Quaestiones Mathematicae
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tqma20

Some Extremal Properties of Section Spaces of Banach Bundles and Their Duals, II

Terje Hoim & D.A.. Robbins

Version of record first published: 12 Nov 2009.

Quaestiones Mathematicae **25**(2002), 57–65.

SOME EXTREMAL PROPERTIES OF SECTION SPACES OF BANACH BUNDLES AND THEIR DUALS, II

Terje Hõim

Dept. of Mathematics, Trinity College, Hartford, CT 06106 USA.
E-Mail terje.hoim@trincoll.edu

D.A. Robbins

Dept. of Mathematics, Trinity College, Hartford, CT 06106 USA.
E-Mail david.robbins@trincoll.edu

Abstract. Let $\pi : \mathcal{E} \to X$ be a real Banach bundle with section space $\Gamma(\pi)$, where X is a compact Hausdorff space. We complete the characterization of weak-* and sequential weak-* points of continuity in the unit ball of $\Gamma(\pi)^*$ for certain classes of bundles (which include the trivial ones), which was begun in an earlier paper. The proofs avoid the use of vector measures.

Mathematics Subject Classification (2000): 46B20, 46E40, 46H25.

Key words: Banach bundle, section space, weak-* points of continuity.

The present paper, a continuation of [10], is motivated by the literature which exists on the various extremal properties of Banach spaces of the form $C(X, E)$ and their duals, where X is a compact Hausdorff space and $C(X, E)$ is the space of continuous E-valued functions on X. As might be expected, the extremal properties of these vector-valued spaces of continuous functions are related quite strongly to the extremal structure of E. Noting that the elements of $C(X, E)$ take their values in a constant space, it is natural to ask what might happen in a situation in which continuous vector-valued functions take their values in spaces which vary with $x \in X$. A reasonable setting for this question is that of section spaces of Banach bundles and their dual spaces.

In this paper, X will denote a compact Hausdorff space, and $\pi : \mathcal{E} \to X$ will denote a Banach bundle (= bundle of Banach spaces) with real fibers $\{E_x : x \in X\}$; we will assume throughout the paper that for all $x \in X$, $E_x \neq \{0\}$. The total space $\mathcal{E}$ carries a topology such that the relative topology on each fiber E_x is its original Banach space topology. We can regard $\mathcal{E}$ as the disjoint union $\dot{\bigcup}\{E_x : x \in X\}$. (Alternatively, we can think of $\mathcal{E}$ as $\bigcup\{\{x\} \times E_x : x \in X\}$; this is the approach of [3], which uses fibered vector spaces.) See e.g. [3] or [8] for details on the construction of Banach bundles; the most important properties for our present purposes are outlined here. We denote by $\Gamma(\pi)$ the space of sections (= continuous choice functions $\sigma : X \to \mathcal{E}$) of the bundle $\pi : \mathcal{E} \to X$; $\Gamma(\pi)$ is a Banach $C(X)$-module under the norm $\|\sigma\| = \sup\{\|\sigma(x)\| : x \in X\}$, pointwise addition, and the operation $(a \cdot \sigma)(x) = a(x)\sigma(x)$ $(a \in C(X), x \in X)$. In particular, see [3] or [8] for details on the topology of the total space $\mathcal{E}$, which is determined by $\Gamma(\pi)$.

In thinking about the section space $\Gamma(\pi)$, perhaps the most important intuitive point to keep in mind is that when $\sigma \in \Gamma(\pi)$, then the values $\sigma(x) \in E_x$ vary continuously over (quite possibly very) different spaces as x varies over X. This is an analogue to a way in which one may think of a space of the form $C(X, E)$ (= space of continuous functions from X to the (real) Banach space E). We can regard $C(X, E)$ as the space of sections of the trivial bundle $\rho : \mathcal{T} = X \times E \to X$, where $X \times E$ is given the product topology. Here, the total space $\mathcal{T} = X \times E = \bigcup\{\{x\} \times E : x \in X\}$ can be thought of as a union of copies of E, and an element $\sigma \in C(X, E)$, which we usually think of as having values which vary continuously over the fixed set E, can be interpreted as a section in $\Gamma(\rho)$ which varies in a very nice way between these copies of E.

The reader may also wish to consult [7] for a discussion of several ways of thinking about bundles.

