SCATTERING OF AN ELECTROMAGNETIC WAVE IN A ONE-DIMENSIONAL MEDIUM WITH AN ARBITRARY INDEX OF REFRACTION

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UDC: 524.1-65

A new method of determining the amplitudes of scattering of a plane electromagnetic wave incident at an angle a onto an arbitrary, isotropic, one-dimensionally inhomogeneous medium of finite thickness is developed. It is shown that this problem reduces to a Cauchy problem for a system of two first-order, linear differential equations.

1. Introduction

It is well known that one of the urgent problems of astrophysics is the determination of the amplitudes of transmission and reflection of an electromagnetic wave by an inhomogeneous layer of a medium of finite thickness. In [1] V. A. Ambartsumian's method of "addition of a layer to a medium" [2] was used and differential equations were obtained describing the scattering of an electromagnetic wave in a one-dimensional, isotropic, inhomogeneous medium. However, the problem was solved for the case in which the wave's electric vector is perpendicular to the plane of incidence, i.e., for a so-called s wave, and when the wave is incident perpendicularly onto the boundary of the medium.

The purpose of the present work is to determine the amplitudes of transmission and reflection of an electromagnetic wave of arbitrary polarization and incident at an arbitrary angle a onto a one-dimensionally isotropic, inhomogeneous layer with an index of refraction $n(x)=\sqrt{\varepsilon(x)}$. All interference effects are taken into account exactly in the equations obtained.

Let us consider light reflection from and transmission through a finite layer of a one-dimensionally isotropic, layered medium with an arbitrary dependence of the dielectric constant, $\varepsilon = \varepsilon(x)$ and a magnetic permeability $\mu = 1$, bounded on both sides by a homogeneous medium with an index of refraction n_0. The plane of incidence of the wave coincides with the (x, y) plane, while the wave is incident at an angle a to the normal to the boundary of the layer, which coincides with the (y, z) plane. In accordance with Maxwell's equations, the electric field $\vec{E}$ and the magnetic field $\vec{H}$ satisfy the wave equations [3]

$$\vec{\nabla}^2 \vec{E}-\frac{\varepsilon}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = -\operatorname{grad}\left(\vec{E}\operatorname{grad}(\ln\varepsilon)\right), \tag{1}$$

$$\vec{\nabla}^2 \vec{H}-\frac{\varepsilon}{c^2}\frac{\partial^2 \vec{H}}{\partial t^2} = \operatorname{rot}\vec{H}\times\operatorname{grad}(\ln\varepsilon), \tag{2}$$

It follows from these equations that s and p waves satisfy different wave equations. In fact, whereas for an s wave (the vector $\vec{E}$ is directed along the z axis) the electric field E_z satisfies the usual uniform wave equation (1) without the right side, for a p wave, with the magnetic field directed along the z axis, the magnetic field H_z satisfies the wave equation (2) with a nonzero right side. The scatterings of the two polarizations will therefore differ fundamentally from each other. The case of normal incidence ($\alpha = 0$) was considered in [1]. As is well known, for an arbitrary wave with both polarizations present, the problem reduces to the separate consideration of scattering for the s and p polarizations. We note that in the case of

Yerevan State University, Armenia; State Engineering University, Armenia. Translated from Astrofizika, Vol. 43, No. 2, pp. 269-276, April-June, 2000. Original article submitted January 7, 2000.

 0571-7256/00/4302-0198$25.00

p polarization, one can eliminate the magnetic field and express the solutions in terms of the electric field $\vec{E}$ but the equations for joining the electromagnetic fields at the boundaries of the layer will be different for the two polarizations.

