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# Elimination of inequality constraints in convex optimization and application to yield design

**S. Turgeman and B. Guessab**

**Abstract** The problem of convex optimization under inequality constraints is considered. It is established, using Slater's hypothesis only, that this problem is equivalent to an unconstrained optimization problem, independent of any penalization coefficient. A consequence of this result concerns operational determination of an upper bound of the exterior penalty coefficient, from which the conventional exterior penalty function is exact. The optimization methods developed find a particularly propitious field of application in yield design. Numerical determination of the tensile strength of a bar with cuts illustrates this point.

**Key words** convex constrained optimization, exact penalty function, minimax optimization, yield design, limit analysis

## 1 Introduction

The optimization problem considered is

$$\mu^* = \min\ f(\mathbf{x})\,,$$

$$g_i(\mathbf{x}) \le 0\,, \quad (i = 1, \dots, m)\,, \tag{1}$$

where $f$ and $g_i$ $(i = 1, \dots, m)$ are convex functions of $I\!R^n$ in $I\!R$.

This problem concerns in particular yield design, a field of application chosen to test the methods of solution presented in this paper.

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The literature relating to solving problem (1) is abundant and highlights two major families of methods: primal methods, operating directly on (1), and dual methods which associate a succession of unconstrained optimization problems to (1). From this standpoint, the study presented here leads to solving methods based on this second family.

A function $H$ of $I\!R^n$ in $I\!R$ is first exhibited admitting a global minimum on $I\!R^n$ equal to $\mu^*$. This result is obtained using Slater's hypothesis only, which appears as a minimum condition in convex optimization.

We know that this hypothesis is sufficient for the penalty function

$$P(\mathbf{x}) := f(\mathbf{x}) + K\|\mathbf{g}^+(\mathbf{x})\|\,, \tag{2}$$

[where $\mathbf{g}^+(\mathbf{x}) \in I\!R^m$ has as components $g_i^+(\mathbf{x}) = \max\{g_i(\mathbf{x}), 0\}$ $(i = 1, \dots, m)$ and $\|\cdot\|$ is a norm of $I\!R^m$] to be exact, i.e. for its minimum to be equal to $\mu^*$ for $K$ sufficiently great but finite $(K \ge K_0 \ge 0)$. Studies concerning penalty functions associated to convex or nonconvex problems are numerous and their purpose is in general to establish sufficient conditions ensuring this exactitude property (Pietrzykowski 1969; Conn 1973; Fletcher 1973; Bertsekas 1975; Han and Mangasarian 1979; Coleman and Conn 1980; Di Pillio and Grippo 1986, for example). They do not, on the other hand, enable an upper bound of $K_0$ to be a priori determined.

The function $H$ unlike the function $P$ does not depend on any penalization coefficient. It does however present the drawback of being nonconvex, which makes seeking its global minimum problematic.

This difficulty is overcome by considering the function $\overline{H} = \max\{H, f\}$ whose global minimum on $I\!R^n$ is also equal to $\mu^*$. The function $\overline{H}$, although nonconvex, presents the advantage of not presenting local minima strictly greater than $\mu^*$. Solving (1) by direct minimization of $\overline{H}$ can therefore be envisaged.

The second point of this study has the purpose of obtaining an operational upper bound of $K_0$. The function $H$ is in fact expressed naturally in the form (2), with the difference that $K$ is to be replaced by a function $\tilde{K}(\mathbf{x})$ of $I\!R^n$ in $I\!R$. It is then shown that knowing a lower bound of $\mu^*$ (which can be determined by minimization of $P$ with

$K$ fixed at a positive value, even low) enables an upper bound $K_1$ of $K_0$ to be effectively computed.

Finally this study leads to two methods of solving (1):

- either by a direct minimization of $\overline{H}$;
- or by a minimization of $P$, $K$ being governed by $K_1$.

These methods find a particularly propitious field of application in the static yield design (or limit analysis) method.

Firstly, a sufficient condition related to the yield criteria can in fact be specified for which Slater's hypothesis is satisfied. Secondly, a point $\tilde{\mathbf{x}}$ can be exhibited, without prior computation, such that $g_i(\tilde{\mathbf{x}}) < 0$ $(i = 1, \ldots, m)$. This enables the function $H$ to be explicited and (1) to be solved by minimization of $\overline{H}$.

