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STRESS DISTRIBUTION IN A TRANSVERSELY ISOTROPIC BODY WITH A THERMALLY INSULATING PARABOLIC CRACK SUBJECTED TO A UNIFORM HEAT FLUX

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An explicit static thermoelastic solution is constructed for an infinite transversely isotropic body containing a thermally insulating parabolic crack in the plane of isotropy. The surface of the crack is free of stress. A uniform thermal flux is incident on the crack perpendicular to its surface. Formulas are obtained for the stress intensity factors near the tip of the crack.

We shall consider an infinite transversely isotropic body with a thermally insulating parabolic crack

$$\frac{y^2}{b^2-a^2}+2x\le b^2 \quad (b>a\ge 0) \tag{1}$$

in its plane of isotropy.

The x and y axes are taken to lie in the plane of isotropy, while the z axis is directed along the anisotropy axis. We shall assume that at a sufficient distance from the crack the body is acted on by a uniform heat flux perpendicular to the plane of the crack.

In solving a problem for the thermally stressed state of this body we shall use a potential-function representation of the general solution of the equations of steady-state thermoelasticity [1, 2], i.e.,

$$u=\frac{\partial\Phi_1}{\partial x}+\frac{\partial\Phi_2}{\partial x}+\frac{\partial\Phi_3}{\partial y}+\frac{\partial\Phi_4}{\partial x};\quad v=\frac{\partial\Phi_1}{\partial y}+\frac{\partial\Phi_2}{\partial y}-\frac{\partial\Phi_3}{\partial x}+\frac{\partial\Phi_4}{\partial x};$$

$$w=k_1\frac{\partial\Phi_1}{\partial z}+k_2\frac{\partial\Phi_2}{\partial z}+k_4\frac{\partial\Phi_4}{\partial z}, \tag{2}$$

where Φ_j $(j=\overline{1,4})$ are functions which satisfy the equations

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\nu_j\frac{\partial^2}{\partial z^2}\right)\Phi_j=0 \quad \left(j=\overline{1,4}\right); \tag{3}$$

$$\frac{\partial^2\Phi_4}{\partial z^2}=k_3T; \tag{4}$$

and

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\nu_4\frac{\partial^2}{\partial z^2}\right)T=0, \tag{5}$$

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and the k_j ($j = \overline{1, 4}$) are constants which depend on the elastic and thermal properties of the material, with

$$k_j = \frac{c_{11}\nu_j - c_{44}}{c_{13} + c_{44}} \quad (j = 1, 2),$$

$$k_4 = \frac{\beta'(c_{44} - \beta'' c_{11}) + \beta\beta''(c_{13} + c_{44})}{\beta(c_{33} - \beta'' c_{44}) - \beta'(c_{13} + c_{44})},$$

$$k_3 = \frac{\beta}{c_{44} + k_4(c_{13} + c_{44}) - \beta'' c_{11}}; \tag{6}$$

$$\beta = (c_{11} + c_{12})\alpha_1 + c_{33}\alpha_2, \; \beta' = 2c_{13}\alpha_1 + c_{33}\alpha_2, \; \beta'' = \nu_4,$$

$$\nu_3 = \frac{2c_{44}}{c_{11} - c_{12}}, \quad \nu_4 = \frac{\lambda'}{\lambda}.$$

Here ν_1 and ν_2 are the roots of the characteristic equation

$$c_{11}c_{44}\nu^2 - \left[c_{44}^2 + c_{33}c_{11} - (c_{13} + c_{44})^2\right]\nu + c_{33}c_{44} = 0 \tag{7}$$

and λ, λ', α_1, α_2, and c_{ij} denote, respectively, the coefficients of thermal conductivity and thermal expansion along the isotropy and anisotropy axes, as well as the elastic constants of the transversely isotropic body.

