# The $P$–$h^2$ relationship in indentation

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(Received 13 October 1999; accepted 25 February 2000)

In this paper we derive an analytical expression for the indentation load–depth relation during loading in an indentation experiment, namely $P = E_r\,(1/\sqrt{C}\,\sqrt{E_r/H} + \epsilon\,\sqrt{\pi/4}\,\sqrt{H/E_r})^{-2}\,(h + \xi)^2$. The advantage over previously used expressions is that no additional empirical constants are necessary. A comparison between the new expression and results from finite element calculations shows excellent agreement.

## I. INTRODUCTION

It has been shown in various investigations[1–3] that the loading curve obtained from depth-sensing indentation experiments can be accurately described by the following relationship:

$$P = \mathrm{K}h^2 \quad . \tag{1}$$

Thus, the load $P$ is equal to a constant K times the square of the indentation depth $h$. This equation has been obtained by experiments (Hainsworth *et al.*, 1996),[1] finite element calculation (Zeng and Rowcliffe, 1996),[2] and dimensional analysis (Cheng and Cheng, 1998).[3] Recently, the relation (1) has been used as a basis for an indenter tip radius and load frame compliance calibration (Sun *et al.*, 1999).[4]

The value of the constant K depends on the indenter tip shape and on the material properties of the indented material. Hainsworth *et al.* (1996)[1] derived the following relation for K:

$$\mathrm{K} = E\left(\Phi\sqrt{\frac{E}{H}} + \Psi\sqrt{\frac{H}{E}}\right)^{-2} \quad . \tag{2}$$

In this equation, $E$ and $H$ are Young's modulus and the hardness, respectively, of the indented material, $\Phi$ and $\Psi$ are two empirical constants, for which the values $\Phi = 0.194$ and $\Psi = 0.903$ have been obtained by Hainsworth *et al.* after analyzing indentation load–displacement curves measured on a broad range of different materials.

We show in this paper that the relation (1) can be derived analytically, giving an explicit expression for K, which is of the form of Eq. (2).

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## II. THEORY

The starting point of our derivation is to write the total indentation depth as the sum of the contact depth $h_c$ and the displacement of the surface at the perimeter of the contact $h_s$:

$$h = h_c + h_s \quad . \tag{3}$$

This is illustrated in Fig. 1.

For $h_c$ we use the following definition of hardness $H$:

$$H = \frac{P}{A_c} = \frac{P}{24.5h_c^2} \quad , \tag{4}$$

where $P$ is the load and $A_c$ is the contact area at that load, which is equivalent to $24.5h_c^2$ in case of a perfect Berkovich (or Vickers) indenter. Hence, for a perfect Berkovich indenter we have

$$h_c = \sqrt{\frac{P}{24.5H}} \quad . \tag{5}$$

The case of a nonperfect indenter will be discussed below.

For $h_s$ we can write (Oliver and Pharr, 1992):[5]

$$h_s = \epsilon\frac{P}{S} \quad , \tag{6}$$

in which $\epsilon$ is a geometrical constant, which takes a value of 0.72 for a conical indenter, and 0.75 for a paraboloid, and $S$ is the unloading contact stiffness at load $P$. The following well-known expression relates $S$ to Young's modulus of the indented material (Oliver and Pharr, 1992):[5]

$$S = \sqrt{\frac{4}{\pi}}\,E_r\sqrt{A_c} \quad , \tag{7}$$

  

in which $E_r$ is the reduced Young's modulus, given by

$$\frac{1}{E_r} = \frac{1-\nu_i^2}{E_i} + \frac{1-\nu_s^2}{E_s} \quad , \tag{8}$$

where $E_s$ and $\nu_s$ are Young's modulus and Poisson's ratio, respectively, of the indented specimen and $E_i$ and $\nu_i$ are those of the indenter.

Combination of Eqs. (4), (6), and (7) yields

$$h_s = \epsilon\sqrt{\frac{\pi}{4}}\,\frac{\sqrt{P\,H}}{E_r} \quad . \tag{9}$$

Finally, we combine the expressions for $h_c$ and $h_s$, Eqs. (5) and (9), by substituting them into Eq. (3). After some rearrangement, the result is the following relation between indentation depth and displacement:

$$P = E_r\left(\frac{1}{\sqrt{24.5}}\sqrt{\frac{E_r}{H}} + \epsilon\sqrt{\frac{\pi}{4}}\sqrt{\frac{H}{E_r}}\right)^{-2} h^2 \quad . \tag{10}$$

Note that the prefactor of $h^2$ in this expression has the form given by Hainsworth *et al.* (Eq. 2), apart from Young's modulus being replaced by the reduced modulus. However, in our formula (10) we have an explicit expression for the constants $\Phi$ and $\Psi$ in Eq. (2). Taking $\epsilon = 0.72$, we find that $\Phi = 0.202$ and $\Psi = 0.638$, which is comparable to the values quoted above from Hainsworth *et al.* The difference probably arises due to the nonperfect indenter used by Hainsworth *et al.* and the fact they use $E$ and not $E_r$ in their relation.