Furthermore, in numerous practical yield design applications, a lower bound of $\mu^*$ can be obtained by simple mechanical reasonings. This lower bound of $\mu^*$ then leads, without prior optimization, to an upper bound of $K_0$. This is the case in the example considered by Andersen *et al.* (1998), consisting in determining the tensile strength of a bar with cuts, where it is shown that $K$ simply has to be taken equal to 0.5 (whereas $\mu^*$ is about one) for the associated penalty function $P(\mathbf{x})$ to be exact.

Numerical solution of this mechanical problem enables a comparison of the performances of the methods of solution presented.

## 2 Elimination of the inequality constraints in convex optimization

The problem (1) is considered and the following is noted:

- $\hat{g}(\mathbf{x})$ the convex function of $I\!R^n$ in $I\!R$ defined by:

$$\hat{g}(\mathbf{x}) := \max_{i=1,\ldots,m} g_i(\mathbf{x}) , \tag{3}$$

- $G$ the domain of the possible solutions:

$$G = \{\mathbf{x} \in I\!R^n \text{ such that } \hat{g}(\mathbf{x}) \le 0\} . \tag{4}$$

It is assumed that:

A1: the problem (1) admits a solution $\mu^*$ achieved in $\mathbf{x}^* \in G$,
A2: (Slater's hypothesis): $\tilde{\mathbf{x}}$ exists such that $\hat{g}(\tilde{\mathbf{x}}) < 0$.

For any $(\lambda, \mathbf{u}) \in I\!R^+ \times I\!R^n$ we have:

$$\hat{g}\left(\frac{\lambda\tilde{\mathbf{x}} + \mathbf{u}}{\lambda + 1}\right) \le \frac{\lambda\hat{g}(\tilde{\mathbf{x}}) + \hat{g}(\mathbf{u})}{\lambda + 1} , \tag{5}$$

due to the convexity of $\hat{g}$.

However:

$$\hat{g}(\mathbf{u}) \le \left\|\mathbf{g}^+(\mathbf{u})\right\|_p , \quad \forall \mathbf{u} \in I\!R^n , \tag{6}$$

where $\|\mathbf{g}^+(\mathbf{u})\|_p$ designates the $p$-norm of the vector $\mathbf{g}^+(\mathbf{u}) \in I\!R^m$.

It results from (5) and (6) that

$$\hat{g}\left(\frac{\lambda\tilde{\mathbf{x}} + \mathbf{u}}{\lambda + 1}\right) \le \frac{\lambda\hat{g}(\tilde{\mathbf{x}}) + \|\mathbf{g}^+(\mathbf{u})\|_p}{\lambda + 1} , \tag{7}$$

We note

$$G_1 = \left\{(\lambda, \mathbf{u}) \in I\!R^+ \times I\!R^n / \hat{g}\left(\frac{\lambda\tilde{\mathbf{x}} + \mathbf{u}}{\lambda + 1}\right) \le 0\right\} , \tag{8}$$

$$G_2 = \left\{(\lambda, \mathbf{u}) \in I\!R^+ \times I\!R^n / \left\|\mathbf{g}^+(\mathbf{u})\right\|_p = -\lambda\hat{g}(\tilde{\mathbf{x}})\right\} , \tag{9}$$

According to (7), these sets are such that

$$G_2 \subset G_1 . \tag{10}$$

We then have, due to convexity of $f$ and according to (10),

$$\mu^* = \min_{(\lambda,\mathbf{u})\in G_1} f\left(\frac{\lambda\tilde{\mathbf{x}} + \mathbf{u}}{\lambda + 1}\right) \le \min_{(\lambda,\mathbf{u})\in G_1} \frac{\lambda f(\tilde{\mathbf{x}}) + f(\mathbf{u})}{\lambda + 1} \le \min_{(\lambda,\mathbf{u})\in G_2} \frac{\lambda f(\tilde{\mathbf{x}}) + f(\mathbf{u})}{\lambda + 1} . \tag{11}$$