Let us introduce the notation $z = \sqrt{\nu_j}\, z_j$. Then the functions $\Phi_j(x, y, z_j)$ ($j = \overline{1, 4}$) and $T(x, y, z_4)$ will be harmonic in the corresponding coordinate systems. We shall construct a solution of the problem with the aid of the potential functions $\Phi_j(x, y, z_j)$ ($j = \overline{1, 4}$). Here we shall use the paraboloidal coordinate system [3]

$$x = \left(\rho_j^2 + \mu_j^2 + \lambda_j^2 - a^2 - b^2\right)/2;$$

$$y = \sqrt{\frac{(\rho_j^2 - a^2)(\mu_j^2 - a^2)(\lambda_j^2 - a^2)}{a^2 - b^2}};$$

$$z = \sqrt{\nu_j}\, z_j = \sqrt{\nu_j}\sqrt{\frac{(\rho_j^2 - b^2)(\mu_j^2 - b^2)(\lambda_j^2 - b^2)}{b^2 - a^2}}$$

$$\left(-\infty \le \lambda_j \le a \le \mu_j \le b \le \rho_j \le \infty; \quad j = \overline{1, 4}\right). \tag{8}$$

At the surface of the paraboloid, ρ_j = const. When $\rho_j = b$, the paraboloid degenerates into the crack (1). Equations (8) imply that the following equations hold at the crack surface ($\rho_j = b$):

$$\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu; \quad \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda. \tag{9}$$

We note also that in the $z = 0$ plane outside the crack the paraboloidal coordinates are

$$\mu_1 = \mu_2 = \mu_3 = \mu_4 = b; \quad \rho_1 = \rho_2 = \rho_3 = \rho_4 = \rho;$$

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda. \tag{10}$$

In this problem, the boundary conditions can be written as

$$\frac{\partial T}{\partial z}=0;\quad \sigma_z=0;\quad \tau_{xz}=0;\quad \tau_{yz}=0\quad \left(\rho_j=b\right). \tag{11}$$

In the $z=0$ plane outside the crack the following conditions must be satisfied:

$$T=0;\quad \sigma_z=0;\quad u=0;\quad v=0\quad \left(\mu_j=b\right). \tag{12}$$

At a sufficient distance from the crack, there are no mechanical forces and the thermal flux must be equal to

$$\frac{\partial T}{\partial z}=Q\quad \left(\rho_4=\infty\right). \tag{13}$$

Determining the temperature field reduces to finding a solution to the steady-state heat conduction equation for a transversely isotropic body:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z_4^2}=0 \tag{14}$$

with boundary conditions (11)–(13) for the temperature. The solution of this boundary value problem has the form

$$T=Q_1 z_4 X_3(\rho_4)+\sqrt{v_4}\, z_4 Q. \tag{15}$$

Here and in the following, we have

$$X_3(\rho)=\int\frac{\rho\, d\rho}{\left(\rho^2-b^2\right)\Delta(\rho)}=-\frac{1}{\rho^2-b^2+\Delta(\rho)};$$

$$\Delta(\rho)=\sqrt{\left(\rho^2-a^2\right)\left(\rho^2-b^2\right)};$$

$$X_{33}(\rho)=\int\frac{\rho\, d\rho}{\left(\rho^2-b^2\right)^2\Delta(\rho)}=-\frac{2\Delta(\rho)+\rho^2-a^2}{3\Delta(\rho)\left[\Delta(\rho)+\rho^2-b^2\right]^2};$$

$$X_2(\rho)=\int\frac{\rho\, d\rho}{\left(\rho^2-a^2\right)\Delta(\rho)}=-\frac{1}{\rho^2-a^2+\Delta(\rho)};$$

$$X_{23}(\rho)=\int\frac{\rho\, d\rho}{\Delta^3(\rho)}=\frac{1}{b^2-a^2}\left[X_3(\rho)-X_2(\rho)\right]. \tag{16}$$

Function (15) obeys Eq. (14) and boundary conditions (12) and (13). From Eqs. (11) and (15) we have

$$\lim_{\rho_4\to b}\frac{\partial T}{\partial z}=\frac{1}{\sqrt{v_4}}Q_1\lim_{\rho_4\to b}\left[X_3(\rho_4)+\frac{z_4^2\Delta(\rho_4)}{\left(\rho_4^2-b^2\right)^2\left(\rho_4^2-\mu_4^2\right)\left(\rho_4^2-\lambda_4^2\right)}\right]+Q=$$

$$=\frac{1}{\sqrt{v_4}}Q_1\frac{1}{b^2-a^2}+Q=0. \tag{17}$$

Here and in the following we use the fact that

$$\frac{\partial \rho_j}{\partial x} = \frac{\Delta^2(\rho_j)}{\rho_j(\rho_j^2 - \mu_j^2)(\rho_j^2 - \lambda_j^2)};$$

$$\frac{\partial \rho_j}{\partial y} = \frac{y\Delta^2(\rho_j)}{\rho_j(\rho_j^2 - \mu_j^2)(\rho_j^2 - \lambda_j^2)(\rho_j^2 - a^2)};$$

$$\frac{\partial \rho_j}{\partial z_j} = \frac{z_j\Delta^2(\rho_j)}{\rho_j(\rho_j^2 - \mu_j^2)(\rho_j^2 - \lambda_j^2)(\rho_j^2 - b^2)}. \tag{18}$$

