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# A non-singular inverse Vitali lemma with applications

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*Abstract*. A proof for a non-singular version of the inverse Vitali lemma is given. The result is used to describe non-singular orbit equivalence within the framework of Rudolph's restricted orbit equivalence and in the construction of an alternative proof of the Hurewicz ergodic theorem.

## 1. *Introduction*

In extending ergodic theory from a measure-preserving to a non-singular setting many of the problems we encounter are technical rather than fundamental. Frequently, once these technical barriers have been overcome, it is straightforward to establish a theorem for non-singular actions by generalizing the corresponding measure-preserving result.

It is in this light that we examine the inverse Vitali lemma. As we show in this paper, a non-singularized version of this result opens the door to a number of possibilities.

While the basic form of the non-singularized proof is taken from the measure-preserving proof given in [**5**], the ideas which allow us to extend the result are due to Halmos [**1**].

The first use of our main result is demonstrated in §3, in which we establish the foundations of a non-singular version of Rudolph's restricted orbit equivalence. This part of our work is still in its early stages and does not yet deal with any form of entropy. However, we are able to define non-singular orderings and sizes, and we can show that much of the basic machinery still works in a non-singular context. In particular, we present a size for non-singular orbit equivalence, which we denote as $m_0$. The inverse Vitali lemma is used to show that $m_0$ is in fact a size and that all orbit equivalent dynamical systems give rise to orderings which are $m_0$ equivalent.

It will be assumed that the reader has some familiarity with [**4**], from which most of our technical work in §3 is adapted. It should be noted that although the restricted orbit equivalence as presented in its original form in [**4**] deals exclusively with $\mathbb{Z}$ actions, the theory has since been extended to the actions of more general groups. (See [**2**] and [**3**].) At this preliminary stage of development we have chosen to stay close to the original format as we believe that in doing so we can better highlight the problems involved in extending the theory to include non-singular actions.

Our second application is an alternate non-Halmos type proof of the Hurewicz ergodic theorem. Now that we have a non-singular inverse Vitali lemma at our disposal, the Ornstein–Weiss style proof of the Birkhoff ergodic theorem given in [**5**] is easily generalized.

## 2. *The inverse Vitali lemma*

Our proof of the non-singular version of the inverse Vitali lemma is based on that given for the standard measure-preserving case in [**5**]. We first state the result.

LEMMA 2.1. (Inverse Vitali lemma) *Let $T$ be an ergodic, non-singular, non-periodic invertible automorphism on $(X, \mathcal{B}, \mu)$. Further suppose that for some measurable $A \subset X$ with $\mu(A) > 0$ there are measurable integer valued functions $i_k(x)$, $j_k(x)$ such that $\lim_{k\to\infty} j_k(x) - i_k(x) = \infty$ for all $x \in A$. Then there is some subset $B \subset A$ such that for all $x \in B$ there are functions $i(x) \le 0 \le j(x)$ where $i(x) = i_k(x)$, $j(x) = j_k(x)$ for some $k$ depending on $x$, so that the orbit intervals $I(x) = \cup_{i(x)}^{j(x)} T^i(x)$ are all pairwise disjoint and*

$$\mu(A \setminus \cup_{x\in B} I(x)) < \epsilon.$$

There is only one real problem in trying to generalize the proof given in [**5**] to the non-singular case. In covering a set with orbit intervals we obviously need to know how much measure we are including with each step. For a measure-preserving transformation $T$ we know that if the length of an orbit interval is extended by a certain factor then its associated mass will also be increased by the same proportion. In our non-singular proof we need some similar control over the Radon derivatives. For this reason we first establish an intuitively obvious fact.

LEMMA 2.2. *Let $\omega_i(x) = (dT^{-i}\mu/d\mu)(x)$, and suppose $\sum_{i=0}^{\infty} \omega_i(x) = \infty$ for $\mu$ almost all $x \in X$. Then*

$$\lim_{n\to\infty} \frac{\omega_n(x)}{\sum_{i=0}^{n-1} \omega_i(x)} = 0$$

*for $\mu$ almost all $x \in X$.*

*Proof.* For a real valued function $q(x)$ and $n \in \mathbb{Z}^+$ let

$$q^n(x) = \sum_{i=0}^{n-1} q(T^i(x))\omega_i(x).$$

Then if $q_c(x) = \omega_1(x) - (1+c)$, we have

$$q_c^n(x) = \omega_n(x) - c\sum_{i=1}^{n-1} \omega_i(x) - (1+c).$$

For $A \subseteq X$ a $T$ invariant set and $c \in \mathbb{R}$, let

$$E_A^+(c) = \bigcup_{n=1}^{\infty} \{x \in A : q_c^n(x) \ge 0\}$$

and

$$E_A^-(c) = \bigcup_{n=1}^{\infty} \{x \in A : q_c^n(x) \le 0\}.$$

Then

$$\int_{E_A^+(c)} q_c(x)\, d\mu \ge 0 \quad \text{and} \quad \int_{E_A^-(c)} -q_c(x)\, d\mu \ge 0.$$

For proof see [**1**]. So

$$\int_{E_A^+(c)} \omega_1(x)\, d\mu \ge (1+c)\mu(E_A^+(c)) \quad \text{and} \quad (1+c)\mu(E_A^-(c)) \ge \int_{E_A^-(c)} \omega_1(x)\, d\mu.$$

Now the sequence

$$a_n(x) = \frac{\omega_n(x) - 1}{\sum_{i=0}^{n-1} \omega_i(x)}$$

is bounded $\mu$-almost everywhere. For if, for some set $B \subseteq X$, $x \in B$ implies that $a_n(x)$ is unbounded then

$$B = \bigcap_{c>0} E_X^+(c) \bigcup \bigcap_{c>0} E_X^-(-c)$$

and so by the above inequalities, $\mu(B) = 0$.