We resolve the components of the amplitudes of the electric field of the incident, reflected, and transmitted waves into projections parallel (p polarization) and perpendicular (s polarization) to the plane of incidence,

$$\vec{E}_{i,r,t} = E^{s}_{i,r,t}\vec{n}_s + E^{p}_{i,r,t}\vec{n}_p, \tag{3}$$

where the indices i, r, and t denote the incident, reflected, and transmitted waves, respectively, while $\vec{n}_s$ and $\vec{n}_p$ are the unit vectors of the s and p polarizations. We introduce the dimensionless amplitudes of transmission $T^{s,p}$ and reflection $R^{s,p}$ as follows:

$$T^{s,p} = \frac{E^{s,p}_t}{E^{s,p}_i} \quad \text{and} \quad R^{s,p} = \frac{E^{s,p}_r}{E^{s,p}_i}. \tag{4}$$

$T^{s,p}$ and $R^{s,p}$ will depend on the thickness d of the inhomogeneous layer and the law of variation of the index of refraction, $n = n(x)$, as well as on the angle of incidence α and the index of refraction n_0.

2. Recursive Equations for $T_N^{s,p}$ and $R_N^{s,p}$

The method being proposed for solving the problem of the scattering of an electromagnetic wave with an arbitrary polarization consists in the following. We divide the layer of medium under consideration into a large number of thin layers with thicknesses d_1, d_2, ..., d_N. If their maximum thickness is small enough, we can assume that e is constant in each layer. Then, in accordance with [4-6], the problem of determining R and T comes down to the problem of calculating the product of second-order matrices

$$\begin{pmatrix} 1/T_N^{*} & -R_N^{*}/T_N^{*} \\ -R_N/T_N & 1/T_N \end{pmatrix} = \prod_{n=N}^{1} \begin{pmatrix} 1/t_n^{*} & -r_n^{*}/t_n^{*} \\ -r_n/t_n & 1/t_n \end{pmatrix}, \tag{5}$$

where T_N and R_N are scattering parameters of the system, N is the number of homogeneous layers, and r and t are the scattering amplitudes of the nth homogeneous layer.

Solving the problem of the scattering of an electromagnetic wave from a homogeneous layer of thickness d and an index of refraction n, we obtain the following well-known expressions for t_s, r_s, t_p, and r_p:

$$1\; t_s = \exp(ik_{0x}d)\left[\cos(k_x d) - \mathrm{i}\frac{\left(k_{0x}^2 + k_x^2\right)}{2k_{0x}k_x}\sin(k_x d)\right], \tag{6}$$

$$r_s/t_s = -i\exp(2ik_{0x}x)\left[\frac{\left(k_{0x}^2 - k_x^2\right)}{2k_{0x}k_x}\sin(k_x d)\right], \tag{7}$$

$$1\; t_p = \exp(ik_{0x}d)\left[\cos(k_x d) - \mathrm{i}\frac{\left(\frac{n}{n_0}\right)^2 k_{0x}^2 + \left(\frac{n_0}{n}\right)^2 k_x^2}{2k_{0x}k_x}\sin(k_x d)\right], \tag{8}$$

$$r_p\; t_p = -i\exp(2ik_{0x}x)\left[\frac{\left(\frac{n}{n_0}\right)^2 k_{0x}^2 - \left(\frac{n_0}{n}\right)^2 k_x^2}{2k_{0x}k_x}\sin(k_x d)\right], \tag{9}$$

where $k_{0x} = (\omega/c)n_0\cos\alpha$, $k_x = (\omega/c)n\cos\beta$, x is the coordinate of the middle of the chosen layer, and α and α are the angles of incidence and refraction, respectively. It is assumed that each layer with an index of refraction n is bordered on both sides by a medium with an index of refraction n_0.

