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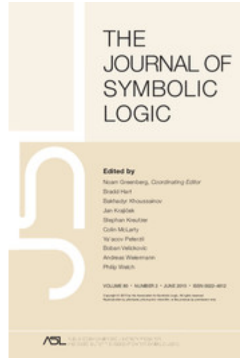

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The Journal of Symbolic Logic / Volume 67 / Issue 03 / September 2002, pp 1126 - 1140
DOI: 10.2178/jsl/1190150153, Published online: 12 March 2014

Link to this article: http://journals.cambridge.org/abstract_S0022481200009464

How to cite this article:
A. Marcja, M. Prest and C. Toffalori (2002). The torsionfree part of the Ziegler spectrum of *RG* when *R* is a Dedekind domain and *G* is a finite group . The Journal of Symbolic Logic, 67, pp 1126-1140 doi:10.2178/jsl/1190150153

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THE JOURNAL OF SYMBOLIC LOGIC
Volume 67, Number 3, Sept. 2002

THE TORSIONFREE PART OF THE ZIEGLER SPECTRUM OF RG WHEN R IS A DEDEKIND DOMAIN AND G IS A FINITE GROUP

A. MARCJA, M. PREST, AND C. TOFFALORI

§1. Introduction. For every ring S with identity, the (right) Ziegler spectrum of S, Zg_S, is the set of (isomorphism classes of) indecomposable pure injective (right) S-modules. The Ziegler topology equips Zg_S with the structure of a topological space. A typical basic open set in this topology is of the form

$$(\varphi/\psi) = \{M \in Zg_S : |\varphi(M) : \varphi(M) \cap \psi(M)| > 1\}$$

where φ and ψ are pp-formulas (with at most one free variable) in the first order language L_S for S-modules; let $[\varphi/\psi]$ denote the closed set $Zg_S - (\varphi/\psi)$. There is an alternative way to introduce the Ziegler topology on Zg_S. For every choice of two f.p. (finitely presented) S-modules A, B and an S-module homomorphism $f : A \to B$, consider the set (f) of the points N in Zg_S such that some S-homomorphism $h : A \to N$ does not factor through f. Take (f) as a basic open set. The resulting topology on Zg_S is, again, the Ziegler topology.

The algebraic and model-theoretic relevance of the Ziegler topology is discussed in [Z], [P] and in many subsequent papers, including [P1], [P2] and [P3], for instance. Here we are interested in the Ziegler spectrum Zg_{RG} of a group ring RG, where R is a Dedekind domain of characteristic 0 (for example R could be the ring $\mathbf{Z}$ of integers) and G is a finite group. In particular we deal with the R-torsionfree points of Zg_{RG}.

The main motivation for this is the study of RG-lattices (i.e., finitely generated R-torsionfree RG-modules). Their model theory has been treated in several papers (see [T], for instance). Here we try to understand their role within the spectrum.

The analysis of the R-torsionfree part of Zg_{RG} is developed in § 2. RG-lattices are directly dealt with in § 3.

In § 2. we show that every R-torsionfree point of the Ziegler spectrum of RG either is a simple KG-module, where K denotes the quotient field of R, or is R-reduced and is then a point of the Ziegler spectrum of $\hat{R}_P G$ for some maximal prime ideal P, where $\hat{R}_P$ denotes the completion of R at P. Fix such a prime P. We show that the topology on the R-torsionfree R-reduced points which are $\hat{R}_P$-modules is the same whether these are considered as points of the spectrum of RG or of $\hat{R}_P G$. We also show that every such point is in the topological closure of the set of such points which are $\hat{R}_P G$-lattices. Then we investigate how these "P-patches" fit into the Ziegler spectrum of RG.

Received September 19, 2000; revised January 8, 2002.

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0022-4812/02/6703-0017/$2.50

In § 3. we show, under the assumption that each factor R/P is finite, that two RG-lattices are elementarily equivalent if and only if they have isomorphic pure-injective hulls; notice that this is the case precisely when they have the same genus [BT].

We refer to [CR] for representation theory, and to [P] for model theory of modules. The notation of these references is adopted. In particular, for every ring S, L_S is the first order language for (right) S-modules; among S-modules, $\leq$ denotes the submodule relation, while $\prec$ means *elementarily embeddable*; f.g. abbreviates finitely generated, and f.p. finitely presented; for a given module M over S, $\hat{M}$ is the pure injective hull of M. We are interested in Dedekind domains of characteristic 0. When R is such a domain, R_P denotes its localization at a given maximal ideal P, $\hat{R}_P$ its P-adic completion; recall that the pure injective hull of R_P as a module over R just equals the P-adic completion, so there is no ambiguity in this notation. When M is a module over the Dedekind domain R, M_P is the localization of M at the maximal ideal P.

§2. **The torsionfree points in the spectrum.** Let R be a Dedekind domain (of characteristic 0) and let G be a finite group. Let K denote the quotient field of R. We assume that the characteristic of K is 0. We study here the Ziegler topology for the R-torsionfree points in Zg_{RG}. First we show that, with only finitely many possible exceptions, any R-torsionfree point of Zg_{RG} lies in $Zg_{\hat{R}_P G}$ for some non-zero prime P of R.

THEOREM 2.1. *Let N be an R-torsionfree point in Zg_{RG}. Then either N is a simple KG-module or there exists some maximal ideal P of R such that $N \in Zg_{\hat{R}_P G}$ and N is R_P-reduced.*

PROOF. Let $N \in Zg_{RG}$ be torsionfree over R.

CLAIM 1. *N is a module over $R_P G$ for some maximal ideal P of R.*

PROOF OF CLAIM 1. $S = End(N)$ is a local ring. Let I denote its unique maximal ideal and $C(S)$ be its center,

$$C(S) = \{a \in S : \forall s \in S, as = sa\}.$$

For every subring C of S, if C contains the identity element 1_S of S and C is closed under $^{-1}$ (in the sense that, when $a \in C$ is a unit in S, a^{-1} is in C, too), then C is a local ring and $I \cap C$ is its maximal ideal. Notice that this applies to $C = C(S)$. As N is R-torsionfree, R embeds into $C(S)$ just by mapping each $r \in R$ into the scalar multiplication by r. Let C be the set of all elements in S of the form ab^{-1} with a, b in the copy of R inside $C(S)$ and b invertible in S. So C is a subring of S containing 1_S and closed under $^{-1}$, and consequently C is local. Moreover C can be embedded as a subring in the quotient field K. Accordingly C is (isomorphic to) either K or R_P for some maximal ideal P of R. Define the action of $R_P G$ on N in the obvious way: given $m \in N$ and $c_g \in C$ for every $g \in G$, put

$$m \sum_{g \in G} c_g g = \sum_{g \in G} c_g(m) g.$$

This equips N with a KG-module or an $R_P G$-module structure extending the original RG-action. Observe that every KG-module is a module also over $R_P G$ for every maximal ideal P of R in a trivial way (just forget $K - R_P$). This concludes the proof of Claim 1. $\dashv$

Notice that N, as an R_PG-module, is still indecomposable and pure injective. For N is indecomposable pure injective over RG and the R_PG-action on N is defined by the RG-action.