Taking account of the definition of $G_2$ [cf. (9)], the variable $\lambda$ can be eliminated in (11), whence

$$\mu^* \le \min_{\mathbf{u}\in I\!R^n} H(\mathbf{u}) , \tag{12}$$

with $H$ the function of $I\!R^n$ in $I\!R$ defined by

$$H(\mathbf{u}) := \frac{f(\tilde{\mathbf{x}})\,\|\mathbf{g}^+(\mathbf{u})\|_p - \hat{g}(\tilde{\mathbf{x}})f(\mathbf{u})}{\|\mathbf{g}^+(\mathbf{u})\|_p - \hat{g}(\tilde{\mathbf{x}})} . \tag{13}$$

As $H(\mathbf{x}^*) = \mu^*$, it then follows:

**Proposition 1.** *The function $H(\mathbf{u})$ defined in (13) admits a global minimum on $I\!R^n$ equal to $\mu^*$.*

It is clear that the function $H$ is not convex which makes $\mu^*$ difficult to obtain by minimization of $H$. To overcome this difficulty, we consider the function $\overline{H}$ of $I\!R^n$ in $I\!R$ defined by

$$\overline{H}(\mathbf{u}) := \max[H(\mathbf{u}), f(\mathbf{u})] = \begin{cases} H(\mathbf{u}) & \text{if } f(\mathbf{u}) \le f(\tilde{\mathbf{x}}) , \\ f(\mathbf{u}) & \text{if not.} \end{cases} \tag{14}$$

**Proposition 2.** *The function $\overline{H}$ defined in (14) admits a global minimum on $I\!R^n$ equal to $\mu^*$ and does not admit local minima strictly greater than $\mu^*$.*

*Proof.* It can be assumed, without any loss of generality, that $f(\tilde{\mathbf{x}}) = 0$ and that $\hat{g}(\tilde{\mathbf{x}}) = -1$.

The function $\overline{H}$ is then such that

$$\overline{H}(u) = \begin{cases} H(\mathbf{u}) = \frac{f(\mathbf{u})}{\|\mathbf{g}^+(\mathbf{u})\|_p + 1} & \text{if } f(\mathbf{u}) \le 0\,, \\ f(\mathbf{u}) & \text{if not}\,. \end{cases} \tag{15}$$

It obviously admits a global minimum on $I\!R^n$ equal to $\mu^*$ as it is greater than the function $H$ and we have $\overline{H}(\mathbf{x}^*) = \mu^*$.

Let us assume that $\overline{H}$ admits in $\tilde{\mathbf{u}} \in I\!R^n$ a local minimum $\overline{H}(\tilde{\mathbf{u}})$ strictly greater than $\mu^*$. A neighbourhood $V(\tilde{\mathbf{u}})$ of $\tilde{\mathbf{u}}$ therefore exists such that

$$\overline{H}(\mathbf{u}) \ge \overline{H}(\tilde{\mathbf{u}})\,, \quad \forall \mathbf{u} \in V(\tilde{\mathbf{u}})\,. \tag{16}$$

If $f(\tilde{\mathbf{u}}) > 0$, (16) implies, taking account of the definition of $\overline{H}$ [cf. (15)]:

$$f(\mathbf{u}) \ge f(\tilde{\mathbf{u}})\,, \quad \forall \mathbf{u} \in V'(\tilde{\mathbf{u}})\,, \tag{17}$$

where $V'(\tilde{\mathbf{u}})$ is a neighborhood of $\tilde{\mathbf{u}}$.

This means that $f$ admits in $\tilde{\mathbf{u}}$ a local minimum $f(\tilde{\mathbf{u}}) > \mu^*$ (since $\mu^* \le 0$), which is impossible due to the convexity of $f$.

The assumption $f(\tilde{\mathbf{u}}) > 0$ is therefore absurd and we necessarily have $f(\tilde{\mathbf{u}}) \le 0$, whence $\overline{H}(\tilde{\mathbf{u}}) = H(\tilde{\mathbf{u}})$.