Note that

$$a_n(T(x)) = \frac{\omega_{n+1}(x) - \omega_1(x)}{\sum_{i=0}^{n-1} \omega_{i+1}(x)}$$

and so $\limsup a_n(x)$ and $\liminf a_n(x)$ are $\mu$-almost everywhere $T$ invariant. Hence if we let

$$A_{\alpha,\beta} = \{x : \liminf a_n(x) < \alpha < \beta < \limsup a_n(x)\}$$

for $\alpha, \beta \in \mathbb{Q}$, then $A_{\alpha,\beta}$ is a $T$ invariant set. So

$$\int_{E^+_{A_{\alpha,\beta}}(c)} \omega_1(x)\, d\mu \ge (1+c)\mu(E^+_{A_{\alpha,\beta}}(c)) \quad \text{and} \quad (1+c)\mu(E^-_{A_{\alpha,\beta}}(c)) \ge \int_{E^-_{A_{\alpha,\beta}}(c)} \omega_1(x)\, d\mu$$

for $c \in \mathbb{R}$.

Now $E_X^-(\alpha) \cap A_{\alpha,\beta} = E^-_{A_{\alpha,\beta}}(\alpha)$, but $A_{\alpha,\beta} \subseteq E_X^-(\alpha)$, so $A_{\alpha,\beta} = E^-_{A_{\alpha,\beta}}(\alpha)$. Similarly, $A_{\alpha,\beta} = E^+_{A_{\alpha,\beta}}(\beta)$. So

$$\int_{A_{\alpha,\beta}} \omega_1(x)\, d\mu \ge (1+\beta)\mu(A_{\alpha,\beta})$$

and

$$(1+\alpha)\mu(A_{\alpha,\beta}) \ge \int_{A_{\alpha,\beta}} \omega_1(x)\, d\mu.$$

But as $\alpha < \beta$ this is not possible unless $\mu(A_{\alpha,\beta}) = 0$. Hence $\lim_{n\to\infty}(\omega_n(x) - 1)/\sum_{i=0}^{n-1} \omega_i(x)$ exists $\mu$-almost everywhere so it is easy to show that

$$\lim_{n\to\infty} \frac{\omega_n(x)}{\sum_{i=0}^{n-1} \omega_i(x)} = 0$$

on a set of full measure. $\square$

Note that this proof could easily be extended to show that $\lim_{n\to\infty} \sum_{i=n}^{n+p} \omega_i(x)/\sum_{i=0}^{n} \omega_i(x) = 0$ for any finite $p \in \mathbb{N}$, or that $\lim_{n\to\infty} \omega_{-n}(x)/\sum_{i=0}^{-n} \omega_i(x) = 0$.

LEMMA 2.3. *Suppose that for $x \in A \subseteq X$, $\mu(A) > 0$ there are bounded integer valued functions $i(x) \le 0 \le j(x)$ such that $N_1 < j(x) - i(x) + 1 < N_2$ where $N_1$ is so large that*

$$\sum_{i-1}^{j+1} \frac{dT^{-r}\mu}{d\mu}(x) < (1+2\epsilon)\sum_{i}^{j} \frac{dT^{-r}\mu}{d\mu}(x)$$

*for all $i, j \in \mathbb{N}$ with $j - i + 1 > N_1$ and $\epsilon < \mu(A)/16$. Then there is subset $A' \subset A$ so that if $I(x) = \bigcup_{i=i_k(x)}^{j_k(x)} T^i(x)$ then $I(x)$ is disjoint from $I(x')$ for $x, x' \in A'$ where $x \ne x'$ and*

$$\mu(A) \le 4\mu\bigg(\bigcup_{x\in A'} I(x)\bigg).$$

*Proof.* Find a subset $X_1 \subset X$ of measure at least $1-\epsilon$, and $N_3 \in \mathbb{N}$ so that for $x \in X_1$,

$$\sum_{r=N_3-N_2}^{N_3} \frac{dT^{-r}\mu}{d\mu}(x) \bigg/ \sum_{r=0}^{N_3} \frac{dT^{-r}\mu}{d\mu}(x) < \epsilon.$$

By using the induced transformation on the set $X_1$ and building a Rokhlin tower of height $N_3$, construct a Kakutani skyscraper for $T$ with base $B \subseteq X_1$ such that each column has height $h(x) \ge N_3$ for all $x \in B$.

For each $x \in A$ there is a $y \in B$ such that $x = T^{p(x)}(y)$ where $0 \le p(x) \le h(y)$ and an interval $[i(x), j(x)]$ to which we ascribe the 'length'

$$l(x) = \sum_{r=i(x)}^{j(x)} \frac{dT^{-(p(x)+r)}}{d\mu}(y).$$

We cover the points in $A$ contained in $P(y) = \cup_{r=0}^{h(y)-N_2} T^r(y)$, $(y \in B)$ by the following method: list all $x \in A \cap P(y)$ in descending order of length $l(x)$ i.e. label the points $x_1, x_2, \dots$, etc, where $l(x_r) \ge l(x_{r+1})$. Start with $x_1$ and cover an interval of points

$$I(x_1) = \cup_{i(x_1)}^{j(x_1)} T^r(x_1)$$

in $P(y)$ with length $l(x_1)$. Next choose $x_s$ where $l(x_s)$ is the largest length remaining with $I(x_s)$ disjoint from $I(x_1)$.

Continue with this method of disjoint covering until no further intervals can be chosen without intersecting with those already selected. For $y \in B$, define $R(y) \subset [0, h(y) - N_2]$ as the set of integers such that $k \in R(y)$ if and only if $T^k(y) \in A$ and $I(T^k(y))$ is chosen as a covering interval.

Let $L(x) = 3l(x)$ and define

$$I'(x) = \sum_{i'(x)}^{j'(x)} T^r(x)$$

where $i'(x)$ is the greatest integer such that

$$\sum_{r=i'(x)}^{i(x)-1} \frac{dT^{-(p+r)}\mu}{d\mu}(y) \geq l(x)$$

and $j'(x)$ is the least integer such that

$$\sum_{r=j(x)+1}^{j'(x)} \frac{dT^{-(p+r)}\mu}{d\mu}(y) \geq l(x)$$

where $x = T^p(y)$ for $y \in B$.

Now

$$A \cap P(y) \subseteq \cup_{k \in R(y)} I'(T^k(y)).$$

For if we suppose otherwise then for some $x' \in P(y)$ we would have $I(x')$ disjoint from all the selected orbit intervals $I(T^k(y))$ $(k \in R(y))$. But this would mean that $I(x')$ would already have been included in our covering, which would be a contradiction.