Now look at N over R_P. As a pure injective module, N decomposes in the following way

$$N = N' \oplus N'',$$

where (up to isomorphism) N' is the pure injective hull of $\hat{R_P}^{(\lambda)}$ for some cardinal λ, and N'' is $K^{(\mu)}$ for some μ. Clearly

$$\cap_n N'P^n = 0.$$

Furthermore, for every $g \in G$, $N''g \leq N''$. For, let $a \in N''$, then $Ka \leq N''$ and $Kag \leq N$. Decompose $ag = a' + a''$ with $a' \in N'$ and $a'' \in N''$. Then $Ka' \leq N'$, so $a' = 0$ and $ag \in N''$. Hence N'' is a submodule of N over R_PG. Moreover N'' is a module also over KG, and by Maschke's Theorem KG is a semisimple artinian ring. Hence N'' is injective over KG.

Claim 2. *N'' is injective over R_PG.*

Proof of Claim 2. This can be shown by a standard argument. The only facts to be used about N'' and G are that N'' is R-torsionfree and that G preserves R-torsion. We give here the details for completeness. Hence let $i : A \rightarrowtail B$ be an embedding of R_PG-modules, f be an R_PG-homomorphism of A into N''. What we have to find is an R_PG-homomorphism h of B into N'' satisfying $hi = f$. Let τA, τB denote the R-torsion R_PG-submodule of A, B respectively; so

$$\tau A = \{a \in A : \exists r \in R, r \neq 0, ra = 0\}$$

and similarly for B (recall that G preserves R-torsion). Consider the R-torsionfree R_PG-modules $A/\tau A$ and $B/\tau B$, with the corresponding projections of A onto $A/\tau A$ and of B onto $B/\tau B$; i defines a natural embedding of $A/\tau A$ in $B/\tau B$, call it i'. As N'' is R-torsionfree, f induces an R_PG-homomorphism f' of $A/\tau A$ into N''. Now tensor up with K. We get a new embedding $i'' = i' \otimes 1$ of $A/\tau A \otimes K$ into $B/\tau B \otimes K$, and the homomorphism $f'' = f' \otimes 1$ of $A/\tau A \otimes K$ in $N'' \otimes K$. But N'' is a KG-module, and so $N'' \otimes K$ is isomorphic to N''. The KG-injectivity of N'' provides a KG-homomorphism h'' of $B/\tau B \otimes K$ into $N'' \otimes K \simeq N''$. Hence we can find h as required: h is just the composition of h'' with the obvious map of B in $B/\tau B \otimes K$.

Consequently N'' is a direct summand of N over R_PG, too. Since N is indecomposable over R_PG, one can deduce that either

(a) N is a KG-module or

(b) $\cap_n NP^n = 0$ (in other words N is reduced over R).

If (a) holds, then N is a simple KG-module. So assume (b). We can see that N is actually an $\hat{R_P}G$-module. For, let $r \in \hat{R_P}$ and suppose $r = lim_n r_n$ where the $r_n \in R_P$ are chosen such that, for every natural n, P^n contains both $r - r_n$ and $r_n - r_{n+1}$. Pick $a \in N$; we know that ar_n is defined for every n. Now recall that P is f.g. because R is Noetherian, so the membership to NP^n, for any natural n, can be expressed by a suitable pp-formula in the language of R_PG-modules. Let $P^n|v$

denote this formula (so meaning $v \in NP^n$). Look at the partial pp-type over N (as a $R_P G$-module)

$$\Gamma(v) = \{P^n | (v - a) : n \in \mathbf{N}\}.$$

$\Gamma(v)$ is finitely satisfied in N, and N is pure injective over $R_P G$. Consequently there is some $c \in N$ realizing $\Gamma(v)$. (b) implies that c is unique. Hence put $ar = c$. This equips N with the required $\hat{R}_P G$-module structure. ⊣

In order to finish the proof of Theorem 2.1, we have to show that N is indecomposable and pure injective also over $\hat{R}_P G$. Indecomposability is clear. Pure injectivity is a consequence of the following general fact which we state as a separate lemma.

LEMMA. *Let $D \to S$ be a homomorphism of unitary rings. Let N be a module over S, hence over D, such that the S-action on N is pp-type definable over D (in the sense that, for every $s \in S$, there is a set $\Gamma_s(v, w)$ of pp-formulas over D such that, for all $a \in N$,*

$$N \models \Gamma_s(a, as) \wedge \forall w(\Gamma_s(0, w) \to w = 0)).$$

If N is pure injective over D, then N is pure injective also over S.

The proof of the lemma is deferred until after we have finished the proof of the theorem.

In our setting, D is $R_P G$, S is $\hat{R}_P G$ and the homomorphism of D in S is the obvious one. It is easy to see that the assumptions of the lemma are satisfied. Indeed, for every $s \in \hat{R}_P$, choose a sequence $(s_n)_n$ in R_P with limit s (so that P^n contains $s - s_n$ for all n), and put

$$\Gamma_s(v, w) = \{P^n | (vs_n - w) : n \in \mathbf{N}\}$$

(again, use the fact that R is Noetherian to deduce that $\Gamma_s(v, w)$ is a partial pp-type). Then Γ_s satisfies the conditions of the lemma because (b) holds. This implies that N is pure injective also over $\hat{R}_P G$.

It remains to prove the lemma.

PROOF OF LEMMA. This follows from some straightforward calculations. Take a pp-formula $\varphi(v_1, \vec{a})$ of L_S with parameters $\vec{a}$ from N,

$$\varphi(v_1, \vec{a}) \quad : \quad \exists v_2 \dots \exists v_n \bigwedge_{1 \le j \le m} \sum_{1 \le i \le n} v_i s_{i,j} = a_j$$

for a suitable m. Consider the system of linear equations

$$\Sigma_\varphi(v_1, \dots, v_n) \quad : \quad \bigwedge_{1 \le j \le m} \sum_{1 \le i \le n} v_i s_{i,j} = a_j$$

corresponding to $\varphi(v_1, \vec{a})$ in the natural way. Now look at the scalars $s_{i,j}$. For every choice of i and j, consider a partial pp-type $\Gamma_{s_{i,j}}(v, w)$ defining the action of $s_{i,j}$, its pp-formulas $\gamma_{i,j}(v, w)$ and the corresponding systems of linear equations (with parameters in N and variables v, w, $\vec{z}_{\gamma_{i,j}}$)

$$\Sigma_{\gamma_{i,j}}(v, w, \vec{z}_{\gamma_{i,j}}).$$

Thus $\Sigma_\varphi(v_1, \dots, v_n)$ determines a system of linear equations in L_D (with possibly infinitely many equations and variables, and parameters from N)

$$(\star) \qquad \left\{ \sum_{1 \le i \le n} w_{i,j} = a_j : 1 \le j \le m \right\} \cup \bigcup_{\gamma_{i,j}} \Sigma_{\gamma_{i,j}}(v_i, w_{i,j}, \vec{z}_{\gamma_{i,j}}).$$

