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MODELING OF THE TELESCOPE MIRROR DEFORMATION PROCESS

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We study the influence of a system of bar supports, which are positioned symmetrically with respect to the telescope axis, on the deformation of the optical surface of the mirror. We show that two circular rows of bar supports do not provide the necessary degree of mirror deformation allowed for practical use. Bibliography: 1 *title.*

Increasing the effectiveness of optical telescopes and improving their informational properties depend to a large extend on the correct choice of a new class of active adaptive mirrors. This gives rise to the necessity of carrying out a theoretical study of physical properties and construction principles for such mirrors, taking into account the conditions imposed by the technical and technological requirements and the operating conditions. This was the topic of a seminar of leading experts of the former Soviet Union countries conducted in Moscow in Spring 1997 and devoted to the relieve problem for the telescope main mirror

The aim of this article is to study the influence of a system of bar supports that are positioned symmetrically with respect to the axis of the telescope main mirror on the deformation of the optical surface of the mirror held in the gravitational field.

The body of the telescope main mirror is a portion of a spherical shell of constant width h bounded by two parallel planes with base radii r_0 and R on the inner and outer parts of the reflecting surface, respectively. The support points are uniformly located in two cylindrical cross sections that are symmetric with respect to the axis of the telescope. This allows one to split the entire space of the telescope into n sections, each of which is in the same stress condition. The number n corresponds to the number of support points in each of the two cross sections (Fig. 1).

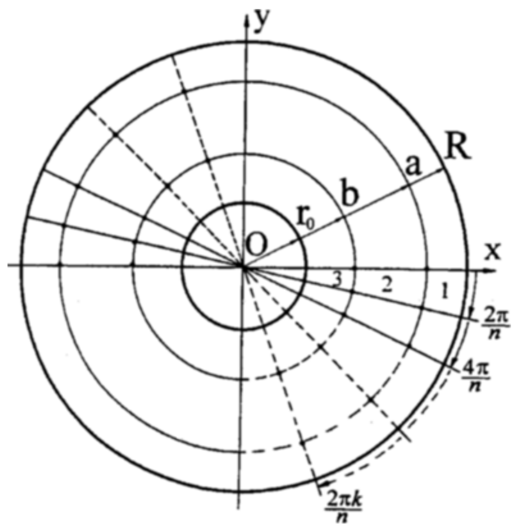

FIG. 1. A diagram of the telescope: 1–3 are zone numbers; • are support points.

Since the curvature of the reflecting surface of the main mirror is small and the thickness of the spherical shell is significantly smaller than the outer diameter of the telescope, the study of the deformation, with

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the first order of accuracy, can be reduced to a study of the bend of a round annular plate caused by its own weight if the plate is supported at the support points, which are symmetric with respect to the center of the plate [1].

We will look for a solution of the problem on the plate bend with the assumption that the bends w of the plate are small compared to its thickness h and that the value of the concentrated force (the supporting force) is not large. The problem will be solved in polar coordinates (r, θ). The direction of the coordinates is shown in Fig. 2.

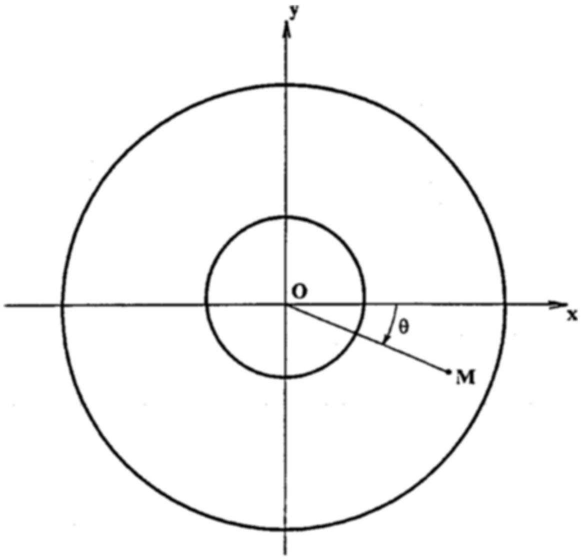

FIG. 2

Then the relation between the polar and Cartesian coordinates is given by

$$r^2 = x^2 + y^2, \qquad \theta = \arctan(y/x).$$

Let us write a differential equation for the surface bend of a transversely stressed round plate in the polar coordinates:

$$\Delta\Delta w = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial 2}{\partial \theta^2}\right)\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) = q/D. \tag{1}$$

Here $q = \rho g h$ is the stress applied to the plate caused by its weight, ρ is the density of the material, g is the acceleration of the gravitational field, $D = Eh^3/[12(1-\sigma^2)]$ is the bend rigidity of the plate, E is the Young modulus, and σ is the Poisson coefficient.

