symmetric* groups (Sec. 6).

We would like to announce here the results of the second author [10] concerning a similar problem, namely the *subnormal* embeddability of the given group H in some group G such that the isomorphic image of H lies in $V(G)$. It turns out that such embeddings always exist. In addition, G can have some additional properties. In particular, [10] contains generalizations of some well-known theorems on embeddings of groups in 2-generated groups, etc.

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2. Notions

For the sake of brevity, we use a special notion:

Definition 1. For a word set V, the nV-embedding ν of the group H into the group G is the monomorphism $\nu : H \to G$ such that

$$\tilde{H} = \nu(H) \triangleleft G \quad \text{and} \quad \tilde{H} \subseteq V(G).$$

The group H is said to be nV-embeddable if there is a group G and an nV-embedding of H into G.

Further, $\mathfrak{V}$ will be used to denote the variety $\mathfrak{V} = \operatorname{var}(F_\infty/V(F_\infty))$ corresponding to the word set V. $\mathfrak{O}$ will be the variety of all groups. $\gamma_c(x_1, \dots, x_c)$ is the word $[x_1, \dots, x_c]$; the word $\delta_n(x_1, \dots, x_{2^n})$ is defined as $\delta_0 = x$ and

$$\delta_{n+1}(x_1, \dots, x_{2^{n+1}}) = [\delta_n(x_1, \dots, x_{2^n}), \delta_n(x_{2^n+1}, \dots, x_{2^{n+1}})].$$

For general background information we refer the reader to [9] or [13]. The book of Hanna Neumann [12] can be used for information on the varieties of groups.

3. General Condition for nV-Embeddability

Let M be an infinite set and S_M be the group of permutations of M. For the case where M is countable, Dark has noted in [4] that the derived group $S_M{}'$ coincides with S_M. The following more general lemma is of some independent interest.

Lemma 1. *For an arbitrary nontrivial word set V and an arbitrary infinite set M the following holds:*

$$V(S_M) = S_M.$$

So S_M is rather trivially nV-embeddable into itself for each nontrivial word set V.

Proof. Let $\aleph_\mu$ be the order of M and let $0 \leq \beta \leq \mu$ be a smaller ordinal number. Let S_β be the subset of permutations of S_M that move in fact not more than $\aleph_\beta$ elements of the set M. As is easy to see, these subsets are in fact subgroups. For example, S_0 is the subgroup of S_M that consists of "ordinary" permutations of set M, i.e., the permutations that move only *finitely* many elements. Let A_M be the subgroup of S_0 consisting of all even permutations on M.

As is shown by Baer [1], all the invariant subgroups of S_M are members of the following series:

$$1 \triangleleft A_M \triangleleft S_0 \triangleleft S_1 \triangleleft \dots \triangleleft S_\beta \triangleleft \dots \triangleleft S_\mu \triangleleft S_M. \tag{1}$$

The verbal subgroups are invariant in the group, so it would be enough to show that the subgroup $V(S_M)$ is *transitive* on the set M and is therefore larger than each proper normal subgroup from (1).

Let m_1 and m_2 be two elements of M. The variety $\mathfrak{V}$ is different from the variety $\mathfrak{O}$ of all groups and therefore there exists some finite group N not belonging to $\mathfrak{V}$ (because the set of all finite groups generates $\mathfrak{O}$). Via the right regular representation, N is embedded into the group S_N. Let

$$N' = N \cup \{m_1, m_2\}.$$

S_N is naturally embeddable into the alternating group $A_{N'}$. Without loss of generality, the order of N' is larger than 4 (if this is not so we simply take $N \times N$ instead of N). So the group $A_{N'}$ is simple and does not belong to V. Therefore $V(A_{N'}) = A_{N'}$. In the transitive group $A_{N'}$ there exists a permutation ρ such that $\rho : m_1 \mapsto m_2$. Thus the permutation we are looking for can be defined as

$$\tilde{\rho} : m \mapsto \begin{cases} \rho(m) & \text{if} \quad m \in N' \\ 1 & \text{if} \quad m \in M \backslash N'. \end{cases}$$

The lemma is proved. □

The following lemma is an analog of Lemma 2 from [5], proved for p-groups.