It is clear that, if a sequence $(b_1, b_2, \dots, b_n) \in N^n$ satisfies $\Sigma_\varphi(v_1, \dots, v_n)$, then, for some suitable $\vec{d}_{\gamma_{i,j}}$ in N, $((b_i)_{1\le i\le n}, (b_i s_{i,j})_{1\le i\le n, 1\le j\le m}, (\vec{d}_{\gamma_{i,j}})_{\gamma_{i,j}})$ satisfies $(\star)$. On the other hand, if $((b_i)_{1\le i\le n}, (c_{i,j})_{1\le i\le n, 1\le j\le m}, (\vec{d}_{\gamma_{i,j}})_{\gamma_{i,j}})$ in N satisfies $(\star)$, then $c_{i,j}$ must equal $b_i s_{i,j}$ for all i and j, and consequently $(b_1, \dots, b_n)$ satisfies $\Sigma_\varphi(v_1, \dots, v_n)$.

Now take a set $\Phi(v_1)$ of pp-formulas $\varphi(v_1)$ in L_S, and assume that $\Phi(v_1)$ is finitely realized in N. Let Σ_Φ be the corresponding set of linear equations

$$\Sigma_\varphi(v_1, v_{2,\varphi}, \dots, v_{n_\varphi,\varphi})$$

when $\varphi(v_1)$ ranges over $\Phi(v_1)$ (notice that n_φ and $v_{2,\varphi}, \dots, v_{n_\varphi,\varphi}$ may change with $\varphi(v_1)$). Form a new system of linear equations Σ'_Φ in L_D, just as before in $(\star)$, where now $\varphi(v_1)$ ranges over $\Phi(v_1)$. Arguing as above we see that Σ'_Φ is finitely satisfiable in N, and so Σ'_Φ is realized in N because N is pure injective over D. Let b_1 be an element of N satisfying Σ'_Φ (as v_1), then b_1 satisfies each formula in $\Phi(v_1)$ as well, and so $\Phi(v_1)$ is realized in N. It follows that N is pure injective also over S. $\dashv$

REMARKS. Fix a maximal ideal P in R.

1. Let $N \in Zg_{\hat{R}_P G}$ be torsionfree and reduced over R. Then $N \in Zg_{RG}$.

By the lemma N is pure injective over RG. Furthermore, suppose $N = N_0 \oplus N_1$ over RG. Accordingly, for $a \in N_0$ and $s \in \hat{R}_P G$, decompose $as = a_0 + a_1$. Since N is torsionfree and reduced over R, the action of s on N is pp-type definable over $R_P G$ and even over RG. Consequently $N \models \Gamma_s(a, a_0 + a_1)$, and hence $N \models \Gamma_s(a, a_0)$ over RG. It follows that $a_1 = 0$. So actually N_0 is an $\hat{R}_P G$-module and similarly for N_1. Then either N_0 or N_1 is 0.

Thus the embedding of RG into $\hat{R}_P G$ induces an inclusion of the R-torsionfree R-reduced part of $Zg_{\hat{R}_P G}$ into the R-torsionfree part of Zg_{RG}. We will show below (2.2) that these inclusions are homeomorphic embeddings.

2. If $N \in Zg_{\hat{R}_P G}$ is R-torsionfree but non-reduced then N may or may not be a point of Zg_{RG}. Of course, N is pure-injective as an RG-module but it might not be indecomposable. For example, take G to be the trivial group, R to be the ring of integers and N to be the p-adic field for some non-zero prime p: as an RG-module N is the direct sum of uncountably many copies of the rationals.

3. Every $\hat{R}_P G$-lattice N is pure injective. In particular, when N is indecomposable (as a lattice), $N \in Zg_{\hat{R}_P G}$.

The latter claim is a consequence of the former, because every direct summand of N must still be an $\hat{R}_P G$-lattice. Now an $\hat{R}_P G$-lattice N is a direct sum of finitely many copies of $\hat{R}_P$ over $\hat{R}_P$. Accordingly fix a basis $e_0, \dots, e_h$ of N over $\hat{R}_P$, and, for every $g \in G$, consider the matrix $A(g)$ with entries in $\hat{R}_P$ such that

$$(e_0, \dots, e_h)A(g) = (e_0 g, \dots, e_h g).$$

Extend linearly A to $\hat{R}_P G$ and get, for every $s \in \hat{R}_P G$, a matrix $A(s)$ still satisfying

$$(e_0, \dots, e_h)A(s) = (e_0 s, \dots, e_h s).$$

Moreover decompose (uniquely) every element $a \in N$ as

$$e_0 a_0 + \ldots + e_h a_h$$

with $a_0, \ldots, a_h$ in $\hat{R}_P$. Notice that a satisfies a pp-formula of $L_{\hat{R}_P G}$

$$\varphi(v_1) \quad : \quad \exists v_2 \ldots \exists v_n \bigwedge_{j \leq q} \sum_{1 \leq k \leq n} v_k s_{j,k} = 0$$

if and only if the sequence $(a_0, \ldots, a_h)$ in $\hat{R}_P$ satisfies over $\hat{R}_P$

$$\varphi'(v_{1,0}, \ldots, v_{1,h}) \quad : \quad \exists v_{2,0} \ldots \exists v_{2,h} \ldots \exists v_{n,0} \ldots \exists v_{n,h}$$
$$\bigwedge_{j \leq q} \bigwedge_{t \leq h} \sum_{1 \leq k \leq n, i \leq h} v_{k,i} A(s_{j,k})_{t,i} = 0.$$

Since $\hat{R}_P$ is pure injective over $\hat{R}_P$, N is pure injective over $\hat{R}_P G$, as claimed.

4. Now assume R/P finite. In this case indecomposable $\hat{R}_P G$-lattices are topologically indistinguishable in $Zg_{\hat{R}_P G}$ if and only if they are isomorphic. Indeed $\simeq$ just equals $\equiv$ for $\hat{R}_P G$-lattices in this setting by the Maranda Theorem (see [BT]). It is also true that if N, N' are (indecomposable) $\hat{R}_P G$-lattices which are elementarily equivalent as RG-modules, then they are isomorphic over $\hat{R}_P G$ (and hence over RG). This is again a consequence of Maranda Theorem ([CR]). In fact, take a positive integer k such that P^k does not include the order of G, then $N \equiv N'$ over RG implies that the finite modules $N/P^k N$ and $N'/P^k N'$ (over RG, or $\hat{R}_P G$) are elementarily equivalent and hence isomorphic. But, just by the Maranda Theorem, this is enough to deduce $N \simeq N'$ over $\hat{R}_P G$.