From Eq. (17) we have

$$Q_1 = -\sqrt{\nu_4}(b^2 - a^2)Q. \tag{19}$$

The temperature field is, therefore, described by function (15), where Q_1 is a constant given by Eq. (19).

We shall now determine the thermally stressed state of the body. If there is a linear temperature distribution

$$T_1 = zQ \tag{20}$$

in an unbounded homogeneous body, then the displacements in the body will be given by

$$u_1 = Ql_1xz;\ \ v_1 = Ql_1yz;\ \ w_1 = \frac{1}{2}\left[l_2z^2 - l_1(x^2 + y^2)\right]. \tag{21}$$

Here the constants l_1 and l_2 are obtained from the system of equations

$$(c_{11} + c_{12})l_1 + c_{13}l_2 = \beta;\ \ 2c_{13}l_1 + c_{33}l_2 = \beta'. \tag{22}$$

In this case, there are no mechanical stresses in the body, i.e.,

$$\sigma_x^{(1)} = \sigma_y^{(1)} = \sigma_z^{(1)} = \tau_{xy}^{(1)} = \tau_{xz}^{(1)} = \tau_{yz}^{(1)} = 0. \tag{23}$$

Now let the following temperature distribution exist within the body:

$$T_2 = Q_1z_4X_3(\rho_4). \tag{24}$$

In order to determine the thermally stressed state of the body, we shall use representation (2) of the solution. Here the components of the stresses are found using

$$\sigma_z^{(2)} = c_{44}\sum_{j=1,2}(1 + k_j)\frac{\partial^2\Phi_j}{\partial z_j^2} + \left[k_3(c_{33}k_4 - \nu_4c_{13}) - \beta'\right]T;$$

$$\tau_{xz}^{(2)} = c_{44}\sum_{j=1,2,4}\frac{1 + k_j}{\sqrt{\nu_j}}\frac{\partial^2\Phi_j}{\partial x\partial z_j} + \frac{c_{44}}{\sqrt{\nu_3}}\frac{\partial^2\Phi_3}{\partial y\partial z_3};$$

$$\tau_{yz}^{(2)} = c_{44} \sum_{j=1,2,4} \frac{1+k_j}{\sqrt{\nu_j}} \frac{\partial^2 \Phi_j}{\partial y \partial z_j} - \frac{c_{44}}{\sqrt{\nu_3}} \frac{\partial^2 \Phi_3}{\partial x \partial z_3};$$

$$\sigma_x^{(2)} = \sum_{j=1,2,4} \left(c_{11} \frac{\partial^2 \Phi_j}{\partial x^2} + c_{12} \frac{\partial^2 \Phi_j}{\partial y^2} + c_{13} k_j \frac{\partial^2 \Phi_j}{\partial z^2} \right) + (c_{11} - c_{12}) \frac{\partial^2 \Phi_3}{\partial x \partial y} - \beta T;$$

$$\sigma_y^{(2)} = \sum_{j=1,2,4} \left(c_{12} \frac{\partial^2 \Phi_j}{\partial x^2} + c_{11} \frac{\partial^2 \Phi_j}{\partial y^2} + c_{13} k_j \frac{\partial^2 \Phi_j}{\partial z^2} \right) - (c_{11} - c_{12}) \frac{\partial^2 \Phi_3}{\partial x \partial y} - \beta T;$$

$$\tau_{xy}^{(2)} = \frac{1}{2} (c_{11} - c_{12}) \left(2 \sum_{j=1,2,4} \frac{\partial^2 \Phi_j}{\partial x \partial y} - \frac{\partial^2 \Phi_3}{\partial x^2} + \frac{\partial^2 \Phi_3}{\partial y^2} \right). \tag{25}$$

The potential function Φ_4 determined using Eqs. (4) and (24) has the form

$$\Phi_4 = \frac{c_4}{3} \int \frac{\tau \, d\tau}{(\tau^2 - b^2) \Delta(\tau)} \left[z_4^3 - 3z^2(\tau) z_4 + 2z^3(\tau) \right] \Bigg|_{\tau=\rho_4}, \tag{26}$$

where

$$z(\tau) = \sqrt{\tau^2 - b^2} \sqrt{\tau^2 - 2x - \frac{y^2}{\tau^2 - a^2}}; \quad z(\rho_j) = z_j;$$

$$c_4 = \frac{1}{2} \nu_4^{3/2} k_3 (a^2 - b^2) Q. \tag{27}$$

It should be noted that for $j = 4$ function (26) satisfies Eq. (3).