We represent the general solution of Eq. (1) as

$$w = w_0 + w_1,$$

where w_0 is a particular solution of Eq. (1), while w_1 is a solution of the homogeneous equation

$$\Delta\Delta w_1 = 0. \tag{2}$$

In this case, it is convenient to write this solution as [1]

$$w_1 = R_0(r) + R_1(r)\cos n\theta, \tag{3}$$

where $R_0(r)$ and $R_1(r)$ are functions of the radial coordinate.

Substituting expressions (3) into Eq. (2) for each of these functions we get the differential equations

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2R_0}{dr^2} + \frac{1}{r}\frac{dR_0}{dr}\right) = 0; \tag{4}$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right)\left(\frac{d^2R_1}{dr^2} + \frac{1}{r}\frac{dR_1}{dr} - \frac{n^2R_1}{r^2}\right) = 0. \tag{5}$$

Equations (4) and (5) are easily integrated. Their general solutions have the form

$$R_0(r) = A_0 + B_0r^2 + C_0\ln r + D_0r^2\ln r;$$
$$R_1(r) = A_1r^n + B_1r^{-n} + C_1r^{n+2} + D_1r^{-n+2}.$$

If $q = \text{const}$, a particular solution of (1) is known:

$$w_0 = \frac{qr^4}{64D}.$$

Then the general solution of (1) can be written as

$$w(r,\theta) = A_0 + B_0r^2 + C_0\ln r + D_0r^2\ln r + \frac{qr^4}{64D} + \left(A_1r^n + B_1r^{-n} + C_1r^{n+2} + D_1r^{-n+2}\right)\cos n\theta. \tag{6}$$

Let us subdivide the region $r \in [r_0, R]$ into three zones: $r \in [a, R]$, $r \in [b, a]$, and $r \in [r_0, b]$. For each of these zones there is a solution of type (6) but with different integration constants. Label these zones with indices 1, 2, and 3 (Fig. 1) and let w_1, w_2, and w_3 be the bends of the plate in the zones. We have

$$w_i = A_{i0} + B_{i0}r^2 + C_{i0}\ln r + D_{i0}r^2\ln r + \frac{qr^4}{64D} + \left(A_{i1}r^n + B_{i1}r^{-n} + C_{i1}r^{n+2} + D_{i1}r^{-n+2}\right)\cos n\theta, \quad i = 1, 2, 3. \tag{7}$$

The cylindrical cross sections $r = b$ and $r = a$, $r_0 < b < a$, $b < a < R$, correspond to the circles where the support points are located. At these points, the cutting force has a discontinuity, since the concentrated force (the supporting force) is applied at these points [1]. This leads to the necessity of subdividing the plate region into three zones with boundaries passing in the cross sections $r = a$ and $r = b$.

Let us write the expressions for the bending and torque moments:

$$M_r = -D\left[\frac{\partial^2 w}{\partial r^2} + \sigma\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right] \tag{8}$$

$$M_{r\theta} = (1-\sigma)D\left(\frac{1}{r}\frac{\partial^2 w}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial w}{\partial\theta}\right) \tag{9}$$

The cutting force in the cross section r is given by the expression

$$Q_r = -D\frac{\partial}{\partial r}(\Delta w). \tag{10}$$

Equation (1) must satisfy the following boundary-value conditions. In the free boundary of the plate, for $r = r_0$ and $r = R$, the bend moment and cutting forces equal zero:

$$(M_r)_{r=r_0} = 0, \qquad V = \left(Q_r - \frac{1}{r}\frac{\partial M_{r\theta}}{\partial\theta}\right)_{r=r_0} = 0, \tag{11}$$

$$(M_r)_{r=R} = 0, \qquad V = \left(Q_r - \frac{1}{r}\frac{\partial M_{r\theta}}{\partial\theta}\right)_{r=R} = 0. \tag{12}$$