Now let us study the Ziegler topology for R-torsionfree points in Zg_{RG}.

THEOREM 2.2. *The set of R-torsionfree R-reduced points in $Zg_{\hat{R}_P G}$ has the same topology whether considered as a subspace of $Zg_{\hat{R}_P G}$ or of Zg_{RG}.*

PROOF. First notice that every pp-formula $\varphi(v)$ in L_{RG} is also a pp-formula in $L_{\hat{R}_P G}$. Consequently every basic open set (φ/ψ) in Zg_{RG} (with $\varphi(v)$ and $\psi(v)$ pp-formulas in L_{RG}) also defines a basic open set in $Zg_{\hat{R}_P G}$ containing the same R-torsionfree R-reduced points.

In order to show the converse implication, we use the alternative definition of Zg_{RG} and $Zg_{\hat{R}_P G}$ recalled in the introduction. Accordingly, take two f.p. modules A and B over $\hat{R}_P G$ and an $\hat{R}_P G$-homomorphism $f : A \to B$, and look at the basic open set (f) of all $N \in Zg_{\hat{R}_P G}$ for which some $\hat{R}_P G$-homomorphism $h : A \to N$ does not factor through f. Since $\hat{R}_P G$ (and RG) are Noetherian, f.p. just means f.g.. Also, there is no loss of generality in assuming A, B to be R-torsionfree and hence $\hat{R}_P G$-torsionfree. Otherwise, just replace A, B with their quotients with respect to τA, τB respectively, and consider the homomorphism f' of $A/\tau A$ in $B/\tau B$ defined by f in the obvious way:

$$f'(a + \tau A) = f(a) + \tau B, \quad \forall a \in A.$$

Notice that $A/\tau A$, $B/\tau B$ are f.g. if A and B are. Furthermore we claim that, for an R-torsionfree $N \in Zg_{\hat{R}_P G}$, $N \in (f)$ if and only if $N \in (f')$. In fact, suppose $N \in (f)$ and let h be an $\hat{R}_P G$-homomorphism of A in N such that h does not factor through f. As N is R-torsionfree, h determines a homomorphism $h' : A/\tau A \to N$. Let $g' : B/\tau B \to N$ satisfy $g' f' = h'$. By composing h' and the projection of B

onto $B/\tau B$, one gets $g : B \to N$ for which $gf = h$. Hence h' witnesses $N \in (f')$. Conversely suppose $N \in (f')$, so there is a homomorphism $h' : A/\tau A \to N$ which does not factor through f'. Let h be the composition of h' and the natural projection of A onto $A/\tau A$. Assume that some homomorphism $g : B \to N$ satisfies $gf = h$; as N is R-torsionfree, g defines a homomorphism $g' : B/\tau B \to N$ by

$$g'(a + \tau B) = g(a) \quad \forall a \in B.$$

One easily checks $g'f' = h'$, which contradicts $N \in (f')$. Hence h witnesses $N \in (f)$.

Therefore we may assume that A, B are $\hat{R}_P G$-lattices.

From the proof of Maranda's Theorem in [CR], 30.14 p. 624, we have the following result. If A, B are lattices over $\hat{R}_P G$, then there is a positive integer n_0 such that, for all $n \geq n_0$, for every $\hat{R}_P G$-homomorphism $g' : A/P^n A \to B/P^n B$, there is some $\hat{R}_P G$-homomorphism $g : A \to B$ such that, if g_n denotes the reduction of g modulo P^n

$$g_n(a + P^n A) = g(a) + P^n B \quad \forall a \in A,$$

then g_n and g' agree modulo $P^{n-n_0}(A/P^n A)$. Observe that g_n makes sense even for arbitrary $\hat{R}_P G$-modules A, B and a homomorphism $g : A \to B$.

Now note that the same result holds if we replace B by any R-torsionfree module C. For, given $g' : A/P^n A \to C/P^n C$, the image of $A/P^n A$ in g' is a f.g. submodule of $C/P^n C$; so we can choose a finitely generated preimage B of this submodule in C. At this point one can apply the above result to obtain $g : A \to B$ and compose g with the embedding of B into N.

Now consider two $\hat{R}_P G$-lattices A, B and a $\hat{R}_P G$-homomorphism $f : A \to B$ and look at the open set (f) in $Zg_{\hat{R}_P G}$. We want to find an open set U in Zg_{RG} such that, for every R-torsionfree R-reduced point N in $Zg_{\hat{R}_P G}$ (hence in Zg_{RG}), $N \in (f)$ in $Zg_{\hat{R}_P G}$ if and only if $N \in U$ in Zg_{RG}. Equivalently we can consider the complement $[f]$ of (f) in $Zg_{\hat{R}_P G}$ - that is, the closed set of all indecomposable pure injectives N for which every morphism $g : A \to N$ factors through f - and find a closed set Y in Zg_{RG} such that, for every R-torsionfree R-reduced point N in $Zg_{\hat{R}_P G}$, $N \in [f]$ in $Zg_{\hat{R}_P G}$ if and only if $N \in Y$ in Zg_{RG}.

Consider the following condition on N: "For every morphism $g' : A/P^n A \to N/P^n N$ there is $h' : B/P^{n-n_0} B \to N/P^{n-n_0} N$ such that $h' f_{n-n_0} = g'_{n-n_0}$", noting that modules such as $A/P^n A$ are finitely presented RG-modules. Then one may check that this condition defines a closed subset, Y_n, of Zg_{RG}. Indeed, the condition may be expressed by the following sentence:

$$\forall \bar{x}(\phi(\bar{x}) \to \exists \bar{y}(\exists \bar{z}\psi'(\bar{y}+\bar{z}) \wedge P^{n-n_0}|\bar{z} \wedge \exists \bar{u}, \bar{v}\rho'(\bar{x}+\bar{u}, \bar{y}+\bar{v}) \wedge P^{n-n_0}|\bar{u} \wedge P^{n-n_0}|\bar{v})).$$

Here ϕ is equivalent to the pp-type in $A/P^n A$ of some fixed generating tuple, $\bar{a}$, for $A/P^n A$, ψ is equivalent to the pp-type in $B/P^{n-n_0} B$ of some fixed generating tuple, $\bar{b}$, for $B/P^{n-n_0} B$ and ρ is equivalent to the pp-type in $B/P^{n-n_0} B$ of the tuple, $(\pi f_{n-n_0}\bar{a}, \bar{b})$ where $\pi : A/P^n A \longrightarrow A/P^{n-n_0} A$ is the canonical projection. Furthermore if $\psi(\bar{y})$ has the form $\exists \bar{w}\theta(\bar{y}, \bar{w})$ with θ quantifier-free then we define $\psi'(\bar{y})$ to be the formula $\exists \bar{w}, \bar{w}'(\theta(\bar{y}, \bar{w} + \bar{w}') \wedge P^{n-n_0}|\bar{w}')$ and in a similar way we define ρ' from ρ. Thus the condition has the form $\forall \bar{x}(\phi(\bar{x}) \to \phi_1(\bar{x}))$ and hence corresponds to a closed subset $[\phi/\phi_1]$ of the Ziegler spectrum.