Lemma 2. *The group H is not nV-embeddable if*

$$V(\mathrm{Aut}\,(H)) \not\supseteq \mathrm{Inn}\,(H).$$

Proof. Let H be nV-embeddable into G. The action of elements of G by conjugations on H defines an isomorphism of $G/C_G(H)$ onto some subgroup of $\mathrm{Aut}\,(H)$. It maps $HC_G(H)/C_G(H)$ onto $\mathrm{Inn}\,(H)$. Forming verbal subgroups is a monotonic operation [12]. Thus if $V(\mathrm{Aut}\,(H))$ does not contain $\mathrm{Inn}\,(H)$, then

$$V(G/C_G(H)) = V(G)C_G(H)/C_G(H) \not\supseteq HC_G(H)/C_G(H).$$

So H is not contained in $V(G)$. □

Let us reformulate the result stated in the introduction in terms of "nV-embeddability."

Theorem 1 (the main theorem). *For a given nontrivial word set V, a given group H is nV-embeddable if and only if*

$$V(\mathrm{Aut}\,(H)) \supseteq \mathrm{Inn}\,(H), \tag{2}$$

and a finite (finitely generated) group is nV-embeddable into some group if and only if it is nV-embeddable into a finite (finitely generated) group.

Proof. Condition (2) is necessary for nV-embeddability by Lemma 2. Let us now assume that (2) holds and construct a group G and an nV-embedding $\nu : H \to G$.

The variety $\mathfrak{V}$ corresponding to V is smaller than $\mathfrak{O}$. As above, there is a finite group $N \notin \mathfrak{V}$. (As we will see, if H is *infinite*, then for the construction of this proof one could take $N = 1$; but we build the construction in general for the purposes of a later modification.) Let L be defined as

$$L = N \times \mathrm{Hol}\,(H)$$

and θ be the "natural" embedding of H into L given by

$$\theta : h \to (1, h) = (1_N, h{\cdot}\varepsilon_{\mathrm{Aut}(H)})$$

(where $\varepsilon_{\mathrm{Aut}(H)}$ is the trivial automorphism of H). The embedding

$$\varphi = \varphi_i, \quad i = 1, 2,$$

must be defined differently for the cases of finite and infinite groups. If H is finite, φ_1 is the well-known embedding of the finite group L into the alternating group $A_{L\cup\{t_1,t_2\}}$, where t_1 and t_2 are two arbitrary elements not from L. If H is infinite, then $\varphi_2 : L \to S_L$ is the right regular representation of L.

Each automorphism of H is a restriction of some inner automorphism of $\mathrm{Hol}\,(H)$. Consequently each automorphism of the image $(\theta\varphi_i)(H)$, $i = 1, 2$, is a restriction of an appropriate inner automorphism of the group $K_i = \varphi_i(L)$, $i = 1, 2$. Therefore, it is correct to define the split extension G of the group K_1 or K_2 by the automorphism group $\mathrm{Aut}\,(H)$:

$$\forall f \in \mathrm{Aut}\,(H), k \in K_i \quad k^f = (\varphi_i(1, f))^{-1}{\cdot}k{\cdot}(\varphi_i(1, f))$$

($i = 1$ or 2). H is embeddable into G,

$$\forall h \in H, \quad \nu : h \mapsto (f_h)^{-1}{\cdot}((\theta\varphi_i)(h)) \in G,$$

where φ_1 or φ_2 stand for the cases of finite or infinite H, and where $f_h \in \mathrm{Aut}\,(H)$ is the inner automorphism corresponding to the element $h \in H$. It is easy to check that ν is a monomorphism.

The subgroup $\tilde{H} = \nu(H)$ is normal in G. Indeed, if $g \in \mathrm{Aut}\,(H)$ and $\kappa \in K_i$, $i = 1, 2$, then

$$\begin{aligned}(\nu(h))^g &= g^{-1}(f_h)^{-1}g{\cdot}((\theta\varphi_i)(h))^g \\ &= (f_{h^g})^{-1}{\cdot}(\theta\varphi_i)(h^g) = \nu(h^g) \in \tilde{H},\end{aligned}$$

since

$$((\theta\varphi_i)(h))^g = \varphi_i(1, h)^g = \varphi_i(1, g)^{-1}\varphi_i(1, h)\varphi_i(1, g) = \varphi_i(1, h^g),$$

and

$$\begin{aligned}(\nu(h))^\kappa &= (f_h)^{-1}{\cdot}(\kappa^{-1})^{(f_h)^{-1}}\varphi_i(1, h)\kappa \\ &= (f_h)^{-1}{\cdot}\varphi_i(1, h)\kappa^{-1}\varphi_i(1, h^{-1})\varphi_i(1, h)\kappa \\ &= \nu(h) \in \tilde{H}.\end{aligned}$$

Next, we note that $\tilde{H} \subseteq V(G)$. Indeed,

$$\forall h \in H \quad (f_h)^{-1} \in \operatorname{Inn}(H) \subseteq V(\operatorname{Aut}(H))$$

and

$$\forall h \in H \quad (\theta\varphi_i)(h) \in V(K_i), \quad i = 1, 2,$$

since $V(K_i) = K_i$. The latter holds for infinite groups because of Lemma 1 and for finite groups because of the fact that the alternating group A_n, $n > 4$, is simple.