Given a Banach bundle $\pi : \mathcal{E} \to X$, the function $x \mapsto \|\sigma(x)\|$ is upper semi-continuous from X to $\mathbb{R}$ for each $\sigma \in \Gamma(\pi)$. If this function is continuous for each $\sigma \in \Gamma(\pi)$, we will call $\pi : \mathcal{E} \to X$ a *continuous* bundle. We call the bundle $\pi : \mathcal{E} \to X$ *separable* if there exists a countable set $\{\sigma_n\} \subset \Gamma(\pi)$ such that $\{\sigma_n(x)\}$ is dense in E_x for each $x \in X$; this can be interpreted as a sort of uniform separability of the fibers E_x.

We denote by $\mathcal{H}$ the space $Hom_X(\Gamma(\pi), C(X))$ of all $C(X)$-module homomorphisms from $\Gamma(\pi)$ to $C(X)$. As noted in [7], we can identify $\mathcal{H}$ with the space of choice functions $H : X \to \dot{\bigcup}\{E_x^* : x \in X\}$ which satisfy the property that the function $x \mapsto \langle \sigma, H \rangle (x) = \langle \sigma(x), H(x) \rangle$ is continuous for each $\sigma \in \Gamma(\pi)$. (Here and henceforth, for typographical convenience, we write E_x^* for $(E_x)^*$). It is the case that $\mathcal{H}$ is a Banach space under the norm $\|H\| = \sup\{\|H(x)\| : x \in X\}$. As in [5], we say that $\mathcal{H}$ *norms* the bundle $\pi : \mathcal{E} \to X$ if for each $x \in X$ and $z \in E_x$ we have

$$\|z\| = \sup\{|\langle z, H(x) \rangle| : H \in \mathcal{H}, \|H\| \leq 1\}.$$

This condition actually happens with fair frequency; see the discussion in [5] following Definition 4.4. We will say that $\mathcal{H}$ is *strongly norming* provided that for each $x \in X$ we have

$$\{H(x) : H \in \mathcal{H}, \|H\| = \|H(x)\|\} = E_x^*.$$

Examples of bundles which are strongly normed by $\mathcal{H}$ include: 1) the trivial bundles $\rho : X \times E \to X$, where E is a Banach space and $\Gamma(\pi)$ is $C(X)$-isometrically isomorphic to $C(X, E)$ (the constant maps from X to E^* will do the job); and 2) the continuous and separable bundles (see [3, Corollary 19.16]).

If Z is a Banach space, denote by B_Z the unit ball of Z. The paper [10] investigates various extremal properties of the unit balls of $B_{\Gamma(\pi)}$ and $B_{\Gamma(\pi)^*}$. Prior work regarding extremal structure in $B_{C(X,E)}$ and $B_{C(X,E)^*}$ (see e.g. [6] or [2]) has used the characterization of $C(X,E)^*$ very strongly. However, to the knowledge of the authors, there is no such concise characterization of $\Gamma(\pi)^*$. The investigation in [10] was handicapped by this lack of nice characterization. Indeed, contrast the two situations:

1) If X is a compact Hausdorff space, and if E is a Banach space, then $C(X,E)^*$ can be isometrically identified with the space $M(X,E^*)$ of all countably additive E^*-valued Borel measures on X, with the variation norm, and action $\langle f,\mu\rangle = \int_X f d\mu$.

2) Let $\pi : \mathcal{E} \to X$ be a separable real bundle and let $\phi \in \Gamma(\pi)^*$. Then there is a regular Borel measure μ on X and a choice function $\eta : X \to \dot{\bigcup}\{E_x^* : x \in X\}$ such that (among other properties) a) $\|\eta(x)\| \leq 1$ for all $x \in X$ and equality holds μ-almost everywhere; b) the function $x \mapsto \langle \sigma(x), \eta(x)\rangle$ is Borel measurable for each $\sigma \in \Gamma(\pi)$; and c) for all $\sigma \in \Gamma(\pi)$ we have $\langle \sigma, \phi\rangle = \int_X \langle \sigma(x), \eta(x)\rangle\, d\mu$. (See [4].)

In particular, whereas in [6] a necessary and sufficient condition for a functional $\phi \in C(X,E)^*$ to be a weak-* point of continuity (or a sequential weak-* point of continuity, in case X is metric) of the unit ball was obtained, in [10] only a sufficient description for $\phi \in \Gamma(\pi)^*$ to be a weak-* point of continuity of the unit ball could be proven. The purpose of this paper is to complete the characterization of weak-* points of continuity and sequential continuity in $B_{\Gamma(\pi)^*}$ for a class of bundles which includes the trivial ones. Our proofs are actually shorter than those of the more special cases addressed in [6], and do not employ vector measures.