At this point it suffices to show that, for every R-torsionfree R-reduced point N in $Zg_{\hat{R}_PG}$, $N \in [f]$ in $Zg_{\hat{R}_PG}$ if and only if $N \in Y_n$ for all n in Zg_{RG}. This will be enough because $Y = \bigcap_{n \geq n_0} Y_n$ is an intersection of sets which are closed in the RG-topology and hence is itself closed in the RG-topology. We proceed to prove our claim.

Suppose first that $N \in [f]$. Let $n \geq n_0$, $g' : A/P^nA \to N/P^nN$. By (the corollary to) Maranda's Theorem there is $g : A \to N$ such that g_n and g' agree modulo $P^{n-n_0}(A/P^nA)$. Since $N \in [f]$, there is $h : B \to N$ such that $hf = g$ and hence such that $h_{n-n_0}f_{n-n_0} = g_{n-n_0} = g'_{n-n_0}$, as required. Therefore, $N \in Y$.

For the converse, suppose that $N \in Y$ and let $g : A \to N$. We must show that g factorizes through f. Choose a finite generating (over $\hat{R}_PG$) sequence $\bar{a}$ for A and a finite generating sequence $\bar{b}$ for B. Let $p(\bar{v}, \bar{w})$ be the pp-type of the tuple $(f(\bar{a}), \bar{b})$ in B. Since N is pure injective, if we can find a realization $\bar{c}$ in N of the pp-type $p(g(\bar{a}), \bar{w})$, then $(f(\bar{a}), \bar{b}) \mapsto (g(\bar{a}), \bar{c})$ will define a morphism h from B to N satisfying $hf = g$. Again using the pure injectivity of N, we can limit ourselves to prove that, for every formula $\varphi(\bar{v}, \bar{w})$ in p, there is $\bar{c}$ in N satisfying $\varphi(g(\bar{a}), \bar{w})$. $\varphi(\bar{v}, \bar{w})$ states that there is a solution $\bar{z}$ for a linear system

$$(\bar{v}, \bar{w}, \bar{z}) \cdot M = 0$$

where M is a matrix with entries in $\hat{R}_PG$. For some $\bar{d} \in B$,

$$(f(\bar{a}), \bar{b}, \bar{d}) \cdot M = 0.$$

Take g' in the condition defining Y_n to be the reduction g_n of g modulo P^n; we obtain a morphism h' (as in that condition) yielding

$$(g(\bar{a}) + P^{n-n_0}N, h'(\bar{b}), h'(\bar{d})) \cdot M = 0.$$

Hence, if $\bar{b'}$ and $\bar{d'}$ denote preimages of $h'(\bar{b})$, $h'(\bar{d})$ in N, then

$$(g(\bar{a}), \bar{b'}, \bar{d'}) \cdot M = 0 \mod P^{n-n_0}N.$$

It follows

$$(g(\bar{a}), \bar{b'}, \bar{d'}) \cdot M = 0 \mod P^kN$$

for every positive integer k. So, since N is R-reduced,

$$(g(\bar{a}), \bar{b'}, \bar{d'}) \cdot M = 0,$$

and we are done. $\dashv$

In the previous proof we showed that if N is an indecomposable R-torsionfree R-reduced pure-injective over $\hat{R}_PG$ and if $f : A \longrightarrow B$ is a morphism between $\hat{R}_PG$-lattices then any given morphism from A to N may be factorised through f provided its reduction modulo k factorises through the reduction of f modulo k for some k (in fact we required a slightly weaker factorisation condition). Of course the value of k will depend on f but we can deduce immediately that there is a uniform bound on k. For in the proof we represent the basic open set (f) in $Zg_{\hat{R}_PG}$ as the union of infinitely many open sets, and hence, because basic open sets are compact, it is a union of just finitely many.

To summarize, the R-torsionfree part of Zg_{RG} contains

- ⋆ for every maximal ideal P of R, a homeomorphic copy of the R-torsionfree R-reduced part of $Zg_{\hat{R}_PG}$;
- ⋆ the simple KG-modules.

Our aim now is to examine the topology on the P-patch, with particular emphasis on the role of the $\hat{R}_P G$-lattices among the (indecomposable) R-torsionfree R-reduced points in $Zg_{\hat{R}_P G}$. We wish also to study the topological relationship between the KG-modules and the P-patch, for a given P. Note that the topology on the set of simple KG-modules is discrete (since KG is of finite representation type or, directly, use the central idempotents of KG to write down isolating neighbourhoods).

THEOREM 2.3. *Let $N \in Zg_{\hat{R}_P G}$ be torsionfree and reduced over R. Then N is in the closure of the set of indecomposable $\hat{R}_P G$-lattices.*

PROOF. N is the direct limit of f.g. $\hat{R}_P G$-submodules N' (see [J], Theorem 2.7). Each N' is R-torsionfree and hence $\hat{R}_P$-torsionfree. It follows that N' is an $\hat{R}_P G$-lattice. Accordingly N is a direct limit of finite direct sums of indecomposable $\hat{R}_P G$-lattices. Recall now that the elementary class of modules which have support on some closed subset of the Ziegler spectrum is closed under both direct limits and direct sum (see, for example, [Roth]). $\dashv$

Let $\mathscr{L}_0$ denote the set of R-torsionfree points of Zg_{RG} which are $\hat{R}_P G$-lattices for some maximal P and let $\mathscr{L}$ denote the set of R-torsionfree R-reduced points of Zg_{RG}.

Thus, the Ziegler closure of $\mathscr{L}_0$ contains $\mathscr{L}$.

THEOREM 2.4. *Let $M \in \mathscr{L}_0$. Then M is isolated in $\mathscr{L}_0$ and closed in $\mathscr{L}$, where we consider $\mathscr{L}$ with the topology induced from Zg_{RG}.*

PROOF. Suppose that M is an $\hat{R}_P G$-lattice for the prime P. Choose a positive integer k such that P^k does not contain the order of G. By the Maranda Theorem, the isomorphism class of M is fully determined by that of $M/P^k M$. Moreover $M/P^k M$ is indecomposable because M is and because $\hat{R}_P$ is complete.