Now observe that if H is finite G is finite, too.

We finish the proof showing that if H is finitely generated, G can be chosen finitely generated, too. The image $\tilde{H} \subseteq V(G)$ is finitely generated by the elements, say, $\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_s$. So for each element $\tilde{h}_i \in V(G)$, $i = 1, \dots, s$, there is a representation

$$\tilde{h}_i = \left(v_1^{(i)}(g_{11}^{(i)}, \dots, g_{1q_1}^{(i)})\right)^{\delta_1^{(i)}} \cdots \left(v_u^{(i)}(g_{u1}^{(i)}, \dots, g_{uq_u}^{(i)})\right)^{\delta_u^{(i)}},$$

where $v_1^{(i)}, \dots, v_u^{(i)}$ are some words from V, elements $\delta_u^{(i)}$ receive values 1 or -1, and where $g_{jk}^{(i)} \in G$. Let G_1 be the subgroup of G generated by this finite set of elements $g_{jk}^{(i)}$. Of course $\tilde{H} \subseteq V(G_1)$ and $\tilde{H} \triangleleft G_1$. $\square$

The construction of the proof could be made much smaller for the case of infinite groups. According to the mentioned result of Baer [1], S_L has no proper normal subgroup larger than S_λ (the ordinality of L is $\aleph_\lambda$). Thus the factor group S_L/S_λ is simple. If we take, while constructing L, a finite group N not from $\mathfrak{V}$, we obtain

$$V(S_L/S_\lambda) = S_L/S_\lambda.$$

Therefore, this factor group can be used in the proof instead of S_M.

4. nV-Embeddability of Abelian Groups

From Theorem 1 the nV-embeddability of abelian groups for nontrivial V follows automatically. But in fact we are able to prove more:

Theorem 2. *Let V be a nontrivial word set and H be an arbitrary nontrivial abelian group. Then*

(1) *H is nV-embeddable into a nilpotent group G;*
(2) *H is nV-embeddable into an abelian group A if and only if V has some consequence of the form $x^n = 1$ ($n \in \mathbb{N}$);*
(3) *if H is finite (finitely generated), the mentioned nilpotent group G or abelian group A can also be chosen finite (finitely generated).*

The proof follows from the following lemmas, in which V is assumed to be nontrivial.

Lemma 3. *The finite abelian p-group H is nV-embeddable into some finite p-group.*

Proof. Slightly modifying the Lemma 1 from [5], for the case of an arbitrary V we obtain for each power $m = p^k$ a finite p-group P such that its verbal subgroup $V(P)$ is of exponent m and is contained in the center of P. We represent H as

$$H = \mathbb{Z}_{p^{k_1}} \times \cdots \times \mathbb{Z}_{p^{k_s}}$$

and find a group P_i with the mentioned property for each exponent $p^{k_1}, \dots, p^{k_s}$. Since the abelian group $V(P_i)$ is of exponent p^{k_i}, it contains a cyclic subgroup isomorphic to $\mathbb{Z}_{p^{k_i}}$ and normal in P_i (it is contained in the center of P_i). Thus the direct product $P_1 \times \cdots \times P_s$ contains a normal subgroup isomorphic to $\mathbb{Z}_{p^{k_1}} \times \cdots \times \mathbb{Z}_{p^{k_s}}$. $\square$

Lemma 4. *The group $\mathbb{Z}$ is nV-embeddable into a finitely generated nilpotent group N.*

Proof. The variety $\mathfrak{V} \neq \mathfrak{O}$ cannot contain all nilpotent groups. Therefore, there is a free nilpotent group $F = F_n(\mathfrak{N}_c)$ of some rank n and some class c such that $F \notin \mathfrak{V}$. The invariant subgroup $V(F)$ of F is not trivial and must have a nontrivial intersection I with the center $C(F)$. The subgroup I of a *finitely generated* and *torsion-free* nilpotent group inherits both these properties. So

$$I = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{l} = \mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{l-1} = \mathbb{Z} \oplus I',$$

where I' is the sum of the "last" $l-1$ copies of $\mathbb{Z}$. I' is normal in F since I' is from the center. Let $N = F/I'$. Clearly:

$$\mathbb{Z} \subseteq V(N) \quad \text{and} \quad \mathbb{Z} \lhd N.$$

The lemma is proved. □

We are already able to see that each finite (finitely generated) abelian group is nV-embeddable into some finite (finitely generated) nilpotent group. But since the classes of groups used in the proof of Lemma 3 are not necessarily bounded, we cannot simply use Lemma 3 for arbitrary infinitely generated abelian groups.