Recall that a point $\phi \in B_{Z^*}$ is called a weak-* point of continuity provided that whenever $\{\phi_\lambda\}$ is a net in B_{Z^*} which converges weak-* to ϕ, then $\{\phi_\lambda\}$ converges in norm to ϕ. The definition of a sequential weak-* point of continuity replaces "nets" by "sequences". Recall also that if $\pi : \mathcal{E} \to X$ is a Banach bundle, and if $x \in X$, then there is an isomorphic injection $j_x : E_x^* \to \Gamma(\pi)^*$ given by (for $\sigma \in \Gamma(\pi)$) $\langle \sigma, j_x(f)\rangle = \langle \sigma(x), f\rangle = \langle \sigma, f \circ ev_x\rangle$, where $ev_x : \Gamma(\pi) \to E_x, \sigma \mapsto \sigma(x) \in E_x$, is the evaluation map. In addition, for each closed set $C \subset X$ and $\phi \in \Gamma(\pi)^*$, there is an L-projection $P_C : \Gamma(\pi)^* \to \Gamma(\pi)^*, P_C(\phi) = \phi_C$. The action of ϕ_C on $\Gamma(\pi)$ is defined as follows: let W run through the system of open neighborhoods of C, and for each such W, let $i_W \in C(X)$, $i_W : X \to [0,1]$, be such that $i_W(C) = 0$ and $i_W(X \setminus C) = 1$ (i.e. $\{i_W\}$ is an approximate identity for the ideal of functions in $C(X)$ which vanish on C). Then

$$\langle \sigma, \phi_C\rangle = \lim_{W \to C} \langle (1 - i_W)\sigma, \phi\rangle .$$

(See [4] or [9].) That is, $\phi_C = \text{weak-* } \lim_{W\to C}(1 - i_W)\phi$. If $C \subset X$ is closed, we write $\phi_{X\setminus C} = \phi - \phi_C$; thus we have $\|\phi\| = \|\phi_C\| + \left\|\phi_{X\setminus C}\right\|$, and ϕ_A makes sense if $A \subset X$ is either open or closed. With a little work, it can also be shown that if $C \subset X$ is closed, and if $U \subset C$ is open, then we can also write $\phi_C = \phi_{C\setminus U} + \phi_U$, with $\|\phi_C\| = \left\|\phi_{C\setminus U}\right\| + \|\phi_U\|$.

In particular, if $x \in X$ and $\phi \in \Gamma(\pi)^*$, we write $\phi_x = \phi_{\{x\}}$, and we note that if $\{x_1, ..., x_m\} \subset C$, then

$$\left\|\sum_{k=1}^{m} \phi_{x_k}\right\| = \sum_{k=1}^{m} \|\phi_{x_k}\| \leq \sum_{x\in C} \|\phi_x\| = \left\|\sum_{x\in C} \phi_x\right\| \leq \|\phi_C\| \leq \|\phi\| .$$

It follows that for $\phi \in \Gamma(\pi)^*$, we have $\{x : \phi_x \neq 0\}$ is countable. We note that for $x \in X$ and $\phi \in \Gamma(\pi)^*$, $\phi_x \in j_x(E_x^*)$. Note especially that if $\Gamma(\pi) \simeq C(X, E)$, and if $\phi \in \Gamma(\pi)^*$ corresponds to $\mu \in M(X, E^*)$, then $\phi_x \in \Gamma(\pi)^*$ corresponds to $\mu(\{x\}) \in E^*$.

From [10, Proposition 7] we have the following:

Proposition 1. *Let $\pi : \mathcal{E} \to X$ be a bundle of real Banach spaces, and let $I \subset X$ be the set of isolated points in X. Suppose that $\phi \in \Gamma(\pi)^*$ has the form $\phi = \sum_{x \in I} \phi_x$, where for each $x \in I$ either $\phi_x = 0$ or $\phi_x / \|\phi_x\|$ is a weak-* point of continuity of $B_{\Gamma(\pi)^*}$, and $\sum_{x \in I} \|\phi_x\| = \|\phi\| = 1$. Then ϕ is a weak-* point of continuity of $B_{\Gamma(\pi)^*}$.*

We now state our first result, which is a partial converse to Proposition 1 that encompasses a large class of bundles. Together, Propositions 1 and 2 generalize Theorem 6 of [6]. Moreover, the proof of Proposition 2, although it uses generally some of the ideas of this result, is somewhat simpler, because it does not use vector measures.