Defining $A'$ as the set

$$\cup_{y \in B} \cup_{k \in R(y)} T^k(y),$$

we can see that as

$$\sum_{r=0}^{h(y)-N_2} \frac{dT^{-r}\mu}{d\mu}(y) \geq (1-\epsilon) \sum_{r=0}^{h(y)} \frac{dT^{-r}\mu}{d\mu}(y)$$

for $y \in B$, then

$$\begin{aligned}\mu(A) - \epsilon &\leq \int_{y \in B} \sum_{k \in R(y)} L(T^k(y))\, d\mu(y) \\ &\leq 3(1+2\epsilon)\mu(\cup_{x \in A'} I(x))\end{aligned}$$

and as $\epsilon < \mu(A)/16$,

$$\mu(A) \leq 4\mu(\cup_{x \in A'} I(x)). \qquad \square$$

Now we come to the real heart of the inverse Vitali argument. The following lemma tells us that we can cover all but $\epsilon$ in measure of a set with disjoint orbit intervals providing we start with a suitable finite set of integer valued function pairs $i_k(x) \leq 0 \leq j_k(x)$. The covering is achieved by an inductive process, using Lemma 2.3 to cover a quarter in measure of what remains uncovered at each stage. The final proof of the inverse Vitali lemma is the construction of an appropriate set of functions from the infinite series $i_k(x), j_k(x)$.

LEMMA 2.4. *Choose $1/2 > \epsilon > 0$ and a large $M \in \mathbb{N}$ such that $\epsilon M/16 > 1$. Suppose we have a set $B \subseteq X$, $\mu(B) > \epsilon/2$ such that for $x \in B$ we have bounded integer valued functions $i_k(x) \leq 0 \leq j_k(x)$, $(k = 1, 2, \ldots, M)$. Let*

$$n_k^{\inf} = \inf_{x \in B}(j_k(x) - i_k(x) + 1) \quad \textit{and} \quad n_k^{\sup} = \sup_{x \in B}(j_k(x) - i_k(x) + 1)$$

*and let* $N_1 < \min_{k\in(1,2,\dots,M)}\{n_k^{\inf}\}$ *be so large that*

$$\sum_{i-1}^{j+1} \frac{dT^{-r}\mu}{d\mu}(x) < \left(1+\frac{\epsilon}{16}\right)\sum_{i}^{j} \frac{dT^{-r}\mu}{d\mu}(x)$$

*for all* $x \in B$ *and all* $i, j$ *with* $i \le 0 \le j$ *and* $j - i + 1 > N_1$*. Further suppose that*

$$\sum_{i_k(x)-n_{k+1}^{\sup}}^{j_k(x)+n_{k+1}^{\sup}} \frac{dT^{-r}\mu}{d\mu}(x) < \left(1+\frac{\epsilon}{2}\right)\sum_{i_k(x)}^{j_k(x)} \frac{dT^{-r}\mu}{d\mu}(x)$$

*for* $k = 1, 2, \dots, M-1$*. Then there is a subset* $B' \subseteq B, \mu(B') > 0$ *and measurable functions* $i(x) \le 0 \le j(x)$ *for* $x \in B'$ *with* $(i(x), j(x)) = (i_k(x), j_k(x))$ *for some* $k$*,* $(k = 1, 2, \dots, M)$ *depending on* $x$ *so that the sets*

$$I(x) = \cup_{r=i(x)}^{j(x)} T^r(x)$$

*are pairwise disjoint for* $x \in B'$ *and*

$$\mu(B \setminus \cup_{x\in B'} I(x)) < \epsilon.$$

*Proof.* First of all we define

$$i'_k(x) = i_k(x) - \sup_{x\in B}(j_{k+1}(x) - i_{k+1}(x) + 1)$$
$$j'_k(x) = j_k(x) + \sup_{x\in B}(j_{k+1}(x) - i_{k+1}(x) + 1)$$

for $k = 1, 2, \dots, M-1$ and $i'_M(x) = i(x)$, $j'_M(x) = j_M(x)$. Using Lemma 2.3 we find a subset $B'_1 \subset B$ with

$$I'_1(x) = \cup_{i=i'_1(x)}^{j'_1(x)} T^i(x)$$

pairwise disjoint and

$$\mu(\cup_{x\in B'_1} I'(x)) \ge \frac{\mu(B)}{4}.$$

Now take

$$B_2 = B \setminus \cup_{x\in B'_1} I'_1(x).$$

If $\mu(B_2) > \epsilon/2$ we can reapply Lemma 2.3 to construct $B'_2 \subseteq B_2$ so that

$$\mu(\cup_{x\in B'_2} I'_2(x)) \ge \frac{\mu(B_2)}{4}.$$

Continue the induction, setting

$$B_k = B \setminus \cup_{i=1}^{k-1} \cup_{x\in B'_i} I'_i(x).$$

As long as $\mu(B_k) > \epsilon/2$ we can find $B'_k \subseteq B_k$ so that

$$\mu(\cup_{x\in B'_k} I'_k(x)) \ge \frac{\mu(B_k)}{4}.$$

We do this up to $M$ times using $M$ function pairs $i'_k(x)$, $j'_k(x)$.

Now $B' = \cup_{k=1}^M B'_k$ is a disjoint union hence it is not hard to show that

$$I(x) = \cup_{i=i(x)}^{j(x)} T^i(x) \quad \text{and} \quad I(x') = \cup_{i=i(x')}^{j(x')} T^i(x')$$

are disjoint for $x \neq x'$. (Here $i(x)$, $j(x)$ are defined to be $i_k(x)$, $j_k(x)$ where $k$ is the unique integer $1 \leq k \leq M$ such that $x \in B'_k$.)

Let $B_{M+1}$ be the remaining piece of $B$ which we have not yet covered with disjoint orbit intervals $I'(x)$, i.e.

$$B_{M+1} = B \setminus \cup_{k=1}^M \cup_{x \in B'_k} I'(x).$$

We wish to prove that $\mu(B_{M+1}) < \epsilon/2$. Suppose $\mu(B_k) \geq \epsilon/2$ for all $k = 1, 2, \ldots, M+1$. Now

$$\begin{aligned}\mu(\cup_{x \in B'_k} I'(x)) &< (1+\epsilon/2)\mu(\cup_{x \in B'_k} I(x)) \\ &< \mu(\cup_{x \in B'_k} I(x)) + \epsilon/2\mu(\cup_{x \in B'_k} I'(x)).\end{aligned}$$

So

$$\begin{aligned}\mu(\cup_{x \in B'_k} I(x)) &\geq (1-\epsilon/2)\mu(\cup_{x \in B'_k} I'(x)) \\ &\geq (1-\epsilon/2)\frac{\mu(B_k)}{4}\end{aligned}$$

for all $k = 1, 2, \ldots, M$ and because of the disjointness of the $B'_k$ and the fact that $\mu(B_{M+1}) \leq \mu(B_r)$ we can say

$$\mu(\cup_{x \in B'} I(x)) \geq M(1-\epsilon/2)\frac{\epsilon}{8} > \frac{M\epsilon}{16} > 1$$

which is a contradiction.