Lemma 5. *Each abelian group H is nV-embeddable into a nilpotent group.*

Proof. In the proof of Lemma 4 we have seen that $\mathbb{Z}$ is nV-embeddable into some N. For each $n \in \mathbb{N}$ the subgroup $n\mathbb{Z}$ is invariant in N and $S = N/n\mathbb{Z}$ contains the normal subgroup $\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$. Clearly $\mathbb{Z}_n \subseteq V(S)$.

As follows from Zorn's lemma, for each element $h \in H$ there exists a maximal subgroup N_h of H such that $h \in H \backslash N_h$. H is contained in the semidirect product

$$\prod_{h \in H} (H/N_h)$$

of (finite or infinite) cyclic groups H/N_h. Each group H/N_h is embeddable in some factor group of the group N that we constructed earlier. These factor groups have some restricted bound for classes of nilpotency. So $\prod_{h \in H}(H/N_h)$ is embeddable into some nilpotent group G (of class $\leq c$). Finally, we note that in our construction each group H/N_h is contained in the *center* of the appropriate nilpotent group. Therefore $\prod_{h \in H}(H/N_h)$ is from the center of G. Thus H is normal in G. □

We are forced to use two parallel constructions for the nV-embeddability of finite abelian groups because the first one provides no common bound for the nilpotency class and the second one does not give nV-embedding of a finite abelian group into a finite nilpotent group.

Lemma 6. *A nontrivial abelian group H is nV-embeddable into an abelian group A if and only if V has some consequence of the form $x^n = 1$, $n \in \mathbb{N}$. If H is finite (finitely generated), A can be chosen finite (finitely generated) as well.*

Proof. If the condition of the lemma holds, then for every group X $X^n \subseteq V(X)$ holds and it is enough to say that:

(a) Each abelian group H is embeddable into a *divisible* abelian group A. So $H \lhd A^n \subseteq V(A) = A$.
(b) The groups $\mathbb{Z}$ and $\mathbb{Z}_m$ are nV-embeddable for $V = \{x^n\}$ since

$$\mathbb{Z} \cong n\mathbb{Z} \quad \text{and} \quad \mathbb{Z}_m \cong n{\cdot}\mathbb{Z}_{mn}.$$

Thus, the finite (finitely generated) abelian group H is nV-embeddable into a finite (finitely generated) abelian group A.

On the other hand, if V has no consequence of the form $x^n = 1$ (for some $n \in \mathbb{N}$), then all its consequences should be commutator words because each word is equivalent to a (possibly trivial) commutator word and to a word of type x^n (possibly $n = 0$). Since now all n's are 0 and A is abelian we have that $V(A) = 1$.

The proof of Theorem 2 is completed. □

5. Some Smaller Embeddings

Theorem 1 does not use the explicit form of V. In the following we obtain some smaller embeddings depending on the words we use. We will consider the "most common" words

$$[x_1, x_2], \quad x^l, \quad \delta_n(x_1, \dots, x_{2^n}), \quad \gamma_c(x_1, \dots, x_c).$$

As we have said, for each power $m = p^k$ there is a finite p-group P such that its verbal subgroup $V(P)$ is of exponent m and lies in the center of P. Then, as is shown in the proof of Theorem 3 in [5], the diagonal

$$D = \operatorname{Diag}(H \operatorname{Wr} P) = \left\{ \prod_{t \in P} t^{-1} h t \,\middle|\, h \in H \right\}$$

of the wreath product $W = H \operatorname{Wr} P$ is contained in the verbal subgroup $V(W)$ of W. The latter is a p-group. And there is an obvious embedding of H into W:

$$\nu : h \mapsto \prod_{t \in P} t^{-1} h t.$$

This construction will be our initial step for further modification (not only for p-groups). If for the given group H we find an appropriate group K such that

$$D = \operatorname{Diag}(H \operatorname{Wr} K) \subseteq V(H \operatorname{Wr} K), \tag{3}$$

then the natural endomorphism

$$\mu : h \mapsto \prod_{k \in K} k^{-1} h k, \quad h \in H,$$

gives an embedding of H into $V(W) = V(H \operatorname{Wr} K)$. The elements of D admit all the conjugations by elements of K. (If H is abelian, this already provides an nV-embedding of H into W.) More generally, the following lemma holds.

