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STEADY-STATE VIBRATIONS OF A THIN TWO-LAYER PLATE WITH DELAMINATION

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We consider the problem of harmonic vibrations of a thin two-layer plate with horizontal crack. The problem is solved with the help of the null-field approach. The influence of the shape of the crack contour on the amplitude-frequency characteristics of plate vibrations is investigated.

Baisa et al. [1] and Duvaut and Lions [4] considered problems of vibrations of a two-layer plate with plane circular crack located either on the interface between the plates [4] or in the upper plate [1]. The solution was found under the assumption that the total half-thickness of the structure is small as compared with the characteristic dimension of the crack. It was shown that the spectrum of flexural vibrations has a resonance character. On the basis of this property, Baisa et al. [2] proposed a resonance vibroacoustic method of nondestructive testing, in which the value of the resonance frequency is the main informative parameter. For a further development of this method, it is necessary to know how the crack shape affects the location of the resonance line. In this work, we undertake an attempt to answer this question.

Consider an infinite two-layer plate with crack in the upper plate (coating) or on the interface between the plates. The mechanical properties of the coating are characterized by the Young modulus E_1, density ρ_1, and Poisson's ratio ν_1. The corresponding quantities for the lower plate (base) are E_0, ρ_0, and ν_0. The crack contour Γ differs from a circle. A concentrated force is applied to the free surface of the coating. The variation in this force with time is described by an exponential multiplier $\exp(-i\omega t)$, where ω is the angular frequency and t is the time. We assume that the opposite lips of the crack do not interact between themselves.

We conditionally draw a cylindrical surface with directrix coinciding with the crack contour and with generatrix perpendicular to the free surface of the plate. In this case, the entire domain occupied by the two-layer plate is divided by this surface and the opposite surfaces of the crack into three parts: Ω_1 is the domain occupied by a homogeneous finite plate over the crack, Ω_2 is the infinite domain occupied by an inhomogeneous plate outside the crack, and Ω_3 is the finite domain occupied by an inhomogeneous plate under the crack (Fig. 1). Assuming that the total half-thickness $h/2$ of the plate is small as compared with the characteristic dimension a of the crack, we obtain the classical equations of flexural vibrations for vertical displacements w_1 in the first plate. For displacements w_2 and w_3 in the two-layer second and third plates, we deduce equations whose form coincides with the classical equations but with averaged values of rigidity and density [5]:

$$d_j \Delta^2 w_j(\boldsymbol{r}) - R_j h_j \omega^2 w_j(\boldsymbol{r}) = p_j, \qquad \boldsymbol{r} \in S_j, \tag{1}$$

where $p_1 = -p_0 \delta(\boldsymbol{r} - \boldsymbol{r}_s)\delta_{1s}$, $p_2 = -p_0 \delta(\boldsymbol{r} - \boldsymbol{r}_s)\delta_{2s}$, $p_3 = 0$, $\delta(x)$ is the Dirac delta function, Δ is the Laplacian operator, S_j are the midsurfaces of the corresponding plates, $\boldsymbol{r}$ is a two-dimensional radius vector, $\boldsymbol{r}_s$, $s = 1, 2$, is the radius vector of the point of application of the force, δ_{js} is the Kronecker symbol, and subscripts $j = \overline{1, 3}$ correspond to the plate numbers. We also use the following notation in Eqs. (1):

$$d_1 = D_1, \qquad d_2 = 4(D_0 + D_2) - \frac{3(D_2 h_0 - D_0 h_2)^2}{D_2 h_0^2 + D_0 h_2^2},$$

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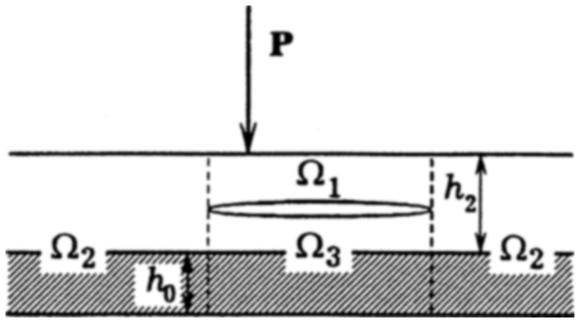