Hence

$$\begin{aligned}\mu(B \setminus \cup_{x \in B'} I(x)) &\leq \mu(B_{M+1}) + \mu(\cup_{x \in B'} I'(x) \setminus I(x)) \\ &< \epsilon/2 + \epsilon/2\mu(\cup_{x \in B'} I(x)) \\ &< \epsilon\end{aligned}$$

as required. □

*Proof of the inverse Vitali lemma.* Choose $\epsilon/2 > 0$ and $M \in \mathbb{N}$ such that $\epsilon M/32 > 1$. We need to construct $M$ function pairs on a set $A_M \subset A$ which satisfy the hypotheses of Lemma 2.4.

First find an $N \in \mathbb{N}$ and a set $A_0 \subset A$ with $\mu(A_0) > \mu(A) - \epsilon/2^2$, such that for any $j - i + 1 > N$, $i \leq 0 \leq j$ and all $x \in A_0$,

$$\sum_{i-1}^{j+1} \frac{dT^{-r}\mu}{d\mu}(x) < \left(1 + \frac{\epsilon}{32}\right) \sum_{i}^{j} \frac{dT^{-r}\mu}{d\mu}(x).$$

Now on a subset $A_1 \subset A_0$ with $\mu(A_1) > \mu(A_0) - \epsilon/2^3$, define bounded functions $\hat{\imath}_1(x) = i_{k(x)}(x)$ and $\hat{\jmath}_1(x) = j_{k(x)}(x)$ where the $k(x)$ are chosen measurably so that $j_{k(x)}(x) - i_{k(x)}(x) + 1 > N$.

Next construct $\hat{\imath}_2(x) = i_{k(x)}$, $\hat{\jmath}_2(x) = j_{k(x)}(x)$ bounded on $A_2 \subset A_1$ with $\mu(A_2) > \mu(A_1) - \epsilon/2^4$ and the $k(x)$ are chosen to ensure $\hat{\jmath}_2(x) - \hat{\imath}_2(x) + 1 > N$ and that for all $x \in A_2$,

$$\sum_{\hat{\imath}_2(x)-n_1^{\sup}}^{\hat{\jmath}_2(x)+n_1^{\sup}} \frac{dT^{-k}\mu}{d\mu}(x) < \left(1 + \frac{\epsilon}{4}\right) \sum_{\hat{\imath}_2(x)}^{\hat{\jmath}_2(x)} \frac{dT^{-k}\mu}{d\mu}(x)$$

where $n_1^{\sup} = \sup_{x \in A_1} \hat{\jmath}_1(x) - \hat{\imath}_1(x) + 1$.

Continuing in this fashion we obtain $M$ bounded function pairs $\hat{\imath}_k(x)$, $\hat{\jmath}_k(x)$ on $A_M$. Now for $x \in A_M$ the $\hat{\imath}_k(x)$, $\hat{\jmath}_k(x)$ listed in reverse order satisfy the required conditions for Lemma 2.4. Hence we can cover all but $\epsilon/2$ in measure of $A_M$ with disjoint orbit intervals $I(x) = \bigcup_{\hat{\imath}(x)}^{\hat{\jmath}(x)} T^k(x)$ for $x \in A' \subset A_M$. But

$$\begin{aligned} \mu(A_M) &> \mu(A) - \epsilon \sum_{k=2}^{M+2} \frac{1}{2^k} \\ &> \mu(A) - \epsilon/2 \end{aligned}$$

and so clearly we are done. □

## 3. *Non-singular orderings and sizes*

To begin our modification of restricted orbit equivalence we start with the definition of an ordering.

*Definition 3.1.* For a non-singular ergodic transformation system $(X, \mathcal{B}, \mu, T)$ an *ordering* will be a map

$$\alpha : X \times X \to \mathbb{Z}$$

where $x' = T^{\alpha(x,x')}(x)$.

We will only be comparing orderings within orbit equivalence classes; that is, if we write $m(\alpha_1, \alpha_2)$ for some size $m$ or any other expression involving the two orderings then we are assuming that there is some invertible orbit map $\phi : X_1 \to X_2$ with $\mu_1 \circ \phi \sim \mu_2$ and

$$\phi(T_1^i(x)) = T_2^{j(i,x)}(\phi(x)).$$

Also, any orbit map between two orderings is assumed to induce an orbit equivalence.

*Definition 3.2.* We say that $\alpha_1$ and $\alpha_2$ *differ by a co-boundary* if there is an orbit map $\phi : X_1 \to X_2$ where

$$\phi(T_1^i(x)) = T_2^{j(i,x)}(\phi(x))$$

and a measurable function $f : X_1 \to \mathbb{Z}$ such that

$$\alpha_1(x, x') - \alpha_2(\phi(x), \phi(x')) = f(x) - f(x').$$

LEMMA 3.3. *The orderings $\alpha_1, \alpha_2$ differ by a co-boundary if and only if $T_1$ and $T_2$ are conjugate.*

The proof of this is an easy extension of the calculation of [**4**, pp. 4–6].