Lemma 7. *If K is chosen so that* (3) *holds and if there exists a subgroup B of the group of automorphisms of H such that $V(B) \supseteq \operatorname{Inn}(H)$, then the embedding*

$$\nu : h \mapsto g_h^{-1} \cdot \prod_{k \in K} k^{-1} h k \cdot 1_K, \quad h \in H,$$

where g_h is the inner automorphism of H corresponding to h, is an nV-embedding of H into the extension of H by B defined by the rule

$$h^g = g(h), \quad k^g = k, \quad \text{where} \quad g \in G, \quad h \in H, \quad k \in K.$$

Of course, B could coincide, in particular, with $\operatorname{Aut}(H)$.

Proof. Clearly, $\tilde{H} = \nu(H) \subseteq V\langle W, B \rangle$. Thus we should just compute that for each $h, t \in H$,

$$\begin{aligned} \nu(h)^t &= t^{-1} \cdot \big(g_h^{-1} \cdot (\prod_{k \in K} k^{-1} h k)\big) \cdot t \\ &= g_h^{-1} \cdot (t^{-1})^{g_h^{-1}} \cdot (\prod_{k \in K} k^{-1} h k) \cdot t = g_h^{-1} \cdot h t^{-1} h^{-1} \cdot (\prod_{k \in K} k^{-1} h k) \cdot t \\ &= g_h^{-1} \cdot (\prod_{k \in K} k^{-1} h k) = \nu(h). \end{aligned}$$

$\square$

This lemma enables us to reduce the process of construction of a small embedding to the construction of embeddings of the type

$$\mu : H \to \operatorname{Diag}(H \operatorname{Wr} K) \subseteq V(H \operatorname{Wr} K).$$

Proposition 1. *The group H is normally embeddable into some group G and $H \subseteq G'$ if and only if* $(\operatorname{Aut}(H))' \supseteq \operatorname{Inn}(H)$.

Proof. Take $K = \mathbb{Z}$. Then $D \subseteq (H \operatorname{Wr} \mathbb{Z})'$ because for each h

$$\prod_{z\in\mathbb{Z}} z^{-1}hz = (\dots h, h, h \dots) = [1_{\mathbb{Z}}, \varphi_h],$$

where $\varphi_h = (\dots h^{-2}, h^{-1}, 1_H, h, h^2 \dots) \in \operatorname{Fun}(\mathbb{Z}, H)$ is the function defined as $\varphi_h = h^z$. $\square$

Proposition 2. *The group H is normally embeddable into some group G and $H \subseteq G^l$ if and only if $(\operatorname{Aut}(H))^l \supseteq \operatorname{Inn}(H)$.*

Proof. Let $K = \mathbb{Z}_l$ and k be the generator of K. Then we have

$$\prod_{z=0}^{l-1} z^{-1}hz = (\underbrace{h, \dots, h}_{l}) = ((h, \underbrace{1, \dots, 1}_{l-1}) \cdot k)^l \in (H \operatorname{Wr} \mathbb{Z}_l)^l.$$

$\square$

Modifying the construction from Proposition 1, we obtain:

Proposition 3. *The group H is normally embeddable into some group G and $H \subseteq \delta_n(G)$ if and only if $\delta_n(\operatorname{Aut}(H)) \supseteq \operatorname{Inn}(H)$.*

Proof. We take

$$Q = \underbrace{(((H \operatorname{Wr} \mathbb{Z}) \operatorname{Wr} \mathbb{Z}) \dots) \operatorname{Wr} \mathbb{Z}}_{s \text{ times}} = W_{(H,s)},$$

where $s = [\log_2 n] + 1$. Let $D_1 = \operatorname{Diag}(H \operatorname{Wr} \mathbb{Z})$. If D_i is already defined, we define D_{i+1} as the subgroup of the group $\operatorname{Diag}\left(W_{(H,i+1)}\right)$ containing elements that have "coordinates" only from D_i (and not from the entire passive group of the wreath product $W_{(H,i+1)}$). So D_s is the set of all s-dimensional matrices over H with the property that all elements of each such matrix are equal.