In case of a nonperfect indenter, the expression changes. In practice, an indenter is never perfect due to some tip rounding; see Fig. 2. In that case, it is reasonable to describe the indenter area function by

$$A_c = C\,(h_c + \xi)^2 \quad . \tag{11}$$

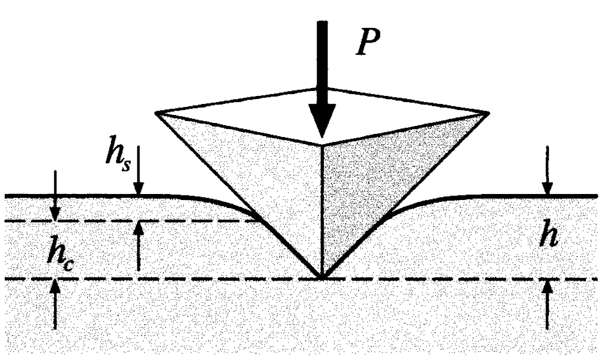


FIG. 1. Total indentation depth $h$ being the sum of the contact depth $h_c$ and $h_s$.

The symbols are explained in Fig. 2. The equation only holds for $h_c > d$. The function given in Eq. (11) has been used successfully by Sun *et al.* (1999)[4] to describe real indenter tips.

Using Eq. (11) in Eq. (4) instead of the perfect Berkovich expression and repeating the analysis outlined above finally leads to

$$P = E_r\left(\frac{1}{\sqrt{C}}\sqrt{\frac{E_r}{H}} + \epsilon\sqrt{\frac{\pi}{4}}\sqrt{\frac{H}{E_r}}\right)^{-2} (h+\xi)^2 \quad . \tag{12}$$

Equation (12) provides an analytical expression which can be used to predict the nanoindentation response of a material if the $E$ and $H$ values of the material are known and if the indenter tip geometry as described by Eq. (11) is known. Moreover, either $E$ or $H$ can be determined if one or the other is known.

Various implicit assumptions have been made in the derivation of Eq. (12). The most important is that the hardness is assumed to be a constant, independent of indentation depth, despite the fact that indentation size effects (ISE) are known to occur. These effects are often related to strain hardening, which is thus not included here. However, as also pointed out by Hainsworth *et al.*, if, during an experiment, the shape of the loading curve alters as the load is changed, the model shows that this effect must be due to a variation in hardness, because Young's modulus might be expected to remain constant with scale (at least for bulk materials rather than for coated systems). Thus subtle changes in the curve shape can help identify whether ISE effects are occurring.

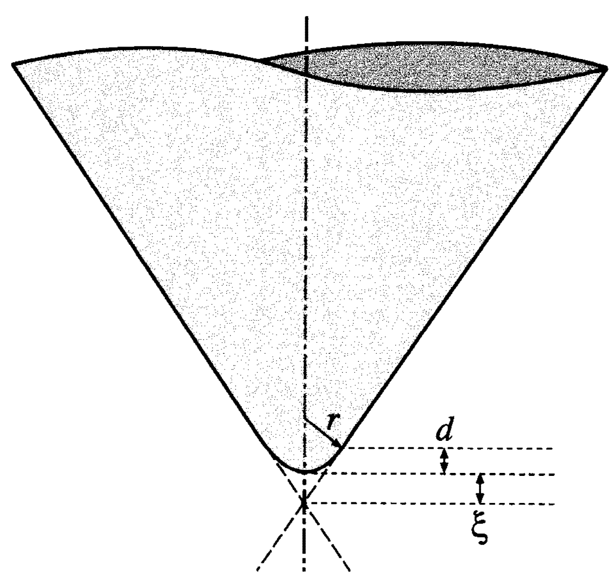


FIG. 2. Schematic diagram showing the geometry of a rounded indenter tip.

Also, the model is partly based on the unloading-stiffness model given by Oliver and Pharr (1992);[5] thus all the assumptions made there, including for example frictionless contact, are also present implicitly in our model.

Note that Eq. (12) is valid only for $h_c > d$. In practice, a good indenter has a tip radius of about 100 nm, which corresponds to a $d$ of less than 6 nm when $C$ in Eq. (11) equals 24.5. Almost always, indentations will be deeper in practice, so the condition $h_c > d$ is not restrictive in practical situations.

## III. FINITE ELEMENT CALCULATION

To verify the model, we carried out finite element calculations. Here the indenter was modeled as a cone with a rounded tip. The half-included angle of the cone was 70.3°, whereas the tip radius was 100 nm. The parameters in Eq. (11) corresponding to this shape are $C = 24.51$ and $\xi = 6.22$ nm. The indenter was assumed to be rigid, and the contact was assumed to be frictionless. The indented material was an elastic–perfectly plastic material with Young's modulus $E = 150$ GPa, Poisson's ratio $\nu = 0.3$, and yield stress $Y = 2.5$ GPa. The calculations were carried out with the FEM package MARC. A plot of the mesh used is shown in Fig. 3. The size of the full mesh was 17.5 × 17.5 μm. We used 8761 axisymmetric quadrilateral linear elements. the size of the elements in the contact area was 1.1 nm. The contact was modeled with the direct constraint method as described in Ref. 6.