*Notation.* Let $\phi : X_1 \to X_2$ be an orbit map for orderings $\alpha_1, \alpha_2$ on $X_1$, $X_2$ respectively, and $x \in X_1$. Define $f_{x,\phi}^{\alpha_1,\alpha_2} : \mathbb{Z} \to \mathbb{Z}$ by

$$f_{x,\phi}^{\alpha_2,\alpha_2}(i) = j$$

whenever $\alpha_2(\phi(x), \phi(T_1^i(x))) = j$. Denote

$$\Pi_{x,\phi,(i,j)}^{\alpha_1,\alpha_2}$$

as the permutation of $(i, j)$ which re-orders the interval in the same order as $f_{x,\phi}^{\alpha_1,\alpha_2}$. We will now discuss non-singular sizes. For a size $m$, the value

$$m(\Pi_{x,\phi,(i,j)}^{\alpha_1,\alpha_2})$$

will no longer depend just on the permutation. It may also depend on the interval $(i, j)$, on $x$, and on the action of $T_1$ on $(X_1, \mathcal{B}_1, \mu_1)$. So the notation is in fact something of a shorthand. Note that if $\Pi_{x,\phi,(i,j)}^{\alpha_1,\alpha_2} = \Pi_{x,\psi,(i,j)}^{\alpha_1,\alpha_3}$ then $m(\Pi_{x,\phi,(i,j)}^{\alpha_1,\alpha_2}) = m(\Pi_{x,\psi,(i,j)}^{\alpha_1,\alpha_3})$.

For a size $m$ (axioms described below) let

$$m(f_{x,\phi}^{\alpha_1,\alpha_2}) = \liminf_{i\to-\infty, j\to\infty} m(\Pi_{x,\phi,(i,j)}^{\alpha_1,\alpha_2}).$$

By axiom ii(a) below this is constant for almost all $x$ by ergodicity; we shall denote it as $m(f_{\phi}^{\alpha_1,\alpha_2})$. When we write $m(\alpha_1, \alpha_2)$ we mean $m(f_{\phi}^{\alpha_1,\alpha_2})$ where $\phi$ is the identity map.

We now present our definition of a size in a non-singular context, generalizing axioms i–vi given in [**4**, pp. 7–8]. Note that all our axioms are stated in terms of $m(f_{\phi}^{\alpha_1,\alpha_2})$ or $m(\Pi_{x,\phi,(i,j)}^{\alpha_1,\alpha_2})$ rather than in terms of $m(\alpha_1, \alpha_2)$. Of course, these axioms may be in need of refinement for future developments, however they are sufficient for our immediate purposes.

*Definition 3.4.* We call $m$ a *size* if it satisfies the following axioms:

i(a) $m(\mathrm{id}) = 0$.

ii(a) $m(\Pi_{T_1(x),\phi,(i,j)}^{\alpha_1,\alpha_2}) = m(\Pi_{x,\phi,(i+1,j+1)}^{\alpha_1,\alpha_2})$.

iii(a) If for orderings $\alpha_1, \alpha_2$ for all $\epsilon > 0$ there exists an orbit map $\phi : X_1 \to X_2$ such that $m(f_{\phi}^{\alpha_1,\alpha_2}) < \epsilon$ then for each $\epsilon > 0$ there exists an orbit map $\psi : X_2 \to X_1$ such that $m(f_{\psi}^{\alpha_2,\alpha_1}) < \epsilon$.

iv(a) For all $\epsilon > 0$ there is a $\delta$ so that if $m(\Pi_{x,\phi,(i,j)}^{\alpha_1,\alpha_2}) < \delta$ then for all but a subset $I \subset (i, j)$ with

$$\sum_I \frac{dT_1^{-k}\mu_1}{d\mu_1}(x) < \epsilon \sum_i^j \frac{dT_1^{-k}\mu_1}{d\mu_1}(x)$$

we have

$$\Pi_{x,\phi}^{\alpha_1,\alpha_2}(k+1) = \Pi_{x,\phi}^{\alpha_1,\alpha_2}(k) + 1.$$

vi(a) For all $\epsilon > 0$ there is a $\delta_1$ so that for each $\alpha_2$ with $m(f_{\phi}^{\alpha_1,\alpha_2}) < \delta_1$ there is a $\delta_2$ such that for all $\alpha_3$ with $m(f_{\psi}^{\alpha_2,\alpha_3}) < \delta_2$ we have $m(f_{\psi\phi}^{\alpha_1,\alpha_3}) < \epsilon$.

We have included in this list of axioms only those which are either technically necessary (i(a), ii(a), iii(a) and vi(a)) for our work here or intuitively desirable (iv(a)). In [**4**], axiom v is needed mainly for the construction of $m$-entropy and its use in the $m$-equivalence theorem. As we have not yet developed a non-singular version of $m$ entropy we have not given an analogue of axiom v.

*Definition 3.5.* We say orderings $\alpha_1$ and $\alpha_2$ are *m-equivalent* (written $\alpha_1\tilde{m}\alpha_2$) if for each $\epsilon > 0$ there is an $\alpha_2^\epsilon$ differing from $\alpha_2$ by a co-boundary such that $m(\alpha_1, \alpha_2^\epsilon) < \epsilon$.

By generalizing the proof of Theorem 2.1 in [**4**] we have the following.

LEMMA 3.6. *Non-singular m-equivalence is an equivalence relation.*

We now present our size for non-singular orbit equivalence.

*Definition 3.7.* (Size for non-singular orbit equivalence) Let

$$m_0(\Pi^{\alpha_1,\alpha_2}_{x,\phi,(i,j)}) = \sum_{k\in I} \frac{dT_1^{-k}\mu_1}{d\mu_1}(x) \bigg/ \sum_{k=i}^{j-1} \frac{dT_1^{-k}\mu_1}{d\mu_1}(x)$$

where $I \subset (i, j-1)$ is such that $k \in I$ implies that

$$\Pi^{\alpha_1,\alpha_2}_{x,\phi}(k+1) \neq \Pi^{\alpha_1,\alpha_2}_{x,\phi}(k) + 1.$$

Our new $m_0$ is still consistent with the measure-preserving size for orbit equivalence given in [**4**]. Clearly $m_0$ satisfies axioms i(a) and iv(a). For axiom ii(a) simply note that

$$\frac{dT^{-n}\mu}{d\mu}(Tx) = \frac{dT^{-(n+1)}\mu}{d\mu}(x) \bigg/ \frac{dT^{-1}\mu}{d\mu}(x).$$

The following three results are modified versions of Lemmas 2.6, 2.7 and 2.8 from [**4**].

LEMMA 3.8. *For orderings $\alpha_1, \alpha_2$, $m_0(f_\phi^{\alpha_1,\alpha_2}) < a$ if and only if there is a set $A \subset X_1$ with $\mu_1(A) > 1 - a$ such that $x \in A$ implies that*

$$f^{\alpha_1,\alpha_2}_{x,\phi}(1) = 1.$$

*Proof.* Let $m_0(f_\phi^{\alpha_1,\alpha_2}) < \bar{a} < a < 1$.