Proposition 2: *Let $\pi : \mathcal{E} \to X$ be a bundle of real Banach spaces for which $\mathcal{H}$ is strongly norming, and suppose that $\phi \in B_{\Gamma(\pi)^*}$ is a weak-* point of continuity. Let $I \subset X$ be the set of isolated points of X. Then $\phi = \sum_{x \in I} \phi_x$, where for each $x \in X$, either $\phi_x = 0$ or $\phi_x / \|\phi_x\|$ is a weak-* point of continuity of $B_{\Gamma(\pi)^*}$, and $\|\phi\| = 1 = \sum_{x \in I} \|\phi_x\|$.*

Proof. We approach the proof of Proposition 2 through a series of lemmas. The first two lemmas will show that if $\phi_x \neq 0$, then $\phi_x / \|\phi_x\|$ is a weak-* point of continuity of $B_{\Gamma(\pi)^*}$, and that for all non-isolated points x in X we have $\phi_x = 0$. Note that Lemma 3 is really Proposition 8 of [10]; we include the proof here for the sake of completeness.

Lemma 3. *Let $\pi : \mathcal{E} \to X$ be a Banach bundle, and let $\phi \in \Gamma(\pi)^*$, $\|\phi\| = 1$, be a weak-* point of continuity of $B_{\Gamma(\pi)^*}$. If $x \in X$, and if $\phi_x \neq 0$, then $\phi_x / \|\phi_x\|$ is a weak-* point of continuity of $B_{\Gamma(\pi)^*}$.*

Proof. Suppose that $\{\psi_\alpha\}$ is a net in $B_{\Gamma(\pi)^*}$ such that $\psi_\alpha \to \phi_x / \|\phi_x\|$ weak-*. We then have $\phi - \phi_x + \|\phi_x\| \psi_\alpha \to \phi$ weak-*, and

$$\|\phi - \phi_x + (\|\phi_x\| \psi_\alpha)\| \leq \|\phi - \phi_x\| + \|\phi_x\| \|\psi_\alpha\| \leq 1,$$

so that $\|\phi - \phi_x + (\|\phi_x\| \psi_\alpha) - \phi\| = \|(\|\phi_x\| \psi_\alpha) - \phi_x\| \to 0$, and thus $\phi_x / \|\phi_x\|$ is a weak-* point of continuity of $B_{\Gamma(\pi)^*}$. □

Lemma 4. *Let $\pi : \mathcal{E} \to X$ be a Banach bundle such that $\mathcal{H}$ is strongly norming. If $\phi \in \Gamma(\pi)^*$, $\|\phi\| = 1$, is a weak-* point of continuity of $B_{\Gamma(\pi)^*}$, and if $x \in X$ is non-isolated, then $\phi_x = 0$.*

Proof. Note first that if $H \in \mathcal{H}$ and $\mu \in M(X)$, then $\mu \circ H \in \Gamma(\pi)^*$, and thus for $\sigma \in \Gamma(\pi)$ we have

$$\langle \sigma, \mu \circ H \rangle = \int_X \langle \sigma(y'), H(y') \rangle \, d\mu.$$

In particular, if $\delta_y \in M(X)$ is the unit point mass at $y \in X$ and if $H \in \mathcal{H}$, we have

$$\langle \sigma, \delta_y \circ H \rangle = \int_X \langle \sigma(y'), H(y') \rangle \, d\delta_y = \langle \sigma(y), H(y) \rangle .$$

Let $\mathcal{S} = \{\delta_y \circ H : \|H\| = \|H(y)\| = 1, H \in \mathcal{H}, y \in X\}$. Then because $\mathcal{H}$ is strongly norming, $\mathcal{S}$ separates points of $\Gamma(\pi)^*$, and, for $\delta_y \circ H \in \mathcal{S}$, we have

$$\|\delta_y \circ H\| = \sup_{\|\sigma\|=1} \left| \int_X \langle \sigma(y'), H(y') \rangle \, d\delta_y \right| = \sup_{\|\sigma\|=1} \{ |\langle \sigma(y), H(y) \rangle| \} = \|H(y)\| = 1.$$

Moreover, each extreme point of $B_{\Gamma(\pi)^*}$ is in $\mathcal{S}$ (see [1]), and so the convex hull $co(\mathcal{S})$ of $\mathcal{S}$ is weak-* dense in $B_{\Gamma(\pi)^*}$.