Let us choose potential functions Φ_j ($j = \overline{1, 3}$) of the form

$$\Phi_j = \left\{ \frac{c_j}{3} \int \frac{\tau \, d\tau}{(\tau^2 - b^2) \Delta(\tau)} \left[z_j^3 - 3z^2(\tau) z_j + 2z^3(\tau) \right] + \right.$$

$$+ \frac{a_j}{2} z_j \int \frac{\tau \, d\tau}{(\tau^2 - b^2) \Delta(\tau)} \left(2x + \frac{y^2}{\tau^2 - a^2} + \frac{z_j^2}{\tau^2 - b^2} - \tau^2 \right) +$$

$$\left. + d_j \int \frac{\tau \, d\tau}{\Delta(\tau)} \left[z_j - z(\tau) \right] \right\} \Bigg|_{\tau=\rho_j} \quad (j = 1, 2);$$

$$\Phi_3 = d_3 y \left\{ \int \frac{\tau \, d\tau}{(\tau^2 - a^2) \Delta(\tau)} \left[z_3 - z(\tau) \right] \right\} \Bigg|_{\tau=\rho_3}. \tag{28}$$

Here a_j, c_j, and d_j are unknown constants.

Equations (26) and (28), together with Eq. (2), imply that

$$u_2 = \Bigg\{ \sum_{j=1,2} a_j z_j X_3(\rho_j) + \frac{1}{3} \sum_{j=1,2,4} c_j \int \frac{\tau\, d\tau}{(\tau^2 - b^2)\Delta(\tau)} \times$$

$$\times \frac{\partial}{\partial x}\left[z_j^3 - 3z^2(\tau) z_j + 2z^3(\tau)\right] - \sum_{j=1,2} d_j \int \frac{\tau\, d\tau}{\Delta(\tau)} \frac{\partial z(\tau)}{\partial x} \Bigg\}_{\tau=\rho_j} +$$

$$+ d_3 \left\{ \int \frac{\tau\, d\tau}{(\tau^2 - a^2)\Delta(\tau)} \left[z_3 - z(\tau) - y \frac{\partial z(\tau)}{\partial y} \right] \right\}_{\tau=\rho_3} ;$$

$$v_2 = \Bigg\{ \sum_{j=1,2} a_j y z_j \int \frac{\tau\, d\tau}{\Delta^3(\tau)} + \frac{1}{3} \sum_{j=1,2,4} c_j \int \frac{\tau\, d\tau}{(\tau^2 - b^2)\Delta(\tau)} \times$$

$$\times \frac{\partial}{\partial y}\left[z_j^3 - 3z^2(\tau) z_j + 2z^3(\tau)\right] - \sum_{j=1,2} d_j \int \frac{\tau\, d\tau}{\Delta(\tau)} \frac{\partial z(\tau)}{\partial y} \Bigg\}_{\tau=\rho_j} +$$

$$+ d_3 y \left\{ \int \frac{\tau\, d\tau}{(\tau^2 - a^2)\Delta(\tau)} \frac{\partial z(\tau)}{\partial x} \right\}_{\tau=\rho_3} ;$$

$$w_2 = \sum_{j=1,2,4} \frac{k_j}{\sqrt{v_j}} \Bigg\{ c_j \int \frac{\tau\, d\tau}{(\tau^2 - b^2)\Delta(\tau)} \left[z_j^2 - z^2(\tau)\right] +$$

$$+ \frac{a_j}{2} \int \frac{\tau\, d\tau}{(\tau^2 - b^2)\Delta(\tau)} \left(2x + \frac{y^2}{\tau^2 - a^2} + \frac{3z_j^2}{\tau^2 - b^2} - \tau^2 \right) + d_j \int \frac{\tau\, d\tau}{\Delta(\tau)} \Bigg\}_{\tau=\rho_j} ; \quad a_4 = d_4 = 0. \tag{29}$$