If we introduce the notation

$$\begin{pmatrix} 1/T^*_{N-1} & -R^*_{N-1}/T^*_{N-1} \\ -R_{N-1}/T_{N-1} & 1/T_{N-1} \end{pmatrix} = \prod_{n=N-1}^{1} \begin{pmatrix} 1/t^*_n & -r^*_n/t^*_n \\ -r_n/t_n & 1/t_n \end{pmatrix} \tag{10}$$

and redefine $1/T_N$ in terms of D_N and R_N^*/T_N^* in terms of $\overline{D}_N$, then Eq. (5) takes the form

$$\begin{pmatrix} D^*_N & -\overline{D}_N \\ -D^*_N & D_N \end{pmatrix} = \begin{pmatrix} 1/t^*_N & -r^*_N/t^*_N \\ -r_N/t_N & 1/t_N \end{pmatrix} \begin{pmatrix} D^*_{N-1} & -D_{N-1} \\ -\overline{D}^*_{N-1} & D_{N-1} \end{pmatrix} \tag{11}$$

which is equivalent to the following system of difference equations written for the s and p polarizations:

$$D_N^{s,p} = \frac{r_N^{s,p}}{t_N^{s,p}} D_{N-1}^{s,p} + \frac{1}{t_N^{s,p}} D_{N-1}^{s,p}, \tag{12}$$

$$\overline{D}_N^{s,p} = \frac{1}{t_N^{s,p\,*}} \overline{D}_{N-1}^{s,p} + \frac{r_N^{s,p\,*}}{t_N^{s,p\,*}} D_{N-1}^{s,p}. \tag{13}$$

Note that Eqs. (12) and (13) with the initial conditions $D_0^{s,p} = 1$ and $D_0^{s,p} = 0$ can be used to solve the problem of light scattering in a layered medium consisting of homogeneous layers.

3. Differential Equations for *T*(*x*) and *R*(*x*)

We introduce the functions $D(x_1) = 1/T(x_1)$ and $\overline{D}(x_1) = R^*(x_1)/T^*(x_1)$, where $T(x_1)$ and $R(x_1)$ are the amplitudes of wave transmission through and reflection from the part of the medium described by a dielectric constant $\varepsilon(x)$, given between the points $0 < x < x_1$ by

$$\varepsilon(x_1, x) = \varepsilon(x)\theta(x)\theta(x_1 - x), \tag{14}$$

where $\theta(x)$ is the Heaviside function. Then the function $\varepsilon(x, x_1 + \Delta x_1)$, i.e., the part of $\varepsilon(x)$ included between the points $x = 0$ and $x = x_1 + \Delta x_1$, where Δx_1 is a small quantity, will look like the function (14) with the addition of a homogeneous layer with the parameters $\varepsilon(x_1)$ and Δx_1 to its right side.

For the amplitudes of transmission and reflection of an infinitely narrow layer, from (6)-(9) we have

$$1/t_s = 1 - i\left[\frac{\omega}{c}\frac{n^2 - n_0^2}{2n_0\cos\alpha}\Delta x_1\right], \tag{15}$$

$$r_s/t_s = -i\left[\frac{\omega}{c}\frac{n^2 - n_0^2}{2n_0\cos\alpha}\Delta x_1 \exp(2ik_{0x}x)\right], \tag{16}$$

$$1/t_p = 1 - i\left[\frac{\omega}{c}\frac{(n^2 - n_0^2)\left(\cos^2\alpha + \frac{n_0^2}{n^2}\sin^2\alpha\right)}{2n_0\cos\alpha}\Delta x_1\right], \tag{17}$$

$$r_p/t_p = -i\exp(2ik_{0x}x)\left[\frac{\omega}{c}\frac{(n^2-n_0^2)\left(\cos^2\alpha-\frac{n_0^2}{n^2}\sin^2\alpha\right)}{2n_0\cos\alpha}\Delta x_1\right]. \tag{18}$$