Now, fix $x \in X$, and suppose that x is non-isolated. Let $\delta_y \circ H \in \mathcal{S}$. For each open neighborhood V of x, choose $x_V \neq x \in V$, and for arbitrary $y \in X$ define $(\delta_y \circ H)^V$ by

$$\begin{aligned} \left\langle \sigma, (\delta_y \circ H)^V \right\rangle &= \textstyle\int_{X \setminus V} \langle \sigma(y'), H(y') \rangle \, d\delta_y + \int_V \langle \sigma(x_V), H(x_V) \rangle \, d\delta_y \\ &= \begin{cases} \langle \sigma(y), H(y) \rangle , & \text{if } y \in X \setminus V \\ \langle \sigma(x_V), H(x_V) \rangle , & \text{if } y \in V. \end{cases} \end{aligned}$$

Note that for $\delta_y \circ H \in \mathcal{S}$ and $\|\sigma\| = 1$ we have

$$\left| \left\langle \sigma, (\delta_y \circ H)^V \right\rangle \right| \leq \|\sigma\| \, \|H\| = 1,$$

so that $(\delta_y \circ H)^V \in B_{\Gamma(\pi)^*}$. We claim that $\delta_y \circ H = \text{weak-*} \lim_{V \to \{x\}} (\delta_y \circ H)^V$.

Let $\sigma \in \Gamma(\pi)$. It is easily checked that for $\sigma \in \Gamma(\pi)$ we have

$$\begin{aligned} \left| \left\langle \sigma, (\delta_y \circ H)^V - (\delta_y \circ H) \right\rangle \right| &= \left| \textstyle\int_V \langle \sigma(y'), H(y') \rangle - \langle \sigma(x_V), H(x_V) \rangle \, d\delta_y \right| \\ &\leq \textstyle\int_V |\langle \sigma(y'), H(y') \rangle - \langle \sigma(x_V), H(x_V) \rangle| \, d\delta_y. \end{aligned}$$

Consider two possibilities:

1) If $y \neq x$, then as $V \to \{x\}$ we will eventually have $y \notin V$, and so in this case the second integral becomes 0.

2) If $y = x$, then let $\varepsilon > 0$ be given. By the continuity of $y' \mapsto \langle \sigma(y'), H((y') \rangle$ we can find a neighborhood V_ε of x such that if $y' \in V \subset V_\varepsilon$ then

$$|\langle \sigma(x), H(x) \rangle - \langle \sigma(y'), H(y') \rangle| < \varepsilon.$$

Since x_V is chosen to be in V, we have

$$\begin{aligned} \textstyle\int_V |\langle \sigma(y'), H(y') \rangle - \langle \sigma(x_V), H(x_V) \rangle| \, d\delta_y &= \textstyle\int_V |\langle \sigma(y'), H(y') \rangle - \langle \sigma(x_V), H(x_V) \rangle| \, d\delta_x \\ &= |\langle \sigma(x), H(x) \rangle - \langle \sigma(x_V), H(x_V) \rangle| \\ &< \varepsilon. \end{aligned}$$

Note also that for any $y \in X$, $H \in \mathcal{H}$, fixed neighborhood V of x, open neighborhood W of x, and $\sigma \in \Gamma(\pi)$ we have

$$\begin{aligned} \left\langle \sigma, ((\delta_y \circ H)^V)_x \right\rangle &= \lim_{W \to \{x\}} (\textstyle\int_{X \setminus V} \langle [(1 - i_W)\sigma](y'), H(y') \rangle \, d\delta_y \\ &\quad + \textstyle\int_V \langle [(1 - i_W)\sigma](x_V), H(x_V) \rangle \, d\delta_y) \\ &= 0, \end{aligned}$$

because the first integrand is 0 if $W \subset V$, and the second integrand is 0 if $x_V \notin W$; both of these will eventually happen as $W \to \{x\}$. Thus, $((\delta_y \circ H)^V)_x = 0$. (Here, we are choosing $i_W \in C(X)$ as in the discussion preceding Proposition 1.)