Fig. 1

$$d_3 = 4(D_3 + D_0) - \frac{3(D_3 h_0 - D_0 h_3)^2}{D_3 h_0^2 + D_0 h_2^2}, \qquad D_j = \frac{E_j h_j^3}{12(1 - \nu_j)}, \qquad j = \overline{0, 3},$$

$$E_1 \equiv E_2 \equiv E_3, \quad \nu_1 \equiv \nu_2 \equiv \nu_3, \quad R_1 = \rho_1, \quad R_2 h_2 = \rho_1 h_2 + \rho_0 h_0, \quad R_3 h_3 = \rho_1 h_3 + \rho_0 h_0.$$

Here, h_1 is the thickness of the plate placed over the crack, h_2 is the coating thickness, h_0 is the base thickness, and $h_3 = h_2 - h_1$.

We integrally satisfy the initial three-dimensional conditions of consistency of solutions in the domains Ω_j. As a result, we obtain [5] the usual conditions for continuity of displacements, angles of rotation, bending moments, and generalized shearing forces at the contour Γ:

$$w_1 = w_2 = w_3, \qquad \gamma_1 = \gamma_2 = \gamma_3,$$

$$M_2 = M_1 + M_3, \qquad F_2 = F_1 + F_3, \qquad r \in \Gamma, \tag{2}$$

where

$$\gamma_j = \left(n_1 \frac{\partial}{\partial r} + n_2 \frac{1}{r} \frac{\partial}{\partial \theta} \right) w_j(r, \theta),$$

$$M_j = n_1^2 M_{r,j} + 2 n_1 n_2 M_{r\theta,j} + n_2^2 M_{\theta,j},$$

$$F_j = Q_j - \left(-n_2 \frac{\partial}{\partial r} + n_1 \frac{1}{r} \frac{\partial}{\partial \theta} \right) M_{\tau,j}, \qquad Q_j = -d_j \Delta \gamma_j(r, \theta),$$

$$M_{\tau,j} = (n_2^2 - n_1^2) M_{r\theta,j} + n_1 n_2 (M_{r,j} - M_{\theta,j}). \tag{3}$$

Here, r and θ are the polar coordinates of the radius vector $\boldsymbol{r}$, and n_1 and n_2 are the direction cosines of the external normal to the crack contour in the polar coordinate system. The relations between the moments $M_{r,j}$, $M_{r\theta,j}$, and $M_{\theta,j}$ and displacements have the form

$$M_{r,j} = -d_j \left[\overline{\nu}_j \Delta + (1 - \overline{\nu}_j) \frac{\partial^2}{\partial r^2} \right] w_j(r, \theta),$$

$$M_{\theta,j} = -d_j\left[\Delta - (1-\bar{\nu}_j)\frac{\partial^2}{\partial r^2}\right]w_j(r,\theta),$$

$$M_{r\theta,j} = -d_j(1-\bar{\nu}_j)\frac{\partial}{r\partial\theta}\left(-\frac{1}{r}+\frac{\partial}{\partial r}\right)w_j(r,\theta), \quad j = \overline{1,3}, \tag{4}$$

where

$$\bar{\nu}_1 = \nu_1, \quad \bar{\nu}_j = \frac{2}{d_j}\left[\left(2+3\frac{z_j}{h_j}\right)\nu_j D_j + \left(2-3\frac{z_j}{h_0}\right)\nu_0 D_0\right], \quad j = 2, 3,$$

$$z_2 = \frac{\kappa h_0^2 - h_2^2}{2(\kappa h_0 + h_2)}, \quad z_3 = \frac{\kappa h_0^2 - h_3^2}{2(\kappa h_0 + h_3)}, \quad \kappa = \frac{E_0(1-\nu_1^2)}{E_1(1-\nu_0^2)}.$$

In order that a solution of the problem be unambiguous, it is necessary also to set a condition at infinity, $w_3 \to 0, \ r \to \infty$, supplemented with the radiation condition [5]. The latter consists of the fact that we preserve only those components in solutions which represent waves propagating from the source of generation of these waves to infinity.