The points $\tilde{\mathbf{u}}$ and $\mathbf{x}^*$ both belong to the convex domain $F_0 = \{\mathbf{x} \in I\!R^n / f(\mathbf{x}) \le 0\}$. A point $\mathbf{u}_0 \in ]\tilde{\mathbf{u}}, \mathbf{x}^*] \cap V(\tilde{\mathbf{u}}) \cap F_0$ consequently exists. This point $\mathbf{u}_0$ can be written as

$$\mathbf{u}_0 = \frac{\tilde{\mathbf{u}} + \lambda \mathbf{x}^*}{\lambda + 1}\,, \quad \lambda > 0\,. \tag{18}$$

It is such that

$$\overline{H}(\mathbf{u}_0) \ge \overline{H}(\tilde{\mathbf{u}})$$

due to (16) and to the fact that $\mathbf{u}_0 \in V(\tilde{\mathbf{u}})$,

$$\Leftrightarrow \quad H(\mathbf{u}_0) \ge H(\tilde{\mathbf{u}})$$

due to the fact that $\mathbf{u}_0 \in F_0$ and to (15),

$$\Rightarrow \quad \frac{f(\tilde{\mathbf{u}}) + \lambda \mu^*}{(\lambda+1)\left(\|\mathbf{g}^+(\mathbf{u}_0)\|_p + 1\right)} \ge \frac{f(\tilde{\mathbf{u}})}{\|\mathbf{g}^+(\tilde{\mathbf{u}}_0)\|_p + 1}$$

due to the convexity of $f$ and (18),

$$\Rightarrow \quad \frac{f(\tilde{\mathbf{u}}) + \lambda \mu^*}{\|\mathbf{g}^+(\tilde{\mathbf{u}})\|_p + \lambda + 1} \ge \frac{f(\tilde{\mathbf{u}})}{\|\mathbf{g}^+(\tilde{\mathbf{u}}_0)\|_p + 1}$$

due to the convexity of $\|\mathbf{g}^+(\mathbf{u})\|_p$ and (18),

$$\Leftrightarrow \quad \mu^* \ge \overline{H}(\tilde{\mathbf{u}})\,,$$

which is in contradiction with the assumption $\overline{H}(\tilde{\mathbf{u}}) > \mu^*$.

## 3 Exact penalty function

The function $H$ defined in (13) enables the function $P(\mathbf{x}) := f(\mathbf{x}) + K\|\mathbf{g}^+(\mathbf{x})\|_p$ to be proved to be exact for $K \ge K_0 \ge 0$ and above all an operational upper bound of $K_0$ to be obtained. The function $H$ can in fact be written in the form

$$H(\mathbf{u}) := f(\mathbf{u}) + \tilde{K}(\mathbf{u})\|\mathbf{g}^+(\mathbf{u})\|_p\,, \tag{19}$$

with $\tilde{K}$, a function of $I\!R^n$ in $I\!R$ defined by

$$\tilde{K}(\mathbf{u}) := \frac{f(\tilde{\mathbf{x}}) - f(\mathbf{u})}{\|\mathbf{g}^+(\mathbf{u})\|_p - \hat{g}(\tilde{\mathbf{x}})}\,. \tag{20}$$

We obviously have, for any strict $\mu_1$ lower bound of $\mu^*$:

$$\mu^* = \min_{\mathbf{u} \in I\!R^n} \left\{ \max[f(\mathbf{u}), \mu_1] + \tilde{K}(\mathbf{u})\|\mathbf{g}^+(\mathbf{u})\|_p \right\}\,. \tag{21}$$

Consequently, for any $\mathbf{u} \in F_1 = \{\mathbf{u} \in I\!R^n \,/\, f(\mathbf{u}) \ge \mu_1\}$, we have

$$\mu^* \le f(\mathbf{u}) + K_1\|\mathbf{g}^+(\mathbf{u})\|_p\,, \tag{22}$$

with

$$K_1 = \frac{f(\tilde{\mathbf{x}}) - \mu_1}{-\hat{g}(\tilde{\mathbf{x}})}\,, \tag{23}$$

due to the fact that $K_1 \ge \max_{\mathbf{u} \in F_1} \tilde{K}(\mathbf{u})$.