Now, $\phi \mapsto \phi_x$ is an L-projection in $\Gamma(\pi)^*$, so we have $(\phi_x)_x = \phi_x$. We can write

$$\phi_x = f_x \circ ev_x = (\delta_x \circ H)_x \text{ (with } \|\phi_x\| = \|H(x)\| = \|H\|)$$

for some $f_x \in E_x^*$ and $H \in \mathcal{H}$ (because $\mathcal{H}$ is strongly norming). If $\phi_x \neq 0$, then Lemma 3 assures us that $\phi_x / \|\phi_x\|$ is a weak-* point of continuity of $B_{\Gamma(\pi)^*}$, and thus since $\phi_x =$ weak-* $\lim_{V \to \{x\}} (\delta_x \circ H)^V$ and $\left\|(\delta_x \circ H)^V\right\| \leq \|\delta_x \circ H\| = \|H(x)\| = \|\phi_x\|$, we have

$$\lim_{V \to \{x\}} \left\|\phi_x - (\delta_x \circ H)^V\right\| = 0.$$

It then follows that

$$\lim_{V \to \{x\}} \left\|(\phi_x)_x - \left((\delta_x \circ H)^V\right)_x\right\| = \|\phi_x\| = 0,$$

a contradiction. Hence, $\phi_x = 0$. □

Remark. Because the set $\mathcal{S}$ described above includes the extreme points of $B_{\Gamma(\pi)^*}$, any element $\phi \in B_{\Gamma(\pi)^*}$ is a weak-* limit of convex combinations from $\mathcal{S}$. The proof of the last lemma shows that if $x \in X$ is fixed, if $\lambda \subset \mathcal{S}$ is finite, and if V denotes an open neighborhood of x, we have $\delta_{y_k} \circ H_k =$ weak-* $\lim_{V \to \{x\}} (\delta_{y_k} \circ H_k)^V$ for each $\delta_{y_k} \circ H_k \in \lambda$. Combining these two facts shows that $\phi \in B_{\Gamma(\pi)^*}$ is a weak-* limit of a net of convex combinations of the form $(\sum_\lambda \delta_{y_k} \circ H_k)^V \in B_{\Gamma(\pi)^*}$, indexed by the open neighborhoods V of x and finite sets $\lambda \subset \mathcal{S}$. Since we are assuming that ϕ is a weak-* point of continuity, ϕ is also a norm limit of the same net.

To finish the proof of Proposition 2 all we need to show is the following:

Lemma 5. *Let $I = \{x \in X : x \text{ is isolated}\}$, and let ϕ be a weak-* point of continuity, with $\|\phi\| = 1$. Then $\left\|\phi_{X \setminus I}\right\| = 0$ and hence $\phi = \sum_{x \in I} \phi_x$ (and $\|\phi\| = \sum_{x \in I} \|\phi_x\|$).*

Proof. If $I \subset X$ is the set of isolated points of X, then I is open, and so $X \setminus I$ is closed. Given $\varepsilon > 0$, we can find $I_0 = \{x_1, ..., x_m\} \subset I$ such that $\sum_{x \notin I_0} \|\phi_x\| < \varepsilon$. Recall that $\phi_x = 0$ if x is non-isolated. By the remark above there is a convex combination $\sum_{k=1}^n \delta_{y_k} \circ H_k$ (where we suppress the scalar coefficients) such that $\|\phi - \sum_{k=1}^n \delta_{y_k} \circ H_k\| < \varepsilon$. Hence,

$$
\begin{aligned}
\|\phi_{X\setminus I}\| &\le \left\|\phi_{X\setminus I} - \left(\sum_{k=1}^n \delta_{y_k}\circ H_k\right)_{X\setminus I}\right\| + \|\left(\sum_{k=1}^n \delta_{y_k}\circ H_k\right)_{X\setminus I} - \\
&\quad - \left(\sum_{k=1}^n \delta_{y_k}\circ H_k\right)_{X\setminus I_0}\| + \left\|\left(\sum_{k=1}^n \delta_{y_k}\circ H_k\right)_{X\setminus I_0}\right\| \\
&= \left\|\phi_{X\setminus I} - \left(\sum_{k=1}^n \delta_{y_k}\circ H_k\right)_{X\setminus I}\right\| + \left\|\left(\sum_{k=1}^n \delta_{y_k}\circ H_k\right)_{I\setminus I_0}\right\| \\
&\quad + \left\|\left(\sum_{k=1}^n \delta_{y_k}\circ H_k\right)_{X\setminus I_0}\right\| \\
&< \|\phi - \left(\sum_{k=1}^n \delta_{y_k}\circ H_k\right)\| + \left\|\sum_{x\in I\setminus I_0}\phi_x - \left(\sum_{k=1}^n \delta_{y_k}\circ H_k\right)_x\right\| \\
&\qquad + \left\|\sum_{x\in I\setminus I_0}\phi_x\right\| + \left\|\left(\sum_{k=1}^n \delta_{y_k}\circ H_k\right)_{X\setminus I_0}\right\|
\end{aligned}
$$