A calculation was made up to an indentation depth of 187 nm. From the curves obtained, we computed the hardness as $H = 8.0$ GPa. This value was found to be constant within 0.2 GPa variation over the whole indentation depth. Figure 4 shows the computed indentation load–displacement curve, along with the relation given in Eq. (12) with the appropriate values substituted for all quantities. Clearly, there is good agreement between the computational results and the model, even though the computations showed a slight pile-up, and this effect is not included in our model (see Fig. 1).

As mentioned before, the model can be used to estimate either $E$ or $H$ of the indented material if one or the other is known. If we assume that $E$ is known and equal to 150 GPa, then a fit to the computed data gives 8.2 GPa. This is close to the value of 8.0 GPa as found directly from the computation.

For comparison, Fig. 5 shows the FEM results together with the model proposed by Hainsworth *et al.*, Eq. (2). Clearly the comparison is not as good as that for the present model. An estimate of $H$ with $E$ fixed to 150 GPa gave $H = 9.5$ GPa, which is a less accurate estimate of the true value of 8.0 GPa than the value found with the present model. We have to bear in mind, however, that the empirical factors $\Phi$ and $\Psi$ found by Hainsworth *et al.* are valid only for the specific indenter they used, as mentioned already before.

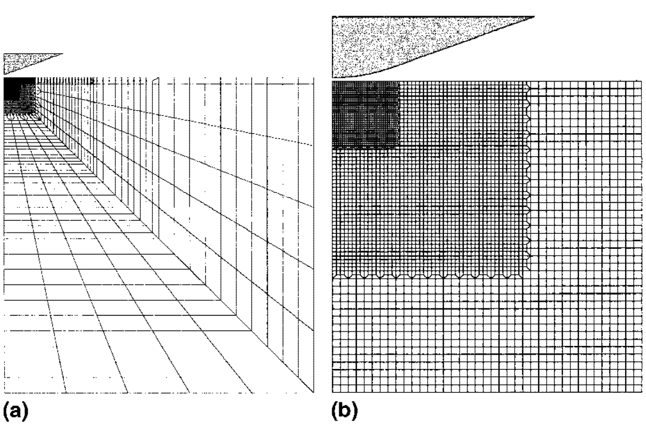


FIG. 3. (a) Full view and (b) detail of the finite element mesh used in the computation. The symmetry axis is on the left side; part of the indenter is visible at the top of the figures.

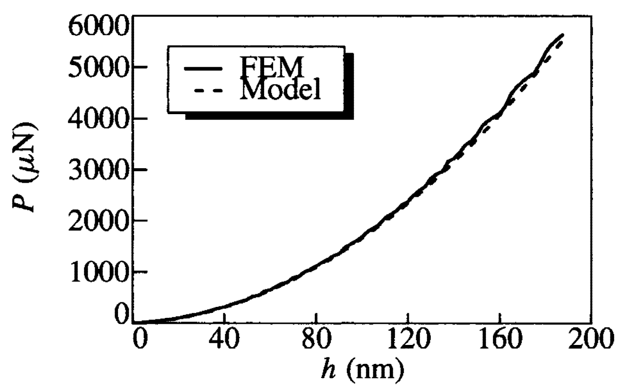


FIG. 4. Indentation load–displacement behavior as computed from FEM and that predicted by the model given by Eq. (12) with the appropriate values substituted for all quantities.

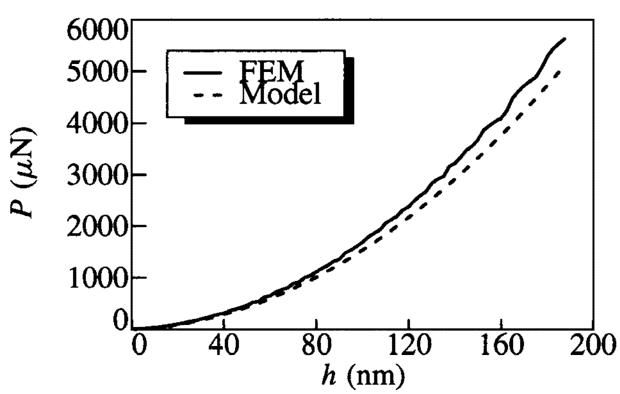


FIG. 5. Indentation load–displacement behavior as computed from FEM and that predicted by the model proposed by Hainsworth *et al.* given by Eq. (2). The appropriate values were substituted for all quantities.

## IV. CONCLUSIONS

In conclusion, we have derived an analytical expression for the load–depth relation during loading in an indentation experiment, namely Eq. (12). The advantage over similar relations derived previously is that our equation contains no additional empirical constants. A comparison between the new expression and results from a finite element calculation has shown excellent agreement for a nonhardening elastic–perfectly plastic material. One interesting issue for future investigations will be the influence of strain-hardening on the $P$–$h^2$ relation in indentation.

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