We construct a solution of problem (1), (2) with the help of the null-field approach [8] (in some works, this approach is identified with the T-matrix method) which, certainly, was first proposed by Barantsev [3] for problems of scattering in media described by the Helmholtz scalar equation. Waterman [9] studied the possibilities of this approach in more detail, having applied it for elastic and electromagnetic media as well. In the present work, we extend the approach to problems of scattering of flexural waves.

As the initial point for the null-field approach, we take the integral representations of displacements in the domain under consideration in terms of displacements and their derivatives at the boundary of this domain. These integral representations are a corollary of the reciprocity theorem [4] and, in our case, have the form

$$-p_0 G_j(r_0, r_s)\delta_{js} + (-1)^j \int_\Gamma \left[M_j \gamma_j^G(r_0, r) - F_j G_j(r_0, r)\right] d\Gamma_r$$

$$- (-1)^j \int_\Gamma \left[\gamma_j M_j^G(r_0, r) - w_j F_j^G(r_0, r)\right] d\Gamma_r = \begin{cases} w_j(r_0), & r_0 \in S_j, \\ 0, & r_0 \notin S_j. \end{cases} \tag{5}$$

Here,

$$G_j(r_0, r) = \frac{1}{8\pi d_j k_j^2}\left[i\pi H_0^{(1)}(k_j|r-r_0|) - 2K_0(k_j|r-r_0|)\right]$$

is the fundamental solution of the equation of flexural vibrations [6], $H_n^{(1)}(x)$ are the Hankel functions of the first kind of order n, $K_n(x)$ are modified Bessel functions of order n, and $k_j = \sqrt[4]{R_j h_j \omega^2 / d_j}$. The quantities with subscript G in relations (5) can be obtained from (3), where one should take G_j instead of w_j.

By using the addition theorems for Hankel and Bessel functions [6], we can expand the fundamental solution of the equation of flexural vibrations in a complete system of functions $\Phi_{il,j}^{\sigma}(r)$ and $\bar{\Phi}_{il,j}^{\sigma}(r)$ orthogonal on a circle:

$$G_j(r, r_0) = \frac{1}{8\pi d_j k_j^2}\sum_{l=0}^{\infty}\sum_{\sigma=1}^{2}\left[i\pi\bar{\Phi}_{1l,j}^{\sigma}(r_<)\Phi_{1l,j}^{\sigma}(r_>) - 2\bar{\Phi}_{2l,j}^{\sigma}(r_<)\Phi_{2l,j}^{\sigma}(r_>)\right], \tag{6}$$

where $r_{\rangle} = r$, $r_{\langle} = r_0$ if $|r| > |r_0|$ and vice versa,

$$\Phi^{\sigma}_{1l,j}(r) = \sqrt{\varepsilon_l}\, H^{(1)}_l(k_j r)\begin{pmatrix}\cos l\theta\\ \sin l\theta\end{pmatrix}, \quad \overline{\Phi}^{\sigma}_{1l,j}(r) = \sqrt{\varepsilon_l}\, J_l(k_j r)\begin{pmatrix}\cos l\theta\\ \sin l\theta\end{pmatrix}, \quad \begin{pmatrix}\sigma = 1\\ \sigma = 2\end{pmatrix},$$

$$\Phi^{\sigma}_{2l,j}(r) = \sqrt{\varepsilon_l}\, K_l(k_j r)\begin{pmatrix}\cos l\theta\\ \sin l\theta\end{pmatrix}, \quad \overline{\Phi}^{\sigma}_{2l,j}(r) = \sqrt{\varepsilon_l}\, I_l(k_j r)\begin{pmatrix}\cos l\theta\\ \sin l\theta\end{pmatrix}, \quad \begin{pmatrix}\sigma = 1\\ \sigma = 2\end{pmatrix}, \tag{7}$$

$\varepsilon_l = 2 - \delta_{l0}$, $J_l(x)$ are the Bessel functions of the first kind, and $I_l(x)$ are the modified Bessel functions.