$$
\begin{aligned}
&< \varepsilon + \sum_{x\in I\setminus I_0}\left\|\phi_x - \left(\sum_{k=1}^n \delta_{y_k}\circ H_k\right)_x\right\| + \left\|\sum_{x\in I\setminus I_0}\phi_x\right\| \\
&\qquad + \left\|\left(\sum_{k=1}^n \delta_{y_k}\circ H_k\right)_{X\setminus I_0}\right\| \\
&< \varepsilon + \|\phi - \left(\sum_{k=1}^n \delta_{y_k}\circ H_k\right)\| + \varepsilon + \left\|\left(\sum_{k=1}^n \delta_{y_k}\circ H_k\right)_{X\setminus I_0}\right\| \\
&< \varepsilon + \varepsilon + \varepsilon + \left\|\left(\sum_{k=1}^n \delta_{y_k}\circ H_k\right)_{X\setminus I_0}\right\|.
\end{aligned}
$$

Finally, note that

$$\sum_{x\in X\setminus I_0}\left\|\phi_x - \left(\sum_{k=1}^n \delta_{y_k}\circ H_k\right)_x\right\| < \left\|\phi - \sum_{k=1}^n \delta_{y_k}\circ H_k\right\| < \varepsilon,$$

and

$$\sum_{x\in X\setminus I_0}\|\phi_x\| < \varepsilon$$

together imply that

$$\sum_{x\in X\setminus I_0}\left\|\left(\sum_{k=1}^n \delta_{y_k}\circ H_k\right)_x\right\| = \left\|\sum_{x\in X\setminus I_0}\left(\sum_{k=1}^n \delta_{y_k}\circ H_k\right)_x\right\| < 2\varepsilon.$$

Now, it is easy to check that for any closed set $C\subset X$ and any $\delta_y\circ H\in\Gamma(\pi)^*$ $(y\in X, H\in\mathcal{H})$, we have

$$(\delta_y\circ H)_C = \begin{cases} \delta_y\circ H, & \text{if } y\in C \\ 0, & \text{if } y\notin C \end{cases}.$$

Thus,

$$
\begin{aligned}
\left\|\left(\sum_{k=1}^n \delta_{y_k}\circ H_k\right)_{X\setminus I_0}\right\| &= \left\|\sum_{y_k\in_{X\setminus I_0}}\left(\sum_{k=1}^n \delta_{y_k}\circ H_k\right)_{y_k}\right\| \\
&\le \left\|\sum_{x\in_{X\setminus I_0}}\left(\sum_{k=1}^n \delta_{y_k}\circ H_k\right)_x\right\|.
\end{aligned}
$$

Hence we have $\left\|\phi_{X\setminus I}\right\| < 5\varepsilon$, and since ε was arbitrary, we have $\phi_{X\setminus I} = 0$. Thus,

$$\phi = \phi_I = \sum_{x\in I}\phi_x = \sum_{\{x\in I:\phi_x\neq 0\}}\phi_x.$$

□

This completes the proof of Proposition 2.

The next corollary follows immediately by combining Propositions 1 and 2.

Corollary 6. *If X has no isolated points, then the unit ball of $\Gamma(\pi)^*$ has no weak-* points of continuity.*

In order to obtain a complete characterization of the weak-* points of sequential continuity of $B_{\Gamma(\pi)^*}$ we restrict ourselves to the strongly norming bundles over a first countable base space X. This generalizes [6, Theorem 9].

Proposition 7. *Let $\pi : \mathcal{E} \to X$ be a bundle of real Banach spaces with X first countable and $\mathcal{H}$ strongly norming. Let $I \subset X$ be the set of isolated points of X. Then $\phi \in B_{\Gamma(\pi)^*}$ is a weak-* point of sequential continuity of $B_{\Gamma(\pi)^*}$ if and only if it has the form $\phi = \sum_{x\in I}\phi_x$ where for each $x \in X$, either $\phi_x = 0$ or $\phi_x/\left\|\phi_x\right\|$ is a weak-* point of continuity of $B_{\Gamma(\pi)^*}$, and $\|\phi\| = 1 = \sum_{x\in I}\left\|\phi_x\right\|$.*

Proof. If $\phi \in B_{\Gamma(\pi)^*}$ has the form $\phi = \sum_{x\in I}\phi_x$ where for each $x \in X$, either $\phi_x = 0$ or $\phi_x/\left\|\phi_x\right\|$ is a weak-* point of continuity of $B_{\Gamma(\pi)^*}$, then the proof that ϕ is a weak-* point of sequential continuity of $B_{\Gamma(\pi)^*}$ follows that given in Proposition 7 of [10] for weak-* points of continuity; no restriction on X is necessary.