Since no external moments are applied to the circles with radii $r = b$ and $r = a$, the continuity conditions on the circles can be written as

$$w_1 = w_2, \qquad \frac{\partial w_1}{\partial r} = \frac{\partial w_2}{\partial r}, \qquad \frac{\partial^2 w_1}{\partial r^2} = \frac{\partial^2 w_2}{\partial r^2} \quad \text{for } r = a; \tag{13}$$

$$w_2 = w_3, \qquad \frac{\partial w_2}{\partial r} = \frac{\partial w_3}{\partial r}, \qquad \frac{\partial^2 w_2}{\partial r^2} = \frac{\partial^2 w_3}{\partial r^2} \quad \text{for } r = b. \tag{14}$$

Moreover, the bends and bending moments are equal to zero at support points for the cross sections $r = b$ and $r = a$:

$$w_1(a;\theta_i) = 0; \qquad M_r(a;\theta_i) = 0; \tag{15}$$

$$w_2(b;\theta_i) = 0 \qquad M_r(b;\theta_i) = 0, \tag{16}$$

where $\theta_i = 2\pi i/n$, $i = 1, 2, \ldots, n$.

Introduce dimensionless variables by the following relations: $\tilde{w} = w/R$, $\tilde{r} = r/R$, $\tilde{a} = a/R$, $\tilde{b} = b/R$, and $\tilde{R} = R/R = 1$. Write the solution of Eq. (1) for each zone in dimensionless form:

$$\tilde{w}_i = A_{i0} + B_{i0}\tilde{r}^2 + C_{i0}\ln\tilde{r} + D_{i0}\tilde{r}^2\ln\tilde{r} + \frac{R_r\tilde{r}^4}{8} + \left(A_{i1}\tilde{r}^n + B_{i1}\tilde{r}^{-n} + C_{i1}\tilde{r}^{n+2} + D_{i1}\tilde{r}^{-n+2}\right)\cos n\theta, \quad i = 1, 2, 3, \tag{17}$$

where $R_r = \rho g h R^3/(8D)$ is a dimensionless variable. In the sequel, we omit the tilde over dimensionless quantities.

Substituting solutions of (17) into boundary-value conditions (11)–(16), we get a system of 24 linear algebraic equations for determining the unknown integration constants. This system is solved by using Gaussian elimination. We solve this system by the Gauss method. The magnitudes of deformation, w_i, $i = 1, 2, 3$, were determined from formulas (17). The calculations were carried out for the following values of the physical and geometric parameters of the problem: $h = 0.1$ m; $R_0 = 9.3333$ m; $R = 0.857$ m; $r_0 = 0.33$ m; $n = 18$: $E = 9.806651010\,\mathrm{N/m^2}$; $r = 2.46103\,\mathrm{kg/m^3}$; $s = 0.236$. The quantities a and b were changing over the study. A numerical experiment was carried out in order to determine stable equilibrium positions of the annular plate so that its surface experiences minimal deformations. The studies showed that the maximal bends of the surface of the telescope mirror do not exceed several tenths of a micrometer. It is clear that to decrease further the deformation to reach one tenth of a micrometer or less, one needs to have three circular rows of supports, to increase the total number of support points, and to determine their optimal location.

Now we pass to the study of the influence of the spherical surface of the mirror on its deformation as compared with the deformation of the round annular plate of constant thickness. Since it is not an easy problem to solve the biharmonic equation for the bend of the surface of a flat gradient spherical shell in spherical coordinates, we rewrite the equation for the bend in polar coordinates.

The equation for the bend in spherical coordinates has the form

$$\Delta\Delta w = q/D, \tag{18}$$

where

$$\Delta w = \frac{1}{\rho^2}\left[\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial w}{\partial\rho}\right) + \frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial w}{\partial\varphi}\right) + \frac{1}{\sin^2\varphi}\frac{\partial^2 w}{\partial\theta^2}\right]$$

is the Laplacian.

Introduce the change of variables

$$r = R_0\sin\varphi, \qquad \theta = \theta, \tag{19}$$

where r and θ are polar coordinate and R_0 is the radius of the spherical surface of the telescope mirror (Fig. 3).