Let us assume that:

$\exists \overline{\mathbf{u}} \in I\!R^n$

such that

$$f(\overline{\mathbf{u}}) + K_1\|\mathbf{g}^+(\overline{\mathbf{u}})\|_p < \mu^*\,. \tag{24}$$

We must have, according to (22), $f(\overline{\mathbf{u}}) < \mu_1$. The function $f$, continuous on the segment $[\overline{\mathbf{u}}, \mathbf{x}^*]$, takes any intermediate value between $f(\overline{\mathbf{u}})$ and $f(\mathbf{x}^*) = \mu_1$ at at least one point of this segment. As $f(\overline{\mathbf{u}}) < \mu_1 < \mu^*$, $\mathbf{u}_1 \in ]\overline{\mathbf{u}}, \mathbf{x}^*[$ therefore exists such that $f(\mathbf{u}_1) = \mu_1$. The point $\mathbf{u}_1 = \alpha\mathbf{x}^* + (1-\alpha)\overline{\mathbf{u}}$ can be written with $\alpha \in ]0,1[$. As $\mathbf{u}_1 \in F_1$, the inequality (22) holds in $\mathbf{u}_1$ and is written, due to the convexity of $f(\mathbf{u})$ and of $\|\mathbf{g}^+(\mathbf{u})\|_p$, as

$$\mu^* \le \alpha f(\mathbf{x}^*) + (1-\alpha)\left[f(\overline{\mathbf{u}}) + K_1\|\mathbf{g}^+(\overline{\mathbf{u}})\|_p\right]\,. \tag{25}$$

The second member of (25) is however strictly smaller than $\mu^*$, on account of (24).

The assumption (24) is therefore absurd and consequently

$$P(\mathbf{u}) \ge \mu^*\,, \quad \forall \mathbf{u} \in I\!R^n\,, \quad \forall K \ge K_1\,. \tag{26}$$

Noting that $P(\mathbf{x}^*) = \mu^*$ for any $K$ we have

$$\mu^* = \min_{\mathbf{u} \in I\!R^n} P(\mathbf{u})\,, \quad \forall K \ge K_1\,. \tag{27}$$

Finally, by making $\mu_1$ tend to $\mu^*$, the following result is obtained:

**Proposition 3.** *The penalty function $P(\mathbf{u})$ is exact for any $K \geq K_0$ with $K_0 = [f(\tilde{\mathbf{x}}) - \mu^*]/[-\hat{g}(\tilde{\mathbf{x}})]$.*

It should be emphasized that any lower bound $\mu_1$ of $\mu^*$ provides an upper bound $K_1$ of $K_0$, with $K_1$ defined in (23).

This result justifies the following algorithm where (1) is solved by minimization of $P(\mathbf{u})$ with $K$ governed by the expression (23):

1. We note $K = K_1 \geq 0$; $\alpha$ a real number different from 1.
2. We solve $P(\mathbf{x}_1) = \min\limits_{\mathbf{x} \in I\!R^n} P(\mathbf{x})$.
3. If $\|\mathbf{g}^+(\mathbf{x}_1)\|_p = 0$ or if $\alpha = 1$ then $f(\mathbf{x}_1) = \mu^*$; if not, we note $K_1 = [f(\tilde{\mathbf{x}}) - f(\mathbf{x}_1)]/[-\alpha \hat{g}(\tilde{\mathbf{x}})]$ with $\alpha \geq 1$ such that $K_1 > K$ and we go back to step 2.

The role of the parameter $\alpha$ is to limit a variation of $K$ which may be too large from one iteration to the other.

## 4 Application to yield design

Solving problem (1) by minimization of $\overline{H}(\mathbf{u})$ or of $P(\mathbf{u})$ (cf. Sect. 3) requires a point $\tilde{\mathbf{x}}$ to be previously determined such that $\hat{g}(\tilde{\mathbf{x}}) < 0$. This point $\tilde{\mathbf{x}}$ can be obtained by solving the unconstrained discrete minimax problem (Turgeman and Guessab 1999):

$$\hat{g}(\tilde{\mathbf{x}}) = \min_{\mathbf{x} \in I\!R^n} \max_{i=1,\dots,m} g_i(\mathbf{x}) =$$

$$\min_{(\eta,\mathbf{x}) \in I\!R \times I\!R^n} \left\{ \eta + \left\| [g(\mathbf{x}) - \eta]^+ \right\|_p \right\} . \qquad (28)$$

Problem (28) can be interrupted as soon as the function $\hat{g}$ becomes strictly negative. We can also choose among the iterates $\mathbf{x}_k$ such as $\hat{g}(\mathbf{x}_k) < 0$ the one which minimizes $K_1$ [cf. (23)] if a lower bound $\mu_1$ of $\mu^*$ is known. However, for convex optimization problems arising from static formulation of yield design (Salençon 1983) or limit analysis, such a point $\tilde{\mathbf{x}}$ can be exhibited without computation for a large class of materials.