The null-field approach received this name because one uses the integral representations (5) to find the unknown quantities at the domain boundary in the case where r_0 is located either inside Γ for exterior domains or outside Γ for interior domains (i.e., the left-hand side in (5) is equal to zero). We choose the origin of the coordinate system inside the domain S_1. Considering Eqs. (5) for the first and third plates, we assume that r_0 is located outside the circle circumscribed around the contour Γ, and, for the second plate, it lies inside the circle inscribed into Γ. By substituting expansion (6) into relations (5) and using the orthogonality of the functions $\Phi^{\sigma}_{il,j}(r_0)$ and $\overline{\Phi}^{\sigma}_{il,j}(r_0)$, we obtain the momental equations for determining the unknown displacements, angles of rotation, shearing forces, and moments at the crack contour:

$$\int_{\Gamma}\left(M_j \gamma^{\sigma}_{il,j} - F_j \Phi^{\sigma}_{il,j} - M^{\sigma}_{il,j}\gamma_j + F^{\sigma}_{il,j} w_j\right) d\Gamma = p_0 p^{\sigma}_{il,j}, \tag{8}$$

where $p^{\sigma}_{il,1} = \overline{\Phi}^{\sigma}_{il,1}(r_s)\delta_{1s}$, $p^{\sigma}_{il,2} = -\Phi^{\sigma}_{il,2}(r_s)\delta_{2s}$, and $p^{\sigma}_{il,3} = 0$, $j = \overline{1, 3}$, i, $\sigma = 1, 2$. We obtain the values of $\gamma^{\sigma}_{il,j}$, $M^{\sigma}_{il,j}$, and $F^{\sigma}_{il,j}$ from the corresponding quantities in relations (3) and (4), setting in them $\overline{\Phi}^{\sigma}_{il,j}$ for $j = 1, 3$ and $\Phi^{\sigma}_{il,j}$ for $j = 2$ instead of w_j. For $j = 1, 3$, it is necessary to replace the functions $\Phi^{\sigma}_{il,j}$ in relations (8) by the functions $\overline{\Phi}^{\sigma}_{il,j}$.

We write down the unknown quantities at the crack contour in the form of expansions in a system of trigonometric functions:

$$\{w_1, \gamma_1, M_1, F_1, M_2, F_2\} = \sum_{m=0}^{\infty} (x^1_{km}\cos m\theta + x^2_{km}\sin m\theta), \qquad k = \overline{1, 6}. \tag{9}$$

Here, w_1 is referred to $a^2 p_0/D_1$, γ_1 to $a p_0/D_1$, M_1 and $\dot{M_2}$ to p_0, and F_1 and F_2 to p_0/a. By substituting expansions (9) into the momental equations (8) and taking the boundary conditions (2) into account, we obtain the following system of linear algebraic equations of the first kind of infinite order for determination of the coefficients of series (9):

$$\sum_{m=0}^{\infty}\sum_{k=1}^{6}\sum_{\sigma'=1}^{2} x^{\sigma'}_{km} Q^{\sigma\sigma'}_{lm,ikj} = p^{\sigma}_{il,j}, \tag{10}$$

where $j, i, \sigma = 1, 2$, $l = \overline{0, \infty}$, and the quantities $Q^{\sigma\sigma'}_{lm,ikj}$ are integrals of the type

$$Q^{\sigma\sigma'}_{lm,i1j} = \int_{\Gamma} F^{\sigma}_{il,j}\begin{pmatrix}\cos m\theta\\ \sin m\theta\end{pmatrix} d\Gamma, \qquad \begin{pmatrix}\sigma' = 1\\ \sigma' = 2\end{pmatrix}.$$