So for almost all $x \in X_1$ we have a sequence of intervals $i_k(x) \to -\infty$, $j_k(x) \to \infty$ such that

$$m_0(\Pi^{\alpha_1,\alpha_2}_{x,\phi,(i_k(x),j_k(x))}) < \bar{a}.$$

Hence for each $k \in \mathbb{N}$, and almost all $x \in X_1$, there is a subset $S_{k,x} \subset (i_k(x), j_k(x))$ with

$$\sum_{i\in S_{k,x}} \frac{dT_1^{-i}\mu_1}{d\mu_1}(x) > (1-\bar{a}) \sum_{i=i_k(x)}^{j_k(x)} \frac{dT_1^{-i}\mu_1}{d\mu_1}(x)$$

such that if $l \in S_{k,x}$ then

$$\Pi^{\alpha_1,\alpha_2}_{x,\phi,(i_k(x),j_k(x))}(l+1) = \Pi^{\alpha_1,\alpha_2}_{x,\phi,(i_k(x),j_k(x))}(l) + 1.$$

Set $\epsilon$ and $N$. Find a set $A' \subset X_1$ with $\mu_1(A') > 1 - \epsilon$ and an $N_1(\gg N)$ so that if $j - i + 1 \geq N_1$ and $I \subset (i, j)$ has $\#I \leq N$ then for $x \in A'$

$$\sum_{i+N}^{j-N} \frac{dT_1^{-l}\mu_1}{d\mu_1}(x) > (1-\epsilon) \sum_{i}^{j} \frac{dT_1^{-l}\mu_1}{d\mu_1}(x).$$

Restricting ourselves to function pairs $i_k(x),\ j_k(x)$ with $j_k(x) - i_k(x) + 1 > N_1$, we apply the inverse Vitali lemma to cover all but $\epsilon$ in measure of $A'$ with disjoint $T_1$ orbit intervals; i.e.

$$C_{N,\epsilon} = \cup_{x \in B_{N,\epsilon}} \cup_{i(x)}^{j(x)} T_1^i(x), \quad B_{N,\epsilon} \subset A'$$

is a disjoint union and $\mu_1(A' \setminus C_{N,\epsilon}) < \epsilon$. Now let

$$A_{N,\epsilon} = \cup_{x \in B_{N,\epsilon}} \cup_{S_{k,x} \cap (i(x)+N, j(x)-N)} T_1^i(x).$$

As $B_{N,\epsilon} \subset A'$ we have

$$\mu_1(\cup_{S_{k,x} \cap (i(x)+N, j(x)-N)} T_1^i(x)) > (1 - \bar{a} - \epsilon)\mu_1(\cup_{i(x)}^{j(x)} T_1^i(x)).$$

So

$$\begin{aligned} \mu_1(A_{N,\epsilon}) &> (1 - \bar{a} - \epsilon)\mu_1(C_{N,\epsilon}) \\ &> 1 - \bar{a} - 3\epsilon. \end{aligned}$$

Put $\epsilon = 2^{-N}$. If $A = \limsup_N A_{N,2^{-N}}$ then $\mu_1(A) \geq 1 - \bar{a} > 1 - a$.

Now $x \in A$ if and only if $x \in A_{N_r,2^{-N_r}}$ for an infinite sequence $\{N_r\}_{r=1}^{\infty}$. If $x = T_1^l(\bar{x})$ for $\bar{x} \in B_{N_r,2^{-N_r}}$, $l \in S_{k,\bar{x}} \cap (i(\bar{x}) + N_r,\ j(\bar{x}) - N_r)$, then putting $u_r = i(\bar{x}) - l$, $v_r = j(\bar{x}) - l$ we have

$$f_{x,\phi,(u_r,v_r)}^{\alpha_1,\alpha_2}(i) = f_{\bar{x},\phi,(i(\bar{x}),j(\bar{x}))}^{\alpha_1,\alpha_2}(l+i) - f_{x,\phi,(i(\bar{x}),j(\bar{x}))}^{\alpha_1,\alpha_2}(l).$$

Hence as $f_{x,\phi,(u_r,v_r)}^{\alpha_1,\alpha_2}(0) = 0$ and $l \in S_{k,\bar{x}}$ we have

$$\Pi_{x,\phi,(u_r,v_r)}^{\alpha_1,\alpha_2}(1) = 1,$$

and as $r \to \infty$

$$\Pi_{x,\phi,(u_r,v_r)}^{\alpha_1,\alpha_2}(1) \to f_{x,\phi}^{\alpha_1,\alpha_2}(1) = 1$$

for $x \in A$ with $\mu_1(A) > 1 - a$.

If such an $A$ exists then if $T_1^k(x) \in A$ then

$$\begin{aligned} f_{T_1^k(x),\phi}^{\alpha_1,\alpha_2}(1) &= f_{x,\phi}^{\alpha_1,\alpha_2}(k+1) - f_{x,\phi}^{\alpha_1,\alpha_2}(k) \\ &= 1. \end{aligned}$$

So

$$\Pi_{x,\phi}^{\alpha_1,\alpha_2}(k+1) = \Pi_{x,\phi}^{\alpha_1,\alpha_2}(k) + 1.$$

But by the Hurewicz ergodic theorem [**1**],

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \chi_A(T_1^k(x)) \frac{dT_1^{-k}\mu_1}{d\mu_1}(x) \Bigg/ \sum_{k=0}^{n-1} \frac{dT_1^{-k}\mu_1}{d\mu_1}(x) = \mu_1(A) > 1 - a.$$

Hence $m_0(f_\phi^{\alpha_1,\alpha_2}) < a$. □

LEMMA 3.9. *The non-singular size $m_0$ satisfies axiom vi(a).*

*Proof.* Choose $\epsilon > 0$. Suppose $m_0(f_\phi^{\alpha_1,\alpha_2}) < \epsilon/2$, and $m_0(f_\psi^{\alpha_2,\alpha_3}) < \delta$ where $\delta$ is chosen so that $\mu_1(\phi^{-1}(E)) < \epsilon/2$ for $\mu_2(E) < \delta$, for all measurable $E \subset X_2$.