Consider $M/P^k M$ as a module over the Artin algebra $(R/P^k)G$. As such, M is f.p. and so forms a clopen set in the Ziegler topology for $(R/P^k)G$ (see 2.9 in [P2]). Hence there are in $L_{(R/P^k)G}$ pp-formulas $\varphi(v)$, $\psi(v)$ and $\varphi_i(v)$, $\psi_i(v)$ (with i in a, finite by compactness of the Ziegler spectrum, set of indices) such that $M/P^k M$ is the only point in

$$(\varphi/\psi) = \bigcap_i [\varphi_i/\psi_i].$$

For every pp-formula $\alpha(v)$ of $L_{(R/P^k G)}$, there is a pp-formula $\alpha'(v)$ of L_{RG} (so in $L_{\hat{R}_P G}$) such that if A is an RG-module and $a \in A$ then $A/p^k A \models \alpha(a + P^k A)$ if and only if $A \models \alpha'(a)$ (recall that P is finitely generated because R is Noetherian). Therefore, inside Zg_{RG}, M belongs to the basic open set (φ'/ψ') as well as to the closed set $\bigcap_i [\varphi_i'/\psi_i']$. We show that M is isolated in $\mathscr{L}_0$ by (φ'/ψ') and that M is the only R-torsionfree non divisible point in a suitable closed set in Zg_{RG} closely related to $\bigcap_i [\varphi_i'/\psi_i']$.

Let N be an R-torsionfree point in Zg_{RG}. If N is a KG-module or a $\hat{R}_Q G$-module for some maximal ideal $Q \neq P$ in R, then $N = P^k N$ and so $\varphi'(N) = \psi'(N)$, that is $N \notin (\varphi'/\psi')$. On the other hand, if N is an $\hat{R}_P G$-lattice then $N \in (\varphi'/\psi')$ if and only if $N/P^k N \in (\varphi/\psi)$ which is so exactly if $N/P^k N \simeq M/P^k M$ which is the case precisely if $N \simeq M$. This proves the first claim.

Moreover, among the R-torsionfree points in Zg_{RG}, the closed set

$$\bigcap_{Q\neq P}[v = v/Q|v]$$

contains only the KG-modules and the $\hat{R}_PG$-modules.

So any point $N \in \mathscr{L}$ in $\bigcap_{Q\neq P}[v = v/Q|v] \cap \bigcap_i[\varphi_i/\psi_i]$ must be an R-torsionfree point of $Zg_{\hat{R}_PG}$ satisfying $N/P^kN \simeq M/P^kM$ and hence, by Maranda's theorem, must be isomorphic to M. Thus M is a closed point of $\mathscr{L}$. $\dashv$

Now let us briefly discuss the isolation statement with respect to $\mathscr{L}$. Let $N \in \mathscr{L}$. Consider the decomposition of N/P^kN as a pure-injective module over $(R/P^k)G$. If $N \in (\varphi'/\psi')$ then it must be that $(\varphi)(N/P^kN) > \psi(N/P^kN)$ and hence, since the pair φ/ψ isolates M/P^kM as a module over A/P^kA, the module M/P^kM must occur as a direct summand of N/P^kN. Can we say any more than this in general?

Now let us treat the relationship with simple KG-modules. Let S be one of these. Fix a non-zero prime P. We know that there is some full R_PG-lattice M such that $S \simeq MK$. In general M is not unique; however the Jordan-Zassenhaus Theorem ([CR], 24.1 p. 534) ensures that only finitely many M's can satisfy $S \simeq MK$ (up to isomorphism). In any case, M must be indecomposable.

LEMMA 2.1. *Let S be a simple KG-module, M be a(n indecomposable) R_PG-lattice satisfying $S \simeq MK$, $\varphi(v)$, $\psi(v)$ be pp-formulas in L_{RG}. Then*

(i) $|\varphi(S) : \psi(S)|$ *is either* 1 *or* ∞,

(ii) $|\varphi(M) : \psi(M)| \geq |\varphi(S) : \psi(S)|$.

PROOF. (i) This is a general fact. For, let $a \in \varphi(S) - \psi(S)$; recall that the characteristic of K is 0 and look at the infinitely many elements ah with $h \in K$. All of them are in $\varphi(S)$; but, for $h \neq h'$ in K, $ah - ah'$ cannot lie in $\psi(S)$, otherwise $a = (a(h - h'))(h - h')^{-1} \in \psi(S)$.

(ii) Assume $S = MK$. Let k be a nonnegative integer and let $a_0, \dots, a_k$ be elements in $\varphi(S)$ such that, for $i < j \leq k$, $a_i - a_j \notin \psi(S)$. Take a suitable $q \in R$ such that $a_iq = a_i'$ is in M and satisfies $\varphi(v)$ in M for every $i \leq k$. Notice that $a_i' - a_j' \in \psi(M)$ implies $a_i - a_j \in \psi(S)$, and this is impossible when $i < j \leq k$. $\dashv$

In particular $|\varphi(S) : \psi(S)| = 1$ when $|\varphi(M) : \psi(M)| < \infty$ and, most notably, $|\varphi(M) : \psi(M)| = \infty$ whenever $|\varphi(S) : \psi(S)| > 1$. Note that M need not be pure-injective and its pure-injective hull need not be indecomposable but we can express this relationship between M and S by saying that S is in the support of M $Supp(M)$ (recall that $Supp(M) = \{N$ indecomposable : N is a direct summand of some M' with $M' \equiv M\}$).

Notice also that, if $S' \not\simeq S$ is another simple KG-module and M' is a(n indecomposable) R_PG-lattice satisfying $M'K \simeq S'$, then S is not in the support of M'. For, let e be the primitive central idempotent of S in KG and let $q \in R$ satisfy $eq = e' \in R_PG$. Put

$$\varphi(v) : ve' = vq, \quad \psi(v) : v = 0,$$

then $|\varphi(S) : \psi(S)| > 1$, as already observed, but $|\varphi(M') : \psi(M')| = 1$.

The relationship with simple KG-modules is clearer if we assume that P does not contain the order of G. Then the $\hat{R}_PG$-lattices form a class of finite representation type. Moreover, in this case, R_PG is a maximal order in KG. This implies that

(i) given S, there is a unique (indecomposable) full lattice M over $R_P G$ such that $MK \simeq S$ (up to isomorphism) (see [CR], ex. 26.11);
(ii) for every $R_P G$-lattice M, M is indecomposable if and only if MK is simple ([CR], 26.12).

Another useful assumption is

($\star$) For every simple KG-module S, the P-adic completion $\hat{S}$ of S is still simple over $\hat{K}G$.

Of course $\hat{K}$ still denotes P-adic completion. The relevance of ($\star$) is discussed in [CR], 30.18; essentially, under ($\star$), $M \to MK$ defines a bijection between isomorphism classes of

- indecomposable $\hat{R}_P G$-lattices (i.e., indecomposable pure injective points in $Zg_{\hat{R}_P G}$) and
- simple KG-modules.

With these assumptions we have the following.