Let $\Omega$ be a mechanical system constituted by $\hat{k}$ materials characterized by strength convex criteria $gr_k(\boldsymbol{\sigma}) \leq 0$ $(k = 1, \dots, \hat{k})$, with $\boldsymbol{\sigma}$ the stress tensor. As this system is subjected to a loading mode dependent on $p$ parameters, the extreme loading $\lambda^* \mathbf{Q}_0$ $(\lambda^* \geq 0)$ in a fixed direction $\mathbf{Q}_0$ $(\mathbf{Q}_0 \in I\!R^p)$ is sought for. Numerical formulation of the static yield design method by finite elements consists in maximizing $\lambda$ (and therefore in minimizing $-\lambda$) with equality linear constraints (which result from static admissibility) and convex inequality constraints (which result from strength conditions). Elimination of the linear constraints leads to solving of a problem of the form (1) with $\mu^* = -\lambda^*$. If the strength criteria are such that

$$gr_k(\mathbf{0}) < 0 , \quad k = 1, \dots, \hat{k} , \qquad (29)$$

this problem complies with Slater's hypothesis as the constraints $g_i(\mathbf{x})$ of (1) are strictly satisfied for

$$\tilde{\mathbf{x}} = \mathbf{0} \in I\!R^n , \qquad (30)$$

and we have in addition $f(\tilde{\mathbf{x}}) = 0$ whence [cf. (14)]

$$\mu^* = \min_{\mathbf{u} \in I\!R^n} \max \left\{ \frac{-\hat{g}(\mathbf{0}) f(\mathbf{u})}{\|\mathbf{g}^+(\mathbf{u})\|_p - \hat{g}(\mathbf{0})}, f(\mathbf{u}) \right\} . \qquad (31)$$

Let us note that the sufficient condition (29) is often satisfied. This is the case when the constituent materials comply for example with von Mises, Tresca or Coulomb (with nonzero cohesion) criteria. Furthermore, for numerous mechanical problems, an upper bound $\lambda_1$ of $\lambda^*$ (i.e. a lower bound $-\lambda_1$ of $\mu^*$) can be obtained by simple mechanical reasonings. This results, if (29) holds, in an immediate upper bound $K_1 = -\lambda_1/\hat{g}(\mathbf{0})$ of $K_0$ [cf. (23)].

As an example, let us consider the plane strain test problem, illustrated in Fig. 1 and described by Andersen *et al.* (1998): uniform tensile strength of a bar with external symmetric cuts.

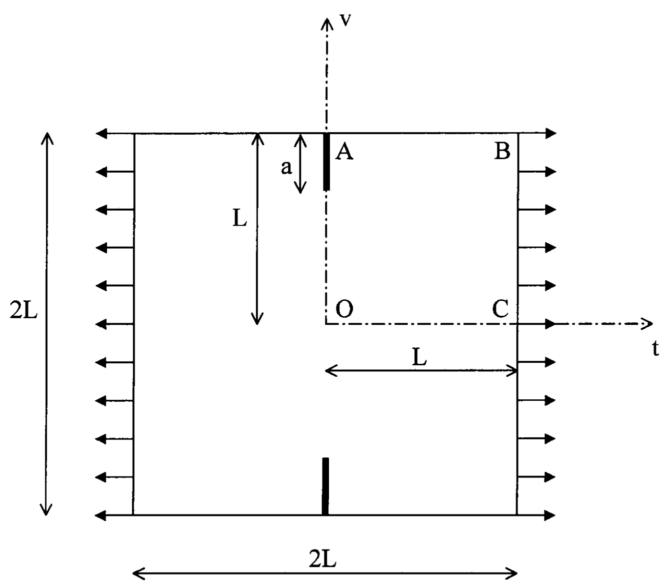


**Fig. 1** Test problem: uniform tensile strength of a bar with symmetric cuts ($a = L/3$)

The bar is formed by a von Mises material of limit shear strength $c$, i.e.