We find the displacement at an arbitrary point of the plate by formulas (5). In particular, if the observation point lies inside the circle inscribed into Γ, then we can expand the fundamental solution in the system of functions (7). In this case, the dimensionless displacements w_1 are

$$w_1(r,\theta) = \frac{d_1}{(k_2a)^2 d_2}\Bigg\{-G_2(r,\theta;r_s,\theta_s)\delta_{1s}$$

$$-\sum_{k=1}^{4}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{\sigma'=1}^{2}\left[i\pi\overline{\Phi}_{1l,1}^{\sigma}(r,\theta)x_{km}^{\sigma'}\overline{Q}_{lm,1k1}^{\sigma\sigma'} - 2\overline{\Phi}_{2l,1}^{\sigma}(r,\theta)x_{km}^{\sigma'}\overline{Q}_{lm,2k1}^{\sigma\sigma'}\right]\Bigg\}, \qquad (11)$$

where $\overline{Q}_{lm,ik1}^{\sigma\sigma'}$ can be obtained from $Q_{lm,ik1}^{\sigma\sigma'}$ if one replaces the integrands $\overline{\Phi}_{il,j}^{\sigma}$ by the functions $\Phi_{il,j}^{\sigma}$. But if the observation point is located outside the circle circumscribed around Γ, then the displacements w_2 have the form

$$w_2(r,\theta) = \frac{d_1}{(k_2a)^2 d_2}\Bigg\{-G_2(r,\theta;r_s,\theta_s)$$

$$+\sum_{k=1}^{6}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{\sigma'=1}^{2}\left[i\pi\Phi_{1l,2}^{\sigma}(r,\theta)x_{km}^{\sigma'}\overline{Q}_{lm,1k2}^{\sigma\sigma'} - 2\Phi_{2l,2}^{\sigma}(r,\theta)x_{km}^{\sigma'}\overline{Q}_{lm,2k2}^{\sigma\sigma'}\right]\delta_{k3}\delta_{k4}\Bigg\}.$$

We carried out a numerical analysis of the problem under consideration in the case where the crack contour Γ is set in the parametric form:

$$r(\beta) = a\sqrt{1+\varepsilon^2+2\varepsilon\cos(N+1)\beta}, \qquad \theta(\beta) = \arctan\frac{\sin\beta-\varepsilon\sin N\beta}{\cos\beta+\varepsilon\cos N\beta},$$

where $\beta\in[0,2\pi]$, $\varepsilon<1$, and N is an integer. Specifying a certain value of N and varying ε, we obtain some families of curves. For example, they are ellipses if $N=1$, triangles for $N=2$, and squares with rounded corners for $N=3$.

We found the solution of system (10) numerically, by the reduction method. In this case, we calculated the integrals $Q_{lm,ikj}^{\sigma\sigma'}$ analytically, using the availability of the small parameter ε, by means of the expansion of integrands in series in ε.

The dependence of the amplitude of transverse displacements w_1 at the point $r=0$ on the dimensionless wave number k_1a which are caused by the action of a concentrated force at the point $r_s=0$ are presented in Figs. 2–4. Elastic parameters were taken such that $E_1/E_0=0.5$, $\rho_1/\rho_0=1$, $\nu_1=0.25$, and $\nu_0=0.33$, and the thicknesses were $h_2/h_0=1.5$ and $h_3=0$.

In the upper right corner of each figure, we schematically depicted the crack contour to which it corresponds. We see that the spectral dependences have a resonance character. The sharp change in the vibration phase by π near the frequencies for which the displacement amplitudes reach their maximal values demonstrates that these frequencies are resonance ones (Fig. 5).

In Fig. 6, the dependence of the dimensionless wave number $k_r = a\sqrt[4]{\rho_1h_1\omega_r^2/D_1}$ on the parameter ε for various N is shown (here, ω_r is the resonance frequency). In this case, we set $E_1/E_0=0.8$, and the other parameters are the same as in Fig. 2. Taking the resonance frequency ω_r from experiments and knowing the thickness and elastic characteristics of the coating, we can easily calculate the characteristic dimension a of the crack with the help of the presented plots. To determine the shape of the crack contour, it is necessary to investigate the dependence of displacements on the angle θ at fixed frequencies. Such dependences for $r/a=0.1$ are presented in Fig. 7. We chose these frequencies in such a way that the amplitudes for different contours should be approximately equal, and the other parameters should be the same as in Fig. 6.