Substituting $D_N^{s,p}=D^{s,p}(x_1+\Delta x_1)$, $\bar{D}_N^{s,p}=\bar{D}^{s,p}(x_1+\Delta x_1)$, and $D_{N-1}^{s,p}=D^{s,p}(x_1)$, $\bar{D}_{N-1}^{s,p}=\bar{D}^{s,p}(x_1)$ into (12) and (13), expanding the resulting equations with respect to the small quantity Δx_1, and replacing x_1 by x, with allowance for (15)-(18) we obtain the following system of linear differential equations:

$$\frac{dD^s}{dx}=iV(x)\left(D^s-\exp(2ik_{0x}x)\bar{D}^s\right), \tag{19}$$

$$\frac{d\bar{D}^s}{dx}=iV(x)\left(\exp(-2ik_{0x}x)D^s-\bar{D}^s\right) \tag{20}$$

for the s polarization and

$$\frac{dD^p}{dx}=iV(x)\left[(a+b)D^p-(a-b)\left(\exp(2ik_{0x}x)\bar{D}^p\right]\right., \tag{21}$$

$$\frac{d\bar{D}^p}{dx}=-iV(x)\left[(a-b)\left(\exp(-2ik_{0x}x)D^p-(a+b)\bar{D}^p\right]\right., \tag{22}$$

for the p polarization. Here

$$V(x)=\frac{\omega}{c}\frac{n_0^2-n^2}{2n_0\cos\alpha},\quad a=\cos^2\alpha,\quad b=\frac{n_0^2}{n^2}\sin^2\alpha. \tag{23}$$

The uniqueness of solutions of the system (19)-(22) is provided by the appropriate initial conditions:

$$D^{s,p}(x)\big|_{x=0}=1,\quad \bar{D}^{s,p}(x)\big|_{x=0}=0. \tag{24}$$

Introducing the notation

$$\begin{aligned} F^{s,p}&=D^{s,p}\exp(-ik_{0x}x)-\bar{D}^{s,p}\exp(ik_{0x}x), \\ \Phi^{s,p}&=i\left(D^{s,p}\exp(-ik_{0x}x)+\bar{D}^{s,p}\exp(ik_{0x}x)\right), \end{aligned} \tag{25}$$

we can write the system (19)-(22) in compact form:

$$\frac{dF^{s,p}}{dx}=\left(2A^{s,p}V-k_{0x}\right)\Phi^{s,p}, \tag{26}$$

$$\frac{d\Phi^{s,p}}{dx}=-\left(2B^{s,p}V-k_{0x}\right)F^{s,p}, \tag{27}$$

$$\begin{aligned} A^s&=0, & A^p&=a, \\ B^s&=1, & B^p&=b. \end{aligned} \tag{28}$$

Let us turn to the solution of the system (26), (27). For convenience, we omit the indices s and p below. We seek the solution of these equations in the form

$$\begin{aligned} F&=H_1+iH_2, \\ \Phi&=N_1+iN_2. \end{aligned} \tag{29}$$

Since the coefficients of Eqs. (26) and (27) are real in the absence of absorption, the pairs of real functions H_1, N_1 and H_2, N_2 will satisfy the same system of equations,

$$\begin{aligned} \frac{dH}{dx}&=(2AV-k_{0x})N, \\ \frac{dN}{dx}&=-(2BV-k_{0x})H, \end{aligned} \tag{30}$$

although the initial conditions for these pairs will be different. In fact, as follows from (24), the functions H and N at the point $x = 0$ have the values

$$H_1(0)=1 \quad \text{and} \quad N_1(0)=0 \tag{31}$$

for the first pair and

$$H_2(0)=0 \quad \text{and} \quad N_2(0)=1 \tag{32}$$

for the second pair.

The problem of determining the amplitudes R and T of light scattering for oblique incidence onto a layer with an arbitrary index of refraction $n = n(x)$ and a thickness d is thus reduced to the integration of the system (30) with the initial conditions (31) and (32).