PROPOSITION 2.1. *Let P be a maximal ideal of R such that $|G| \notin P$. Assume ($\star$). Then each indecomposable $\hat{R}_P G$-lattice is isolated in $\mathscr{L}$; furthermore every simple KG-module S is in the topological closure of a unique indecomposable $\hat{R}_P G$-lattice.*

PROOF. We have already seen that, as the $|G| \notin P$, $\hat{R}_P G$-lattices form a class of finite representation type, hence each indecomposable $\hat{R}_P G$-lattice is isolated in $Zg_{\hat{R}_P G}$ and hence, by 2.4, there are no other R-torsionfree R-reduced points in the P-patch of the spectrum of RG. $|G| \notin P$ also implies that every simple KG-module S can be expressed as MK for a unique indecomposable $R_P G$-lattice M. We know that M is an elementary substructure of $\hat{M}$ -a $\hat{R}_P G$-lattice- (see [T1]). Under ($\star$), $\hat{M}$ is indecomposable, and every indecomposable $\hat{R}_P G$-lattice can be obtained in this way for a suitable M. Therefore we have that S is in the closure of $\hat{M}$ and $\hat{M}$ is the unique $\hat{R}_P G$-lattice with this property. ⊣

Another approach to proving the previous proposition would be to use Theorem 26.20.(i) in [CR]. This ensures that $R_P G$ is a finite direct sum of maximal orders $R_P Ge$ in central simple algebras KGe (here e ranges over the central primitive idempotents of KG, and KGe over the corresponding Wedderburn components). In particular, every indecomposable pure injective module over $R_P G$ can be viewed as a(n indecomposable pure injective) module over some direct summand $R_P Ge$, and so $Zg_{R_P G}$ is the disjoint union of the closed and open subspaces $Zg_{R_P Ge}$ indexed by the elements e. At this point, one can observe that each $R_P Ge$ is a *PI* Dedekind prime ring since it is embedded in the ring of $n \times n$ matrices, $n = |G|$, over the commutative ring R_P, and so is Morita equivalent to some *PI* Dedekind domain. The Morita equivalence provides a homeomorphism between the corresponding Ziegler spaces, and the points of the spectrum over a *PI* Dedekind domain are fully classified in [P1].

What can we say about the relationship between simple KG-modules and indecomposable $\hat{R}_P G$-lattices when P is a maximal ideal of R and we discard our assumptions ($\star$) and $|G| \notin P$? First notice that, when $\hat{R}_P G$-lattices are a class of finite representation type and so there are only finitely many indecomposable $\hat{R}_P G$-lattices up to isomorphism, then they form a closed set and exhaust the points in the P-patch.

Now consider a simple KG-module S. S is totally transcendental, because it is definable in K, viewed as a vector space over itself, or, alternatively, because KG is semisimple artinian. How can we axiomatize the first order theory of S in L_{RG}? Let e_S denote the primitive central idempotent of KG corresponding to S; consider the L_{RG}-sentences characterizing the R-torsionfree R-divisible RG-modules on which e_S acts identically. Accordingly the models of these sentences are KG-modules which are just direct sums of copies of S. So the corresponding theory in categorical in each power $> |K|$, and consequently complete. Hence it equals the theory of S and equips it with an explicit axiomatization. Its models include S, as well as, for every P, its P-adic completion $\hat{S}$ (an elementary extension of S in L_{R_PG}), and the $\hat{K}G$-simple direct summands of $\hat{S}$.

PROPOSITION 2.2. *Let P be a maximal ideal of R, S be a simple KG-module. Then S is in the topological closure of an indecomposable $\hat{R}_PG$-lattice N if and only if S is a direct summand of $N\hat{K}$.*

PROOF. Form $N\hat{K}$; this is a direct sum of simple KG-modules. Notice that

$$S|N\hat{K}$$

if and only if

$$\text{some } a \in N \text{ satisfies } a = ae_S \neq 0.$$

The direction from the right to the left is clear. Conversely, if $S|N\hat{K}$, then there is some $a = ae_S \neq 0$ in $N\hat{K}$ and a suitable $\hat{R}_P$-multiple of a lies in N.

Now assume $S|N\hat{K}$. Let $\varphi(v)$ and $\psi(v)$ be pp-formulas in L_{RG} such that $|\varphi(S) : \psi(S)| > 1$. Consequently $|\varphi(S) : \psi(S)| = \infty$, and the same is true for $N\hat{K}$. Now adapt Lemma 2.1.(*ii*) and deduce $|\varphi(N) : \psi(N)| \geq |\varphi(N\hat{K}) : \psi(N\hat{K})|$, so $|\varphi(N) : \psi(N)| = \infty$. The key point here is to find a suitable q directly in R instead of $\hat{R}_P$; this is possible because the maximal ideal of $\hat{R}_P$ is $P\hat{R}_P$, so is principal with a generator π in R. In conclusion $N \oplus S \equiv N$, and S is in the closure of N.

When $S \nmid N\hat{K}$, and $q \in R$ satisfies $qe_S \in RG$, then

$$(vqe_S = vq/v = 0)$$

includes S and excludes N. ⊣

§3. **RG-lattices.** In this section we discuss briefly the behaviour of (indecomposable) RG-lattices within the topological space Zg_{RG}. Of course, a lattice is not necessarily pure injective and so need not be a point of Zg_{RG}. But every RG-module, in particular every RG-lattice, M does determine a closed set in the Ziegler spectrum, namely its support $Supp(M)$. Every closed subset of Zg_{RG} has this form. Of course, elementarily equivalent modules M, M' have the same support, but the converse is, in general, false even for lattices, for example if there is a direct summand N of M with $N \equiv N \oplus N$ then, although the presence of N as a direct summand of M can be detected, the multiplicity of N in M cannot be determined from the support of M. In fact, every closed C in Zg_{RG} is the support of some M, but, in general, only the elementary equivalence class of $M^{\aleph_0}$ is fully determined by C.

Recall the algebraic characterization of elementary equivalence between RG-lattices given in [BT] when R/P is finite for every maximal ideal P of R. Under this assumption, two RG-lattices are elementarily equivalent if and only if they are in the same genus, i.e., they admit isomorphic localizations at every maximal ideal P, or, equivalently, at every maximal ideal P containing $|G|$. Here we extend that result. Recall that the finiteness condition on the R/P's is certainly satisfied when K is a global field. We need the following lemma (which may be of independent interest).

LEMMA 3.1. *Let M be a RG-lattice. Then $\hat{M} \simeq \prod_P \hat{M}_P$ where P ranges over the maximal ideals of R.*

PROOF. First of all, notice that

$$\hat{M} \equiv M \equiv \oplus_P M_P \equiv \prod_P M_P \equiv \prod_P \hat{M}_P$$

in L_{RG}. This can be shown by arguing as in [MT], Proposition 2. Moreover the canonical embedding of $\prod_P M_P$ in $\prod_P \hat{M}_P$ is pure, hence elementary because $M_P \prec \hat{M}_P$ for every P. There is a natural embedding, from M into $\prod_P M_P$, sending an element $a \in M$ into the constant sequence $(a)_P \in \prod_P M_P$. By proceeding as in [P], 2.$\mathbf{Z}$5, one sees that this embedding also is pure, hence elementary. For, let $\varphi(\vec{v})$ be a pp-formula in L_{RG}, $\vec{a}$ be a finite sequence in M such that $M \models \neg\varphi(\vec{a})$. Then $I = \{r \in R : M \models \varphi(\vec{a}r)\}$ is a proper ideal of R and hence $I \subseteq P$ for some maximal ideal P. According to [MT], $\varphi(M_P) = \varphi(M)_P$. Consequently $M_P \models \neg\varphi(\vec{a})$, otherwise there would be some $s \notin I$ for which $M \models \varphi(\vec{a}s)$. It follows that $\prod_P \hat{M}_P \models \neg\varphi((\vec{a})_P)$.