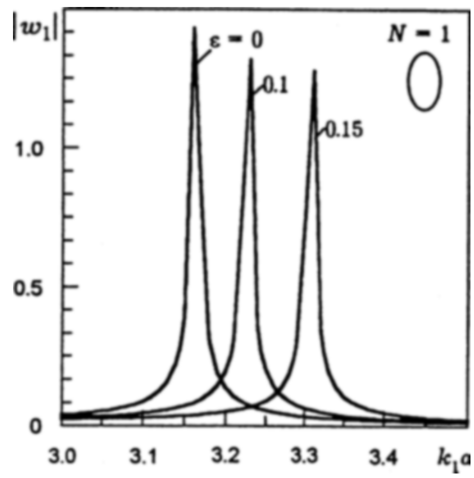


Fig. 2

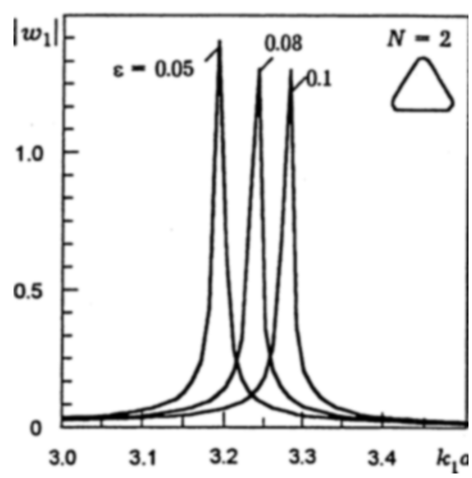
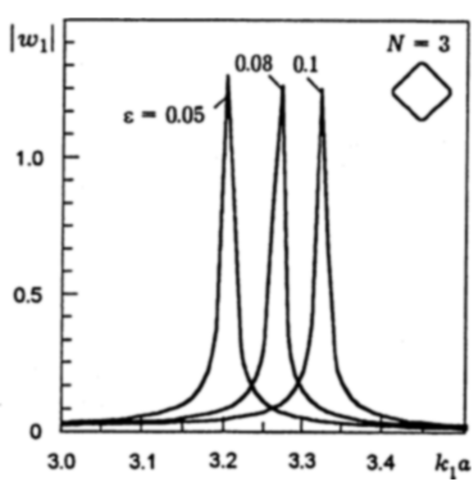


Fig. 3



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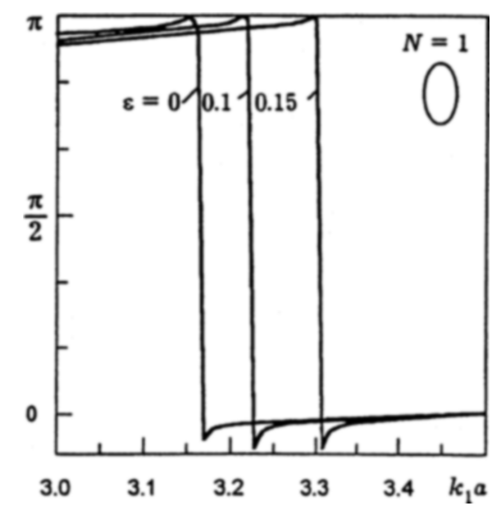