From (29) and (25) we can obtain equations for R and T that express them in terms of the values of the functions $N_{1,2}$ and $H_{1,2}$ at the point $x = d$. We thus have

$$\frac{1}{T}=\frac{1}{2}\exp(ik_{0x}d)\left[(H_1+N_2)-i(N_1-H_2)\right], \tag{33}$$

$$\frac{R}{T}=-\frac{1}{2}\exp(ik_{0x}d)\left[(H_1-N_2)-i(N_1+H_2)\right], \tag{34}$$

or

$$T=\frac{2}{\rho_T}\exp\left[i(\varphi_T-k_{0x}d)\right], \tag{35}$$

$$R=-\frac{\rho_R}{\rho_T}\exp\left[i(\varphi_T-\varphi_R)\right], \tag{36}$$

where

$$\rho_T=\sqrt{\left[H_1(d)+N_2(d)\right]^2+\left[N_1(d)-H_2(d)\right]^2},\quad \varphi_T=\operatorname{arctg}\left[\frac{N_1(d)-H_2(d)}{N_2(d)+H_1(d)}\right], \tag{37}$$

$$\rho_R=\sqrt{\left[H_1(d)-N_2(d)\right]^2+\left[N_1(d)+H_2(d)\right]^2},\quad \varphi_R=\operatorname{arctg}\left[\frac{N_1(d)+H_2(d)}{H_1(d)-N_2(d)}\right]. \tag{38}$$

We finally show that the solutions obtained satisfy the condition of conservation of flux density of electromagnetic energy,

$$R^2+T^2=1, \tag{39}$$

for any d. In fact, substituting the solutions (35) and (36) into (39), with allowance for (37) and (38) we can write it in the form

$$H_1(d)N_2(d)-H_2(d)N_1(d)=1. \tag{40}$$

It follows from the initial conditions (31) and (32) that the condition (40) is satisfied at the point $x = 0$. On the other hand, in accordance with Eq. (30), the derivative of the expression $H_1(x)N_2 - H_2(x)N_1(x)$ vanishes. The condition (40) is therefore satisfied at the point $x = d$.

4. Conclusion

As an example of the application of the proposed method, let us consider the case of determining the amplitudes of scattering of an electromagnetic wave in an isotropic homogeneous layer of finite thickness d.

If we assume that we have the equation

$$(2AV-k_{0x})(2BV-k_{0x})=k_x^2, \tag{41}$$

for both polarizations of the electromagnetic wave, and that the coefficients of the system (30) are constants, then for the case under consideration we can, instead of Eqs. (30), obtain the same equations for N and H,

$$\left(\frac{d^2}{dx^2}+k_x^2\right)\begin{pmatrix}H\\N\end{pmatrix}=0, \tag{42}$$

which coincide with the wave equation for the homogeneous medium under consideration. Solutions of Eqs. (42) satisfying the initial conditions (31) and (32) have the form

$$\begin{aligned} &H_1(x)=\cos(k_x x), \quad H_2(x)=\frac{2AV-k_{0x}}{k_x}\sin(k_x x),\\ &N_1(x)=-\frac{2BV-k_{0x}}{k_x}\sin(k_x x), \quad N_2(x)=\cos(k_x x). \end{aligned} \tag{43}$$

Substituting the values of the functions H and N at the point $x=d$ into Eqs. (33) and (34), we obtain the amplitudes of transmission and reflection for the s and p polarizations, which coincide with Eqs. (6)-(9).

To end with, we note a fundamental advantage of our proposed method over the well-known methods. In our case the problem of the scattering of an electromagnetic wave in a one-dimensional, inhomogeneous medium is reduced to the solution of a system of two first-order, linear differential equations with given initial conditions. It is far more productive to use these equations when they have analytical solutions or when one must perform numerical integrations than to use special programs that are required to apply the matrix transfer method or the dynamical theory of scattering of electromagnetic waves. We note that such simplification is achieved because the boundary conditions of the scattering problem in our method are contained in the proposed system of equations, written directly for scattering amplitudes, and the problem of finding them is reduced to a Cauchy problem for those equations with given initial conditions.

The authors wish to thank D. Badalyan and O. Eritsian for a discussion of the results obtained.

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