In conclusion, there is a pure (and even elementary) embedding of M into the pure injective module $\prod_P \hat{M}_P$. Therefore $\hat{M}$ can be embedded as a pure submodule, hence as a direct summand, in $\prod_P \hat{M}_P$ over RG. Without loss of generality we can identify $\hat{M}$ with its isomorphic copy in $\prod_P \hat{M}_P$. Put

$$\prod_P \hat{M}_P = \hat{M} \oplus N$$

for some N. This decomposition holds also over R. But, over R, $\prod_P \hat{M}_P$ is just the pure injective hull of M (consider the Baur-Garavaglia-Monk invariants). But then, by [P], 4.10.(d), for every $a \in \prod_P \hat{M}_P$ with $a \neq 0$, there are an element $b \in M$ and a pp-formula $\varphi(v, w)$ in L_R such that

$$\prod_P \hat{M}_P \models \varphi(a, b) \wedge \neg\varphi(a, 0).$$

Fix $a \in N$, $a \neq 0$. Therefore $\prod_P \hat{M}_P$ satisfies $\varphi(a, b)$, hence $\varphi(0, b)$ by projecting onto $\hat{M}$, and so $\varphi(a, 0)$ (a contradiction). Hence $N = 0$. ⊣

PROPOSITION 3.1. *Assume R/P finite for every maximal ideal P of R. Let M, M' be RG-lattices. Then the following conditions are equivalent*:

(1) $M \equiv M'$;

(2) *M and M' are in the same genus*;

(3) $\hat{M} \simeq \hat{M}'$;

(4) $\oplus_{|G| \in P} \hat{M}_P \simeq \oplus_{|G| \in P} \hat{M}'_P$.

PROOF. (1) $\Leftrightarrow$ (2) is known (see [BT]), and (3) $\Rightarrow$ (1) is trivial. Also (2) $\Rightarrow$ (3), (4) is clear from the lemma; for, if M and M' are in the same genus, then

$\hat{M}_P \simeq \hat{M}'_P$ for every maximal ideal P, and hence $\prod_P \hat{M}_P \simeq \prod_P \hat{M}'_P$ as well as $\oplus_{|G|\in P}\hat{M}_P \simeq \oplus_{|G|\in P}\hat{M}'_P$. So it suffices to show (4) $\Rightarrow$ (2). Accordingly assume $\oplus_{|G|\in P}\hat{M}_P \simeq \oplus_{|G|\in P}\hat{M}'_P$. The isomorphism (over RG) is preserved if we pass to the quotients with respect to P^k where P is any maximal ideal containing the order of G and P^k does not include $|G|$. The Maranda Theorem implies $\hat{M}_P \simeq \hat{M}'_P$ for every P containing $|G|$ over $\hat{R}_P G$. By [CR], 30.27, M and M' are in the same genus. $\dashv$

Now take two indecomposable RG-lattices M and M' with $M \not\equiv M'$. We assume the following condition on multiplicities:

($\star\star$) For every maximal ideal P of R containing $|G|$, both $\hat{M}_P$ and $\hat{M}'_P$ have no indecomposable pure injective direct summand of multiplicity > 1.

We show

Proposition 3.2. *Assume again R/P finite for every maximal ideal P of R. Let $M \not\equiv M'$ be two indecomposable RG-lattices satisfying ($\star\star$). Then $Supp(M) \not\subseteq Supp(M')$ (and $Supp(M') \not\subseteq Supp(M)$, of course). Furthermore there are pp-formulas $\varphi(v)$, $\psi(v)$, $\varphi'(v)$, $\psi'(v)$ in L_{RG} such that*

$$|\varphi(M) : \psi(M)| > 1, \quad |\varphi(M') : \psi(M')| = 1,$$

$$|\varphi'(M) : \psi'(M)| = 1, \quad |\varphi'(M') : \psi'(M')| > 1.$$

Proof. There exists a maximal ideal P containing the order of G and an indecomposable $\hat{R}_P G$-lattice N such that N is a direct summand of $\hat{M}_P$, but N is not a direct summand of $\hat{M}'_P$. Otherwise, owing to the assumption on multiplicities and the fact that both $\hat{M}_P$ and $\hat{M}'_P$ are $\hat{R}_P G$-lattices, $\hat{M}_P$ is a direct summand of $\hat{M}'_P$ for every P. So there is some lattice M'' in the genus of M such that $M''|M'$ (see [CR], 31.12). Since M' is indecomposable, this implies $M'' \simeq M'$ and hence M and M' in the same genus, in other words $M \equiv M'$ (a contradiction).

So consider P and N. We know that N is isolated among $\hat{R}_P G$-lattices in Zg_{RG}; so there are two pp-formulas in L_{RG} $\varphi(v) \geq \psi(v)$ such that N is the only indecomposable $\hat{R}_P G$-lattice in the basic open set (φ/ψ). Notice that $(v = v/P|v)$ includes the indecomposable R-torsionfree $\hat{R}_P G$-modules in Zg_{RG}, in particular N, but excludes the other R-torsionfree points ($P|v$ is a pp-formula because P is f.g.). Of course we can suppose $(\varphi/\psi) \subseteq (v = v/v \in P)$. Clearly

$$|\varphi(\hat{M}) : \psi(\hat{M})| > 1,$$

so the same is true also for M. Assume that M' and hence $\hat{M}'$ satisfy the same property. By recalling Lemma 3.1 and $(\varphi/\psi) \subseteq (v = v/v \in P)$, we deduce that the same is true for $\hat{M}'_P$. Then some indecomposable $\hat{R}_P G$-direct summand of $\hat{M}'_P$ is in (φ/ψ). $M' \equiv \hat{M}'$ implies that $\hat{M}'_P$ is $\hat{R}_P$-torsionfree and f.g., so $\hat{M}'_P$ is an $\hat{R}_P G$-lattice, as are its indecomposable direct summands. It follows that $N|\hat{M}'_P$, and this is a contradiction.

By reversing the roles of M and M', we find $\varphi'(v)$ and $\psi'(v)$ as required.

It remains to show that N is in $Supp(M) - Supp(M')$. $N \in Supp(M)$ is clear. Assume $N \in Supp(M')$, so $N|M''$ for some $M'' \equiv M'$. Therefore

$$|\varphi(M'') : \psi(M'')| > 1, \text{ and } |\varphi(M') : \psi(M')| > 1;$$

but this contradicts what was observed before. ⊣

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