$$gr(\boldsymbol{\sigma}) = \left( \frac{\sigma_{tt} - \sigma_{vv}}{c} \right)^2 + 4 \left( \frac{\sigma_{tv}}{c} \right)^2 - 4 .$$

The loading parameter $\lambda$ can be defined in adimensional manner by $\lambda = \frac{\sigma_{tt}}{c}$. The extreme loading sought for $\lambda^*$ is lower than the extreme loading $\lambda_1$ of a bar without cuts. As $\lambda_1$ is known and equal to 2 and (29) holds (with in addition $\hat{g}(\mathbf{0}) = -4$), we have $K_1 = 0.5$, whence

$$\mu^* = \min_{\mathbf{u} \in I\!R^n} \left[ f(\mathbf{u}) + 0.5 \|\mathbf{g}^+(\mathbf{u})\|_p \right] . \qquad (32)$$

This example is not atypical. For a large number of mechanical problems (in particular plasticity homogenization problems), simple reasonings can lead to determination of an operational upper bound of $K_0$ as in (32).

Problems (31) and (32) corresponding to a division into triangular finite elements of the quarter OABC of the volume $V$ of the bar are solved, on account of the material and loading symmetries. For a mesh with 5184 finite elements the number of variables of problems (31) and (32) is equal to 1214 (the stress fields considered are constant on each finite element and discontinuous between two adjacent finite elements). The corresponding approximation $\lambda^{*-}$ of $\lambda^*$ obtained is: $\lambda^{*-} = 1.3790$ for problem (31); $\lambda^{*-} = 1.3789$ for problem (32). By Andersen and Christiansen (1998), a computation of $\lambda^*$ is performed using the kinematic limit analysis method which gives an upper bound $\lambda^{*+} = 1.3894$ of $\lambda^*$.

Problems (31) and (32) lead to very close $\lambda^{*-}$ values (which is not surprising) with however a large number of iterations for (32). Comparison between the values $\lambda^{*-}$ and $\lambda^{*+}$ leads one to think that the optimization methods developed, which present the advantage of simplicity, are efficient at least for the type of problems considered.

## 5
## Conclusions

It has been shown that, using Slater's hypothesis only, the solution $\mu^*$ of problem (1) is the unconstrained global minimum of a function $H$ or of a function $\overline{H} = \max(H, f)$. The functions $H$ and $\overline{H}$ are independent of any penalization coefficient and totally explicited when a point $\tilde{\mathbf{x}}$ strictly verifying the inequality constraints of problem (1) is known.

The function $\overline{H}$ presents the property of not admitting local minima strictly greater than $\mu^*$. This property enables its global minimum to be effectively computed.

The function $H$ enables an upper bound $K_1$ of $K_0$ to be determined, a value starting from which the exterior penalty function $P$ is exact. The upper bound $K_1$ can effectively be computed when $\tilde{\mathbf{x}}$ and a lower bound $\mu_1$ of $\mu^*$ are known.

We therefore have two methods available for solving (1) by minimization of $\overline{H}$ or of $P$ with $K$ governed by $K_1$.

These methods find a particularly propitious field of application in the static yield design method. If the zero stress tensor strictly verifies the strength conditions of the constituent materials of the mechanical system considered, a point $\tilde{\mathbf{x}}$ can in fact be exhibited without computation. This condition, although it is not general, is very often achieved. Moreover, simple mechanical reasonings are sufficient, for a large number of problems, to determine lower bounds of $\mu^*$ and subsequently to explicit exact penalty functions (without having to perform iterative computation of the penalty coefficient).

The example of determination of the tensile strength of a bar with cuts is an illustration of this. For this problem, $K$ simply has to be fixed at 0.5 for $P$ to be exact. Minimization of the functions $\overline{H}$ and $P$ leads to identical results the precision of which can be estimated by comparison with those of the literature resulting from a kinematic approach.

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