By the previous lemma we have sets $S_1 \subset X_1$ and $S_2 \subset X_2$ with $\mu_1(S_1) > 1-\epsilon/2$ and $\mu_2(S_2) > 1-\delta$ such that

$$f_{x,\psi\circ\phi}^{\alpha_1,\alpha_3}(1) = f_{\phi(x),\psi}^{\alpha_2,\alpha_3}(f_{x,\phi}^{\alpha_1,\alpha_2}(1)) = 1$$

for $x \in S_1 \cap \phi^{-1}(S_2) \subset X_1$. Hence $m_0(f_{\psi\circ\phi}^{\alpha_1,\alpha_3}) < \epsilon$. □

LEMMA 3.10. *For all $\alpha_1, \alpha_2$ and for all $\epsilon > 0$ we can find an orbit map $\bar{\phi}$ such that $m_0(f_{\bar{\phi}^{-1}}^{\alpha_2,\alpha_1}) < \epsilon$. Hence $m_0$ satisfies axiom iii(a) and further*

$$\alpha_1 \tilde{m}_0 \alpha_2.$$

*Proof.* Fix $\epsilon > 0$. Choose a $\delta > 0$ such that for any measurable set $E \subset X_1$ if $\mu_1(E) > 1-\delta$ then $\mu_2(\phi(E)) > 1-\epsilon/3$.

Choose $N_0$ so large that for a set $\phi(A)$ with $\mu_2(\phi(A)) > 1-\epsilon/3$ if $x \in \phi(A)$ then

$$|\alpha_1(\phi^{-1}(x), \phi^{-1}(T_2(x)))| < N_0.$$

Now choose $N(\gg N_0)$ so that on a set $B \subset X_1$ with $\mu_1(B) > 1-\delta/4$, for all $x \in B$, $N_1 > N$ and for any index set $I \subset [0, N_1-1]$ with $\#I \le N_0$ we have

$$\sum_I \frac{dT_1^{-i}\mu_1}{d\mu_1}(x) < \frac{\delta}{8}\sum_{i=0}^{N_1-1}\frac{dT_1^{-i}\mu_1}{d\mu_1}(x).$$

Take the induced transformation $T_B$ where

$$T_B(x) = T_1^{r_B(x)}(x) \quad \text{for } x \in B.$$

Build a Rokhlin Tower of height $N_1$ covering all but $\delta/4$ of $B$; i.e.

$$\cup_{i=0}^{N_1-1} T_B^i(C) \subset B, \quad C \subset B$$

is a disjoint union and

$$\frac{\mu_1(\cup_{i=0}^{N_1-1} T_B^i(C))}{\mu_1(B)} > 1-\delta/4.$$

We now use this $T_B$ tower as the basis for a series of $T_1$ towers of differing heights.

Partition $C$ into sets $C_p$ where

$$C_p = \{x \in C : T_B^{N_1-1}(x) = T_1^{p-1}(x)\}.$$

The towers $\cup_{i=0}^{p-1} T_1^i(C_p)$ are disjoint unions by the disjointness of $\cup_{i=0}^{N_1-1} T_B(C)$. Further

$$\cup_{i=0}^{p-1} T_1^i(C_p) \cap \cup_{i=0}^{q-1} T_1^i(C_q) = \emptyset \quad \text{a.e.}$$

if $p \ne q$.

Now we construct a new orbit map $\bar{\phi}$ between $T_1$ and $T_2$ as follows.

(i) If $x \notin \cup_{p \geq N_1} \cup_{i=0}^{p-1} T_1^i(C_p)$ then let

$$\bar{\phi}(x) = \phi(x).$$

(ii) Let $x \in \cup_{i=0}^{p-1} T_1^i(C_p)$ for some $p \geq N_1$.

For each orbit segment $\cup_{i=0}^{p-1} T_1^i(z)$, $z \in C_p$, take

$$\phi(\cup_{i=0}^{p-1} T_1^i(z)) = \cup_{i=0}^{p-1} T_2^{j(i,z)}(\phi(x))$$

and rearrange the right-hand side of this expression so that the $j(i, z)$ are in ascending order. Then let $\bar{\phi}(T_1^i(z))$ be the $i$th element in this rearranged sequence.

We define our new ordering:

$$\alpha_1^\epsilon : X_2 \times X_2 \to \mathbb{Z}$$

by

$$\alpha_1^\epsilon(y, y') = \alpha_1(\bar{\phi}^{-1}(y), \bar{\phi}^{-1}(y')).$$

Clearly $\alpha_1$ and $\alpha_1^\epsilon$ differ by a coboundary. Let

$$S = \phi(A) \cap \phi(\cup_{p \geq N_1} \cup_{i=N_0}^{p-N_0-1} T_1^i(C_p)) \cap \bar{\phi}(\cup_{p \geq N_1} \cup_{i=0}^{p-2} T_1^i(C_p)).$$

Now

$$\begin{aligned} f_{y,id}^{\alpha_2,\alpha_1^\epsilon}(1) &= \alpha_1^\epsilon(y, T_2(y)) \\ &= \alpha_1(\bar{\phi}^{-1}(y), \bar{\phi}^{-1}(T_2(y))). \end{aligned}$$

We shall prove that this expression is equal to 1 for $y \in S$.

If $y \in S$ then $y \in \phi(A)$ so

$$|\alpha_1(\phi^{-1}(y), \phi^{-1}(T_2(y)))| < N_0.$$

Thus

$$\phi^{-1}(T_2(y)) = T_1^k(\phi^{-1}(y))$$

where $|k| < N_0$ so as $\phi^{-1}(y) = T_1^j(z)$ for some $j$, $N_0 \leq j \leq p - N_0$, $z \in C_p$, some $p \geq N_1$. It follows that

$$T_2(y) \in \phi(\cup_{i=0}^{p-1} T_1^i(z)).$$

Let $x = \bar{\phi}^{-1}(y)$. Then $x = T_1^i(z)$ for some $i$, $0 \leq i \leq p - 2$.

Hence $T_1(x) \in \cup_{i=0}^{p-1} T_1^i(z)$, from which we deduce

$$\alpha_1(\bar{\phi}^{-1}(y), \bar{\phi}^{-1}(T_2(y))) = 1,$$

for all $y \in S$.

All that remains to prove is that the set $S$ is sufficiently large.