Let $\mu : H \to D_s$ be the endomorphism mapping each $h \in H$ onto the matrix all elements of which are h. Clearly each matrix $\mu(h)$ will remain unchanged under conjugations with elements of all copies of the group $\mathbb{Z}$ from Q (for such conjugations will just exchange multidimensional "columns" of the matrix). As in the proof of Lemma 7, we can take as G the extension of Q by $\operatorname{Aut}(H)$ defining that each automorphism f acts trivially over the copies of $\mathbb{Z}$ and acts as $h^f = f(h)$ over H. Then the family of elements $\tilde{H} = \{(g_h)^{-1}\mu(h) |\ h \in H\}$ will form a *normal* subgroup of G. It remains to understand that $D_s \subseteq \delta_n(Q)$ and thus $H \cong \tilde{H} \subseteq \delta_n(Q)$. $\square$

We could use this very same construction for the word $\gamma_c(x_1, \dots, x_c)$ as well, because for an arbitrary group G $\delta_i(G) \leq \gamma_{2^i}(G)$ holds, and we could simply use Proposition 3 for the value $n = [\log_2 c] + 1$. But for γ_c we will build something much smaller, namely the embedding into $H \operatorname{Wr} \mathbb{Z}$.

Proposition 4. *The group H is normally embeddable into some group G and $H \subseteq \gamma_c(G)$ if and only if $\gamma_c(\operatorname{Aut}(H)) \supseteq \operatorname{Inn}(H)$.*

Proof. Let $G = H \operatorname{Wr} \mathbb{Z}$. For an arbitrary set of integers $\{b_i | i \in \mathbb{Z}\}$, the system of equations

$$-x_{i-1} + x_i = b_i, \quad i \in \mathbb{Z},$$

has a solution in integers. Namely, we could take an arbitrary integer value $x_0 = a_0$ and continue:

$$x_1 = a_1 = b_1 + a_0, \quad x_2 = a_2 = b_2 + a_1, \quad x_3 = a_3 = b_3 + a_2, \dots,$$
$$x_{-1} = a_{-1} = a_0 - b_0, \quad x_{-2} = a_{-2} = a_{-1} - b_{-1}, \quad x_{-3} = a_{-3} = a_2 - b_2, \dots$$

Then the following holds:

$$(\dots, h^{b_{-2}}, h^{b_{-1}}, h^{b_0}, h^{b_1}, h^{b_2}, \dots) = [1_{\mathbb{Z}}, (\dots, h^{a_{-2}}, h^{a_{-1}}, h^{a_0}, h^{a_1}, h^{a_2}, \dots)],$$

where each coordinate h^{b_i} or h^{a_i} is placed in the position with index i.

We have already seen that

$$(\dots h, h, h \dots) = [1_{\mathbb{Z}}, \varphi_h] = [1_{\mathbb{Z}}, (\dots, h^{-2}, h^{-1}, 1_h, h, h^2, \dots)].$$

Taking $b_i = i$, $i \in \mathbb{Z}$, we can find integers $a_i \in \mathbb{Z}$ such that

$$\varphi_h = [1_{\mathbb{Z}}, (\dots, h^{a_{-1}}, h^{a_0}, h^{a_1}, \dots)] = [1_{\mathbb{Z}}, \varphi'_h].$$

Continuing this process c times, we obtain

$$\prod_{z\in\mathbb{Z}} z^{-1}hz = (\dots h, h, h \dots) = [1_{\mathbb{Z}}, [1_{\mathbb{Z}}, \dots, [1_{\mathbb{Z}}, \varphi''_h] \dots]] \in \gamma_c(G)$$

for some φ''_h. □

Now we can prove the following theorem.

Theorem 3. *Let V consist of one of the words $[x_1, x_2]$, x^l, $\delta_n(x_1, \dots, x_{2^n})$, $\gamma_c(x_1, \dots, x_c)$. Then the solvable group H is nV-embeddable into some solvable group G if and only if the group* Aut (H) *contains a solvable subgroup B such that $V(B) \supseteq$* Inn (H).

Proof. Assume that the mentioned subgroup B exists. First we note that all the groups $H \operatorname{Wr} \mathbb{Z}$, $H \operatorname{Wr} \mathbb{Z}_l$, and $W_{(H,s)}$ constructed for the embedding μ in the proofs of Propositions 1–4 are solvable provided that H is solvable. Thus their extensions by B should be solvable as well. We define these extensions as follows. An automorphism $g \in B$ conjugates the elements $h \in H$ as $h^g = g(h)$ and leaves unchanged all the elements of active groups $\mathbb{Z}$ or $\mathbb{Z}_l$ of wreath products $H \operatorname{Wr} \mathbb{Z}$, $H \operatorname{Wr} \mathbb{Z}_l$, or $W_{(H,s)}$. The nV-embedding of H into the appropriate extension G is given by the following rule: $\nu : h \mapsto g_n^{-1}\mu(h)$, where g_h is the inner automorphism of H corresponding to h. Clearly $H \cong \nu(H)$ is normal in G and $\nu(H) \subseteq V(G)$.