Given that

$$\int \left\{ \frac{\partial z(\tau)}{\partial x} + \frac{1}{\tau^2 - a^2} \frac{\partial}{\partial y}[y z(\tau)] \right\} \frac{\tau\, d\tau}{\Delta(\tau)} \Bigg|_{\tau=\rho_j} = -\frac{z_j}{\Delta(\rho_j)}, \tag{30}$$

and taking

$$\sum_{j=1,2} d_j = d_3; \quad \sum_{j=1,2,4} c_j = 0, \tag{31}$$

from Eqs. (29) we obtain

$$u_2 = 0; \quad v_2 = 0 \quad \text{for} \quad \mu_j = b. \tag{32}$$

From Eqs. (25), (26), (28), and (24), we find the stress components to be

$$\sigma_z^{(2)} = c_{44} \sum_{j=1,2} \left(1+k_j\right)\left\{2c_j z_j X_3\left(\rho_j\right) + a_j\left[3z_j X_{33}\left(\rho_j\right) + \right.\right.$$

$$\left.\left. + \frac{z_j^2 \rho_j}{\left(\rho_j^2 - b^2\right)^2 \Delta\left(\rho_j\right)} \frac{\partial \rho_j}{\partial z_j}\right] + d_j \frac{\rho_j}{\Delta\left(\rho_j\right)} \frac{\partial \rho_j}{\partial z_j}\right\} -$$

$$-\left[k_3\left(c_{33}k_4 - \nu_4 c_{13}\right) - \beta'\right]\sqrt{\nu_4}\left(b^2 - a^2\right) Q z_4 X_3\left(\rho_4\right);$$

$$\tau_{xz}^{(2)} = c_{44} \sum_{j=1,2,4} \frac{1+k_j}{\sqrt{\nu_j}}\left\{2c_j X_0\left(\rho_j\right) + a_j\left[X_3\left(\rho_j\right) + \right.\right.$$

$$\left.\left. + \frac{z_j^2 \rho_j}{\left(\rho_j^2 - b^2\right)^2 \Delta\left(\rho_j\right)} \frac{\partial \rho_j}{\partial x}\right] + d_j \frac{\rho_j}{\Delta\left(\rho_j\right)} \frac{\partial \rho_j}{\partial x}\right\} +$$

$$+ \frac{c_{44}}{\sqrt{\nu_3}} d_3\left[X_2\left(\rho_3\right) + y \frac{\rho_3}{\left(\rho_3^2 - a^2\right)\Delta\left(\rho_3\right)} \frac{\partial \rho_3}{\partial y}\right];$$

$$\tau_{yz}^{(2)} = c_{44} \sum_{j=1,2,4} \frac{1+k_j}{\sqrt{\nu_j}}\left\{2c_j y X_2\left(\rho_j\right) + a_j\left[y X_{23}\left(\rho_j\right) + \right.\right.$$

$$\left.\left. + \frac{z_j^2 \rho_j}{\left(\rho_j^2 - b^2\right)^2 \Delta\left(\rho_j\right)} \frac{\partial \rho_j}{\partial y}\right] + d_j \frac{\rho_j}{\Delta\left(\rho_j\right)} \frac{\partial \rho_j}{\partial y}\right\} - \frac{c_{44}}{\sqrt{\nu_3}} y d_3 \frac{\rho_3}{\left(\rho_3^2 - a^2\right)\Delta\left(\rho_3\right)} \frac{\partial \rho_3}{\partial x}. \quad (33)$$

Here we have taken $a_4 = d_4 = 0$.

We shall now determine stress (33) at the surface of a parabolic crack ($\rho_j = b$). Here we assume that

$$\sum_{j=1,2} \left(1+k_j\right) c_j = \frac{1}{2c_{44}}\left[k_3\left(c_{33}k_4 - \nu_4 c_{13}\right) - \beta'\right]\sqrt{\nu_4}\left(b^2 - a^2\right) Q;$$

$$\sum_{j=1,2} \left(1+k_j\right) a_j = 0; \quad \sum_{j=1,2} \left(1+k_j\right) d_j = 0. \quad (34)$$