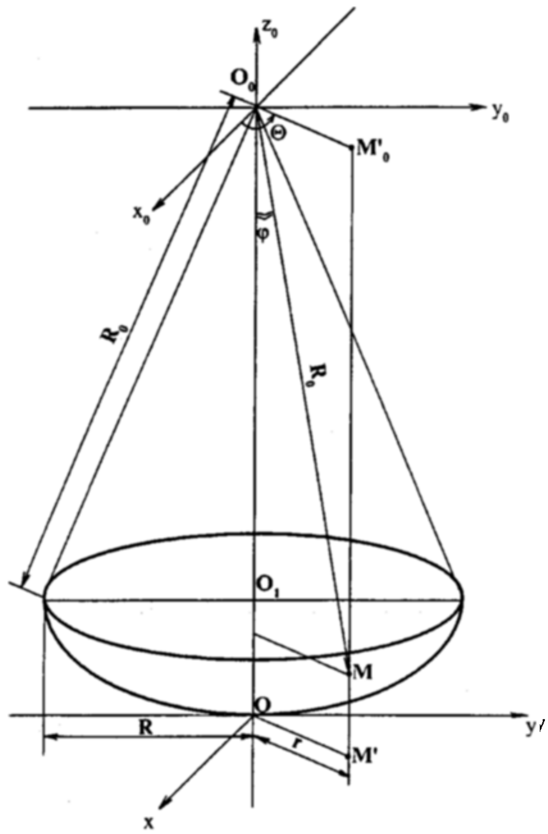

FIG. 3

Since we are looking for an equation for the bend of the reflecting surface of the telescope mirror, Eq. (18) does not depend on r. Passing to variables (19) in Eq. (18), we get

$$\left[\frac{\cos\varphi}{r}\frac{\partial}{\partial r}\left(r\cos\varphi\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]\left[\frac{\cos\varphi}{r}\frac{\partial}{\partial r}\left(r\cos\varphi\frac{\partial w}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right]=\frac{d}{D}, \tag{20}$$

where $\cos\varphi=\sqrt{1-r^2/R_0^2}$, $0\le r\le R$.

Equation (20) describes in polar coordinates the magnitude of deviation of points on the surface of the spherical shell from the initial stress-free state in the direction parallel to the telescope axis.

As $R_0\to\infty$, Eq. (20) becomes an equation for the bend of the plate in polar coordinates, since $\lim_{R_0\to\infty}\sqrt{1-r^2/R_0^2}=1$. The maximal value of the ratio (r^2/R^2) equals $\epsilon=R^2/R_0^2$, and, in the case under consideration, it is of the order of $\sim10^{-2}$. Therefore, we can consider ϵ as a small parameter. Then the differential equation for the bend of the spherical mirror in polar coordinates, if only the first-order approximation is considered, is as follows:

$$\left[(1-\epsilon)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]\left[(1-\epsilon)\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right]=q/D. \tag{21}$$

It is easy to see from Eq. (21) that the influence of the spherical shape on the deformation of the telescope mirror does not exceed 2ϵ times the deformation of the annular plate.

It should be noted that the method used in the paper is applicable for solving problems on the bend of an annular plate that is stressed by its weight if a system of concentrated forces is applied symmetrically with respect to the center of the plate and located evenly in three or more concentric circles.

As a result of the calculations, it was found that by using a system of supports evenly distributed on two concentric circles, it is not possible to obtain a value for the transversal deformation of the telescope mirror that would be less than the given value of $w = 0.05\,\mu\text{m}$, which is admissible for practical use. This leads to the need for studying the deformation in the case of three circular rows of supports.

The calculations carried out for the case of three circular rows of supports show that it is possible to achieve the maximal bend of the telescope mirror less than $0.01\,\mu\text{m}$. Due to the cumbersome exposition, we do not give the setting of this problem.

Finally, we note that the given method for solving the problem differs from the known method of Timoshenko [1], although it uses its main points. For example, the region under consideration is subdivided into zones depending on the location of the concentrated forces (the supporting forces). On the boundaries of the zones, the cutting forces have a discontinuity which is a boundary-value condition and includes the supporting forces, which, in our case, it is not possible to calculate. The boundary-value conditions of the discontinuity of the cutting forces were replaced by the conditions that the values of the bends and bending moments at support points equal zero, which allowed one to obtain a solution of the problem. An advantage of the method used in the article is that we choose a form of the solution that takes into account the uniqueness of the solution in n sectors, since the supports are located symmetrically with respect to the vertex of the telescope mirror, and this is a significant distinction between the solution given in this paper and the one found in [1].

References

1. S. P. Timoshenko, *Plates and Shells* [in Russian], Nauka, Moscow (1966).