On the other hand, if there is a solvable group G and an nV-embedding of H into G, we have, as in the proof of Lemma 2, an embedding of $G/C(H)$ into Aut (H) such that $V(G/C(H)) \supseteq$ Inn (H). So we take $B = G/C(H)$. □

If V_1 and V_2 are two word sets, V_2 is the consequence of V_1, and if some group H is nV_2-embeddable, it is nV_1-embeddable as well because for each group G $V_2(G) \subseteq V_1(G)$ holds. Therefore the following holds:

Corollary 1. *If V is a word set that has at least one consequence of one of the forms $[x_1, x_2]$, x^l, $\delta_n(x_1, \dots, x_{2^n})$, or $\gamma_c(x_1, \dots, x_c)$, then the solvable group H is nV-embeddable into some solvable group G if the group* Aut (H) *contains a solvable subgroup B such that the verbal subgroup of B corresponding to the appropriate word contains* Inn (H).

We could also obtain *bounds* for the solvable length $l(G)$ of the group G for the situations considered, since $l(G) \leq l(X) + l(B)$, where X is one of the groups $H \operatorname{Wr} \mathbb{Z}$, $H \operatorname{Wr} \mathbb{Z}_l$, or $W_{(H,s)}$; and because $l(X) \leq l(H) + 1$ (for wreath products $H \operatorname{Wr} \mathbb{Z}$ or $H \operatorname{Wr} \mathbb{Z}_l$), or $l(X) \leq l(H) + [\log_2 n] + 1$ for the wreath product $W_{(H,s)}$.

We have much less information on nV-embeddability of *nilpotent* groups into *nilpotent* groups. This case is more difficult since the construction of nilpotent groups from cyclic components is more dependent (than that of solvable groups) on the *way* we build the appropriate extensions. The following result gives an illustration.

Theorem 4. *Let V consist of the word $[x_1, x_2]$. Then the nilpotent group H is nV-embeddable into some nilpotent group G if and only if there exists a nilpotent subgroup B of* Aut (H) *such that*

(1) $V(B) \supseteq$ Inn (H),

(2) *the extension of H by B (as by an operator group) is nilpotent.*

Proof. Assume that the conditions of the theorem hold. Consider the extension of the wreath product $H \operatorname{Wr} \mathbb{Z}$ by B, where the elements of B act trivially over $\mathbb{Z}$ and where the action of these elements over the base group $H^{\mathbb{Z}}$ is deduced from the function of B over H. Let φ_h be the element mentioned in the proof of Proposition 1 and let $\theta_h = \prod_{z\in\mathbb{Z}} z^{-1}hz$. We saw that $\theta_h = [1_{\mathbb{Z}}, \varphi_h] \in (H \operatorname{Wr} \mathbb{Z})'$. Therefore the embedding

$$\nu : H \to \langle 1_{\mathbb{Z}}, \varphi_h, B |\ h \in H \rangle = G,$$

given by the rule $\nu : h \mapsto \theta_h$ is a monomorphism of H onto $\nu(H) = \mathrm{Diag}\,(H \,\mathrm{Wr}\, \mathbb{Z})$. $\nu(H)$ is normal in G and lies in the derived group of the latter. So it would be enough to show that G is nilpotent. Let

$$U = \langle \theta_h, \varphi_h, B \mid h \in H \rangle.$$

It is easy to compute that $U \triangleleft G$. The derived group U' is characteristic in U and normal in G.

U' is nilpotent and, according to the criterion of Hall, it would be enough to show that the factor group G/U' is nilpotent (namely, of class 2). In fact, $G = \langle 1_{\mathbb{Z}}, U \rangle$. Elements of U are commutative modulo U' and $\mathbb{Z}$ is a commutative group; thus (because of *commutator identities*) it is enough to show that

$$[1_{\mathbb{Z}}, [1_{\mathbb{Z}}, bw]] \in U'$$

for each $b \in B$ and each $w \in \{\theta_h, \varphi_h \mid h \in H\}$. Indeed,

$$[1_{\mathbb{Z}}, bw] = 1_{\mathbb{Z}}^{-1} w^{-1} b^{-1} 1_{\mathbb{Z}} bw = 1_{\mathbb{Z}}^{-1} w^{-1} 1_{\mathbb{Z}}^{b} w = [1_{\mathbb{Z}}, w].$$

The latter is already equal to 1 if $w = \theta_h$. If $w = \varphi_h$, we have $[1_{\mathbb{Z}}, w] = \theta_h$. So in any case $[1_{\mathbb{Z}}, [1_{\mathbb{Z}}, bw]] = 1$.