Conversely, suppose $\phi \in B_{\Gamma(\pi)^*}$ is a weak-* point of sequential continuity of $B_{\Gamma(\pi)^*}$. Let $I \subset X$ be the set of isolated points of X. As in Lemma 4, it can be shown that $\phi_x = 0$ for each $x \in X\setminus I$, as follows: since X is first countable, each point $x \in X$ has a countable base for its neighborhood system. In particular, if $x \in X\setminus I$ is fixed, let $\{V_n\}$ denote a countable fundamental system of neighborhoods of x. For each V_n choose $x_n \neq x \in V_n$, and as in Lemma 4, for $y \in X$ and $H \in \mathcal{H}$ define $\left(\delta_y \circ H\right)^n$ by

$$\begin{aligned}\langle\sigma,(\delta_y\circ H)^n\rangle &= \textstyle\int_{X\setminus V_n}\langle\sigma(y'),H(y')\rangle\,d\delta_y + \int_{V_n}\langle\sigma(x_V),H(x_V)\rangle\,d\delta_y \\ &= \begin{cases}\langle\sigma(y),H(y)\rangle\,, & \text{if } y\in X\setminus V_n\\ \langle\sigma(x_V),H(x_V)\rangle\,, & \text{if } y\in V_n.\end{cases}\end{aligned}$$

As in the proof of Lemma 4, one can show that $\left(\delta_y \circ H\right)^n \in B_{\Gamma(\pi)^*}$ and that $\left(\delta_y \circ H\right)^n$ converges weak-* to $\delta_y \circ H$ as $V_n \to \{x\}$. Arguments similar to those in the remark preceding Lemma 5 now show that there is a sequence of convex combinations from $\mathcal{S}$ which converges to ϕ weak-*, and hence in norm, since ϕ is a weak-* point of sequential continuity. To complete the proof of Proposition 7, follow along the same lines as the proof of Lemma 5 to show that ϕ has the desired form $\phi = \sum_{x\in I}\phi_x$. □

COROLLARY 8. *Let $\pi : \mathcal{E} \to X$ be a bundle of real Banach spaces with X first countable and $\mathcal{H}$ strongly norming. If X has no isolated points, then the unit ball of $\Gamma(\pi)^*$ has no weak-* points of sequential continuity.*

REFERENCES

1. F. CUNNINGHAM, JR. AND N.M. ROY, Extreme functionals on an upper semicontinuous function space, *Proc. Amer. Math. Soc.* **42** (1974), 461–465.

2. P.N. DOWLING, Z. HU AND M.A. SMITH, *Extremal structure of the unit ball of* $C(K, X)$, Banach Spaces (Merida, 1992), Contemp. Math. 144 (Providence: American Mathematical Society), 1993, 81–85.

3. G. GIERZ, *Bundles of topological vector spaces and their duality*, Lect. Notes in Math. Vol. 955, Springer-Verlag, Berlin, 1982.

4. ———, Integral representations of linear functionals on spaces of sections in Banach bundles, *Rocky. Mt. Math. J.* **17** (1987), 545–554.

5. A.E. GUTMAN AND A.V. KOPTEV, On the notion of the dual of a Banach bundle, *Siberian Adv. Math.* **9** (1999), 46–98.

6. Z. HU AND M.A. SMITH, *On the extremal structure of the unit ball of the space* $C(K, X)^*$, Function spaces (Edwardsville, IL, 1994), Lect. Notes Pure Appl. Math. Vol. 172, Dekker, New York, 1995, 205–222.

7. A.V. KOPTEV AND D.A. ROBBINS, A note on the space of weakly continuous sections of a Banach bundle, *Siberian Adv. Math.* **12** (2002), 51–62.

8. J.W. KITCHEN AND D.A. ROBBINS, *Gelfand representation of Banach modules*, Dissertationes Math. (Rozprawy Mat.) Vol. 203, 1982, 47 pp.

9. ———, Integral operators on the section space of a Banach bundle, *International J. Math. and Math. Sci.* **16** (1993), 449–458.

10. D.A. ROBBINS, Some extremal properties of section spaces of Banach bundles and their duals, *International J. Math. and Math. Sci.* **29** (2002), 563–572.

Received 9 September 2002