Given the limits

$$\lim_{\rho_j \to b}\left[y X_{23}\left(\rho_j\right) + \frac{z_j^2 \rho_j}{\left(\rho_j^2 - b^2\right)^2 \Delta\left(\rho_j\right)} \frac{\partial \rho_j}{\partial y}\right] = \frac{2y}{\left(b^2 - a^2\right)^2};$$

$$\lim_{\rho_j \to b}\left[X_3\left(\rho_j\right) + \frac{z_j^2 \rho_j}{\left(\rho_j^2 - b^2\right)^2 \Delta\left(\rho_j\right)} \frac{\partial \rho_j}{\partial x}\right] = \frac{1}{b^2 - a^2}, \quad (35)$$

and Eqs. (34), we find ($\rho_j = b$)

$$\sigma_z^{(2)} = 0;$$

$$\tau_{xz}^{(2)} = c_{44} \sum_{j=1,2,4} \frac{1+k_j}{\sqrt{\nu_j}} \left[2c_j X_0(b) + \frac{a_j}{b^2 - a^2} \right] - \frac{c_{44}}{\sqrt{\nu_3}\left(b^2 - a^2\right)} d_3;$$

$$\tau_{yz}^{(2)} = \frac{2c_{44}}{b^2 - a^2} y \sum_{j=1,2,4} \frac{1+k_j}{\sqrt{\nu_j}} \left(\frac{a_j}{b^2 - a^2} - c_j \right). \tag{36}$$

Equations (36) and boundary conditions (11) imply that

$$2X_0(b) \sum_{j=1,2,4} \frac{1+k_j}{\sqrt{\nu_j}} c_j + \frac{1}{b^2 - a^2} \sum_{j=1,2} \frac{1+k_j}{\sqrt{\nu_j}} a_j - \frac{d_3}{\sqrt{\nu_3}\left(b^2 - a^2\right)} = 0;$$

$$\sum_{j=1,2,4} \frac{1+k_j}{\sqrt{\nu_j}} c_j - \frac{1}{b^2 - a^2} \sum_{j=1,2} \frac{1+k_j}{\sqrt{\nu_j}} a_j = 0. \tag{37}$$

We also give the values of the stresses at the $z = 0$ plane outside the crack, i.e., for $\mu_j = b$,

$$\sigma_z^{(2)} = 0; \quad \tau_{xz}^{(2)} = c_{44} \sum_{j=1,2,4} \frac{1+k_j}{\sqrt{\nu_j}} \left[2c_j X_0(\rho) + a_j X_3(\rho) + d_j \frac{\rho^2 - a^2}{\left(\rho^2 - \lambda^2\right)\Delta(\rho)} \right] +$$

$$+ \frac{c_{44}}{\sqrt{\nu_3}} d_3 \left[X_2(\rho) + \frac{y^2}{\left(\rho^2 - a^2\right)\Delta(\rho)} \frac{1}{\rho^2 - \lambda^2} \right];$$

$$\tau_{yz}^{(2)} = y c_{44} \sum_{j=1,2,4} \frac{1+k_j}{\sqrt{\nu_j}} \left[2c_j X_2(\rho) + a_j X_{23}(\rho) + \right.$$

$$\left. + d_j \frac{1}{\left(\rho^2 - \lambda^2\right)\Delta(\rho)} \right] - \frac{c_{44}}{\sqrt{\nu_3}} y d_3 \frac{1}{\left(\rho^2 - \lambda^2\right)\Delta(\rho)}. \tag{38}$$

The stressed-deformed state of a transversely isotropic body with a thermally insulating parabolic crack acted on by a uniform thermal flux (13) is, therefore, given by the sum of the displacements in Eqs. (21) and (29):

$$u = u_1 + u_2; \quad v = v_1 + v_2; \quad w = w_1 + w_2. \tag{39}$$

These displacements satisfy conditions (12) if algebraic equations (31) are satisfied. The stress components in the body are given by Eqs. (25), with the potential functions Φ_j ($j = \overline{1,4}$) given by Eqs. (26) and (28). Boundary conditions (11) and (12) for the forces are met if algebraic equations (34) are satisfied. The unknown constants c_1, c_2, a_1, a_2, d_1, d_2, and d_3 are found using the system of equations (31), (34), and (37). The constant c_4 is given by Eq. (27).