Now assume that for the group H there is a nilpotent G and an appropriate embedding. The existence of the mentioned nilpotent subgroup B can be shown as was done for the case of solvable groups in the proof of the previous theorem.

Let, therefore, $B = G/C(H)$ and show that the split extension K of H by the operator group B is nilpotent, i.e., show that all the values of the commutator word $[x_1, \dots, x_s]$ over K will be equal to 1 for some sufficiently large s. Since $K = \langle H, B \rangle$, it is enough to consider the situation where each x_i, $i = 1, \dots, s$, belongs to one of the subgroups B or H. Let τ_g be the automorphism of H corresponding to the element $g \cdot C(H) \in G/C(H)$. Clearly $h^{\tau} = g^{-1}hg$. Thus

$$\begin{aligned}[h, \tau_g] &= \tau_g^{-1}(h^{-1})^{\tau_g^{-1}} h\, \tau_g = \tau_g^{-1}\tau_g((h^{-1})^{\tau_g^{-1}} h)^{\tau_g} \\ &= (gh^{-1}g^{-1}h)^{\tau_g} = g^{-1}gh^{-1}g^{-1}h\, g = [h, g] \in G'.\end{aligned}$$

Continuing this process, we will obtain elements from $\gamma_3(G)$, $\gamma_4(G)$, etc. And G is a nilpotent group. The only way to avoid this situation is to take all the x_i's from B. But the group B is nilpotent as well. □

In analogy with Corollary 1, we get the following corollary.

Corollary 2. *If V is a word set that has a consequence of the form $[x_1, x_2]$, then the solvable group H is nV-embeddable into some solvable group G if the group* $\mathrm{Aut}\,(H)$ *contains a solvable subgroup B such that* $B' \supseteq \mathrm{Inn}\,(H)$.

It is not our aim, but some other properties can also be deduced from the constructions built. For example, if the group H is noetherian, the groups G obtained for words $[x_1, x_2]$ and $\gamma_c(x_1, \dots, x_c)$ can be noetherian, too.

6. nV-Embeddability of Symmetric Groups

Considering the normal embeddings of groups we also have to give some explicit examples of nV-embeddability or *not* nV-embeddability of groups. As we mentioned in the introduction, the found criterion (Theorem 1) can be successfully used in order to concern nV-embeddability of wide classes of groups. In this paper, however, we consider only the nV-embeddability of symmetric groups.

Theorem 5. *Let V be an arbitrary nontrivial word set. Then*

(1) *the symmetric groups S_2 and S_1 are nV-embeddable;*
(2) *the groups S_i for $i \in \mathbb{N}$, $i \geq 3$, and $i \neq 6$ are nV-embeddable if and only if $S_i = V(S_i)$;*
(3) *the group S_6 is nV-embeddable if and only if*

$$\mathrm{Inn}\,(S_6) \subseteq V\langle \mathrm{Inn}\,(S_6)\,, \omega\rangle,$$

where ω is an arbitrary outer automorphism of S_6;
(4) *for each infinite set M the group S_M is nV-embeddable.*

Proof. (1) The groups S_2 and S_1 are cyclic and they are nV-embeddable by Theorem 2.

(2) The groups S_i for integers $i \geq 3$ and $i \neq 6$ are all complete. So $C(S_i) = 1$, $\mathrm{Inn}\,(S_i) \cong S_i \cong \mathrm{Aut}\,(S_i)$.

(3) The group S_6 is not complete. Its group of automorphisms contains S_6 as a normal subgroup of index 2. An arbitrary outer automorphism ω of S_6 is a representer of the nontrivial element of the factor group

$$\mathrm{Out}(S_6) \cong \mathrm{Aut}\,(S_6)\,/S_6 \cong \mathbb{Z}_2,$$

and since S_6 has a trivial center, the condition $\mathrm{Inn}\,(S_6) \subseteq V\langle\,\mathrm{Inn}\,(S_6)\,,\omega\rangle$ is equivalent to our criterion of nV-embeddability. For details and the explicit form of ω, see [8, 11].

(4) For an infinite M the group S_M is always nV-embeddable because of Lemma 1. □

In particular, we can use the fact that $S_i' = A_i$ and $A_i < S_i$ for $i \geq 3$ to obtain the following corollary.

Corollary 3. *No finite symmetric groups S_i can be normally embedded for $i \geq 3$ into a group G such that the image of S_i is contained in the commutator subgroup $[G, G]$.*

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