We know that

$$\mu_1(\cup_{i=N_0}^{p-N_0-1} T_1^i(C_p)) > (1 - \delta/4)\mu_1(\cup_{i=0}^{p-1} T_1^i(C_p))$$

as $C_p \subset B$, and so

$$\mu_1(\cup_{p \geq N_1} \cup_{i=N_0}^{p-N_0-1} T_1^i(C_p)) > 1 - \delta.$$

Hence

$$\mu_2(\phi(\cup_{p\geq N_1}\cup_{i=N_0}^{p-N_0-1} T_1^i(C_p))) > 1-\epsilon/3.$$

Also for $z \in C_p$,

$$\bar{\phi}(\cup_{i=0}^{p-2} T_1^i(z)) = \phi(\cup_{i=0}^{p-1} T_1^i(z) \setminus T_1^{k(z)}(z))$$

for some $k(z) \in [0, 1, \ldots, p-1]$. Hence, as

$$\mu_1(\cup_{z\in C_p}\cup_{i=0}^{p-1} T_1^i(z) \setminus T_1^{k(z)}(z)) > (1-\delta/4)\mu_1(\cup_{i=0}^{p-1} T_1^i(C_p)),$$

we have

$$\mu_1(\cup_{p\geq N_1}\cup_{z\in C_p}\cup_{i=0}^{p-1} T_1^i(z) \setminus T_1^{k(z)}(z)) > (1-\delta/4)(1-\delta/4)\mu_1(B) > 1-\delta.$$

So

$$\begin{aligned}\mu_2(\bar{\phi}(\cup_{p\geq N_1}\cup_{i=0}^{p-2} T_1^i(C_p))) &= \mu_2(\phi(\cup_{p\geq N_1}\cup_{z\in C_p}\cup_{i=0}^{p-1} T_1^i(z) \setminus T_1^{k(z)}(z))) \\ &> 1-\epsilon/3.\end{aligned}$$

Lastly, of course, we have chosen $\phi(A)$ so that $\mu_2(\phi(A)) > 1-\epsilon/3$. Hence $\mu_2(S) > 1-\epsilon$. □

## 4. *The Hurewicz ergodic theorem*

THEOREM 4.1. *Suppose $T$ is non-singular on the Lebesgue probability space $(X, \mathcal{B}, \mu)$ and let $f \in L^1(\mu)$. Then if*

$$A_n(f,x) = \sum_{i=0}^{n-1} f(T^i(x))\omega_i(x) \Big/ \sum_{i=0}^{n-1} \omega_i(x)$$

*then $A_n(f,x)$ converges to a finite limit for almost all $x \in X$. Further if $A_n(f,x) \to f^\star(x)$ then $\int_E f^\star \, d\mu = \int_E f \, d\mu$ for all $T$-invariant sets $E$.*

*Proof.* First we deal with periodic points. If $X_n$ is the set of points in $X$ with period $n$ then it is easy to show that

$$f^\star(x) = \lim_{k\to\infty} A_k(f,x) = \sum_{i=0}^{n-1} f(T^i(x))\omega_i(x) \Big/ \sum_{i=0}^{n-1} \omega_i(x)$$

for all $x \in X_n$.

As each $X_n$ is an invariant set we can deal with the points in $X_\infty$ by assuming that $T$ is aperiodic. Let $\hat{f}(x)$ be defined as $\lim_{n\to\infty}\sup A_n(f,x)$. As $\hat{f}(T(x)) = \hat{f}(x)$ we will assume that $\hat{f} \geq 0$. Let $E$ be a $T$-invariant set. We will examine $\hat{f}$ on the sets $E_M = \{x \in E : \hat{f}(x) \leq M\}$ and $E_\infty = \{x \in E : \hat{f}(x) = \infty\}$. For any $M > 0$ and $\epsilon > 0$ we can define functions $i_k(x) = 0$ and $j_k(x)$ so that

(1) $A_{j_k(x)+1}(f,x) > 1/\epsilon$, for $x \in E_\infty$ and
(2) $|A_{j_k(x)+1}(f,x) - \hat{f}(x)| < \epsilon$ for $x \in E_M$.

By the inverse Vitali lemma there are functions $i(x) = i_{k(x)}(x)$, $j(x) = j_{k(x)}(x)$ and a set $F \subset X$ with $\mu(F) > 0$ such that the intervals $I(x) = \cup_{i(x)}^{j(x)} T^i(x)$ are disjoint for $x \in F$ and $\mu(B) = \mu(\cup_{x \in F} I(x)) > 1 - \epsilon$. Now

$$\begin{aligned}\mu(E_\infty \cap B) &= \int_{E_\infty \cap F} \sum_{i=0}^{j(x)+1} \omega_i(x)\, d\mu \\ &\le \epsilon \int_{E_\infty \cap F} A_{j(x)+1}(f, x) \sum_{i=0}^{j(x)+1} \omega_i(x)\, d\mu \\ &= \epsilon \int_{E_\infty \cap B} f\, d\mu.\end{aligned}$$

So letting $\epsilon \to 0$ we see that $\mu(E_\infty) = 0$. For each $E_M$ and $\epsilon > 0$ we have

$$\begin{aligned}\left| \int_{E_M \cap B} \hat{f} - f\, d\mu \right| &\le \left| \int_{E_M \cap F} (\hat{f}(x) - A_{j(x)+1}(f, x)) \sum_{i=0}^{j(x)} \omega_i(x)\, d\mu \right| \\ &\le \int_{E_M \cap F} |\hat{f}(x) - A_{j(x)+1}(f, x)| \sum_{i=0}^{j(x)} \omega_i(x)\, d\mu \\ &< \epsilon \int_{E_M \cap F} \sum_{i=0}^{j(x)} \omega_i(x)\, d\mu \\ &\le \epsilon \mu(E_M \cap B) \le \epsilon.\end{aligned}$$

Again, let $\epsilon \to 0$ and $M \to \infty$ we get $\int_E \hat{f}\, d\mu = \int_E f\, d\mu$ for all $T$-invariant sets $E$. Repeating the same procedure for $\liminf A_n(f, x)$ we see that $\lim_{n \to \infty} A_n(f, x) = f^\star$ exists as a finite limit and further that $\int_E f^\star\, d\mu = \int_E f\, d\mu$ for all $T$-invariant sets $E$. □

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