Fig. 5

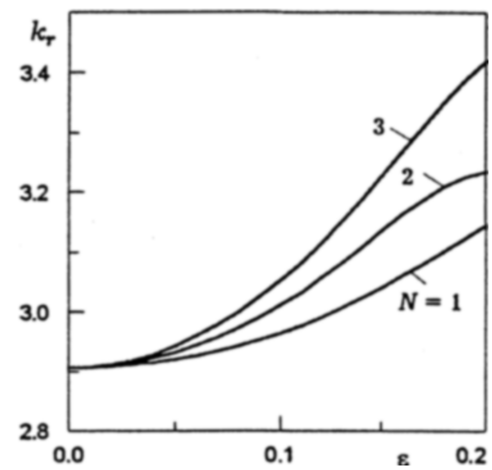

Fig. 6

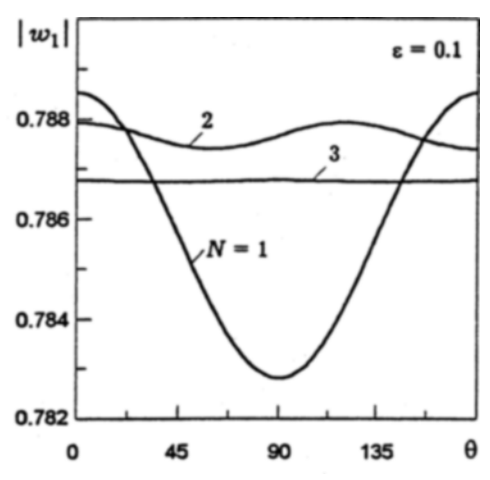

Fig. 7

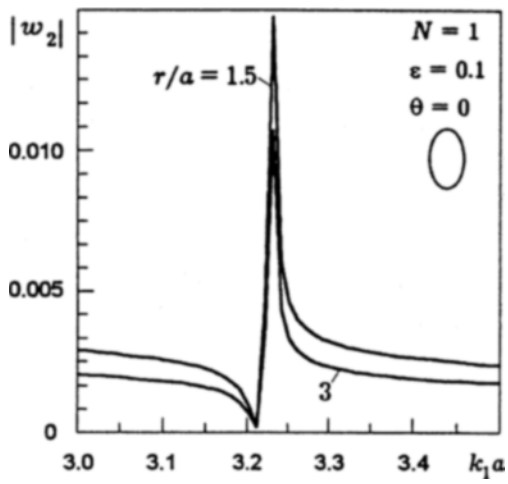

Fig. 8

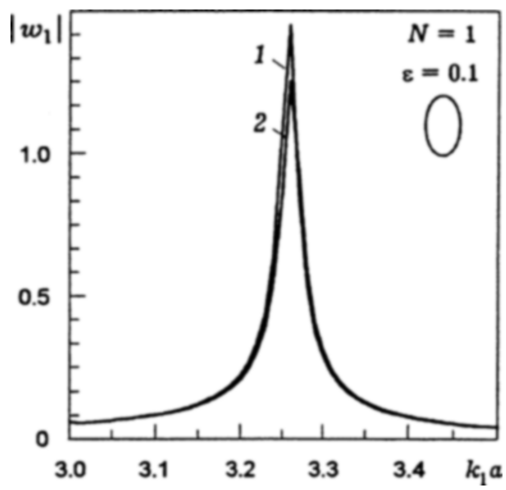

Fig. 9

In Fig. 8, the spectral character of the amplitude w_2 of displacements is shown in the case where the force is applied at the point $r_s = 0$ and the elastic and geometric parameters are the same as in Fig. 2. Note that Fig. 8 includes the coordinates of the observation point. In this case, the resonance character of the vibrations is observed as well, but the amplitudes are much smaller than those in Fig. 2.

Equations (1) in this work are valid only in the cases where the moduli of elasticity and densities of the coating and base insignificantly differ from each other [5]. However, despite this circumstance, we calculated the vibration spectrum for $E_1/E_0 = 10^{-4}$, $\rho_1/\rho_0 = 0.03$, $\nu_1 = 0.45$, $\nu_0 = 0.33$, $h_2/h_0 = 1.5$, and $h_3 = 0$ according to relations (11). The results of these calculations for an elliptic crack and $r = r_s = 0$ are depicted in Fig. 9 (curve 1). It is known [1] that, for such ratios between the characteristics of the coating and base, flexural vibrations of the plate placed over the crack must correspond to vibrations of a rigidly restrained plate, the contour of which coincides with the crack contour. For this reason, we presented the displacement amplitude of an elliptic rigidly restrained plate (curve 2), calculated according to the results in [7], in the same figure. In this case, to avoid the appearance of infinite values near the natural frequency of vibrations, we assumed that the energy absorption takes place in the material of the plate. The comparison of these curves shows that the relations obtained in this work give a correct value of the resonance frequency even for such materials.

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