In solving problems for the limiting equilibrium state of a body with a crack, it is important to know the stress intensity factors, which are found using the formulas

$$K_I = \sqrt{2\pi}\lim_{l\to 0}\sqrt{l}\,\sigma_z;$$

$$K_{II} = \sqrt{2\pi}\lim_{l\to 0}\sqrt{l}\left(\tau_{xz}n_x + \tau_{yz}n_y\right);$$

$$K_{III} = \sqrt{2\pi}\lim_{l\to 0}\sqrt{l}\left(-\tau_{xz}n_y + \tau_{yz}n_x\right), \tag{40}$$

where σ_z, τ_{xz}, and τ_{yz} are the components of the stresses along an extension of the crack into the xy plane; l is the distance to the normal to the edge of the crack; n_x and n_y are the direction cosines of the normal to the parabola ($\rho_j = \mu_j = b$), which are calculated using the formulas

$$n_x = \frac{\sqrt{b^2 - a^2}}{\sqrt{b^2 - \lambda^2}}; \quad n_y = \frac{y}{\sqrt{\left(b^2 - a^2\right)\left(b^2 - \lambda^2\right)}}. \tag{41}$$

Given that along the crack contour ($\rho_j = \mu_j = b$)

$$2x_0 = \lambda^2 - a^2 + b^2; \;\; y_0^2 = \left(b^2 - a^2\right)\left(a^2 - \lambda^2\right); \;\; z_0 = 0, \tag{42}$$

and that [4]

$$\lim_{l\to 0}\frac{\sqrt{l}}{\sqrt{\rho^2 - b^2}} = \frac{\sqrt[4]{b^2 - \lambda^2}}{\sqrt{2}\sqrt[4]{b^2 - a^2}} = \frac{1}{\sqrt{2}}\left[1 + \frac{y_0^2}{\left(b^2 - a^2\right)^2}\right]^{1/4}, \tag{43}$$

as well, for the stress intensity factors at the crack tip we find

$$K_I = 0;$$

$$K_{II} = \frac{\sqrt{\pi}c_{44}}{\sqrt{b^2 - a^2}}\left[1 + \frac{y_0^2}{\left(b^2 - a^2\right)^2}\right]^{3/4}\sum_{j=1,2}\frac{1 + k_j}{\sqrt{\nu_j}}\left\{-a_j + d_j\left[1 + \frac{y_0^2}{\left(b^2 - a^2\right)^2}\right]^{-1}\right\};$$

$$K_{III} = -\frac{\sqrt{\pi}}{\sqrt{\nu_3}}y_0 c_{44} d_3\left(b^2 - a^2\right)^{-3/2}\left[1 + \frac{y_0^2}{\left(b^2 - a^2\right)^2}\right]^{-1/4}, \tag{44}$$

where

$$\sum_{j=1,2}\frac{1 + k_j}{\sqrt{\nu_j}}a_j = -\sqrt{\nu_4}Q\left(b^2 - a^2\right)^2\left\{\frac{k_3\left(c_{33}k_4 - \nu_4 c_{13}\right) - \beta'}{2c_{44}\left(k_1 - k_2\right)}\left[\frac{1 + k_2}{\sqrt{\nu_2}} - \right.\right.$$

$$\left.\left. - \frac{1 + k_1}{\sqrt{\nu_1}}\right] + \frac{1}{2}\nu_4 k_3\left[\frac{\left(1 + k_1\right)\left(1 + k_2\right)}{k_2 - k_1}\left(\frac{1}{\sqrt{\nu_2}} - \frac{1}{\sqrt{\nu_1}}\right) + \frac{1 + k_4}{\sqrt{\nu_4}}\right]\right\};$$

$$\sum_{j=1,2}\frac{1+k_j}{\sqrt{\nu_j}}d_j=\frac{(1+k_1)(1+k_2)\sqrt{\nu_3}[1+2X_0(b)]}{k_1-k_2}\left(\frac{1}{\sqrt{\nu_2}}-\frac{1}{\sqrt{\nu_1}}\right)\sum_{j=1,2}\frac{1+k_j}{\sqrt{\nu_j}}a_j;$$

$$d_3=\sqrt{\nu_3}[1+2X_0(b)]\sum_{j=1,2}\frac{1+k_j}{\sqrt{\nu_j}}a_j. \tag{45}$$

We have, therefore, constructed an explicit static thermoelastic solution for an infinite transversely isotropic body containing a thermally insulating parabolic crack in its plane of symmetry. The crack surface is force free. A uniform heat flux is incident on the crack perpendicular to its surface. Formulas have been obtained for calculating the stress intensity factors.

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