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Asymptotic Statistics of Zeroes for the Lamé Ensemble

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Abstract: The Lamé polynomials naturally arise when separating variables in Laplace's equation in elliptic coordinates. The products of these polynomials form a class of spherical harmonics, which are joint eigenfunctions of a quantum completely integrable (QCI) system of commuting, second-order differential operators $P_0 = \Delta, P_1, \ldots, P_{N-1}$ acting on $C^\infty(\mathbb{S}^N)$. These operators naturally depend on parameters and thus constitute an ensemble. In this paper, we compute the limiting level-spacings distributions for the zeroes of the Lamé polynomials in various thermodynamic, asymptotic regimes. We give results both in the mean and pointwise, for an asymptotically full set of values of the parameters.

1. Introduction

Fix $N+1$ distinct, positive real numbers $0 < \alpha_0 < \cdots < \alpha_N$. Given Cartesian coordinates $(z_1, \ldots, z_{N+1}) \in \mathbb{R}^{N+1}$, consider the partial differential operators

$$P_k := \sum_{i<j} S_k^{ij}(\alpha_0, \ldots, \alpha_N) \left(z_i \frac{\partial}{\partial z_j} - z_j \frac{\partial}{\partial z_i} \right)^2; \quad k = 0, \ldots, N-1 \tag{1}$$

acting on $C^\infty(\mathbb{S}^N)$. Here, S_k^{ij} denotes the k^{th} elementary symmetric polynomial in the α parameters with α_i and α_j deleted. It is easy to check that

$$[P_i, P_j] = 0 \quad \text{for all } i, j = 0, \ldots, N-1.$$

Consequently, since the P_j's are jointly elliptic, they possess a Hilbert basis of joint eigenfunctions. Since P_0 is just the constant curvature spherical Laplacian, these eigenfunctions form a class of spherical harmonics, the so-called *generalized Lamé harmonics*

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[T1]. These systems constitute important examples of quantum completely integrable systems and they have as complex analogues the Gaudin spin-chains of various types [HW, K]. The purpose of this note is to derive asymptotic formulas for the level-spacings distributions of the zeroes of these spherical harmonics. To describe our results in more detail, we begin by noting that in terms of appropriate (elliptic-spherical [HW, T1, T2]) parametrizing coordinates $(u_1, \dots, u_N) \in (\alpha_0, \alpha_1) \times \cdots \times (\alpha_{N-1}, \alpha_N)$ on $\mathbb{S}^N$, and up to constant multiples, the joint eigenfunctions of $P_0, \dots, P_{N-1}$ can be written in the form:

$$\psi(u_1, \dots, u_N) = \prod_{j=1}^{N} S_m(\sqrt{u_j - \alpha_0}, \dots, \sqrt{u_j - \alpha_N}) \cdot \phi(u_j).$$

Here, ϕ is a polynomial, and $S_m(x_0, \dots, x_N); m = 0, \dots, N+1$ denotes the m^{th} elementary symmetric function on $N+1$ variables. Furthermore, the function $\psi(x) := S_m(\sqrt{x - \alpha_0}, \dots, \sqrt{x - \alpha_N}) \cdot \phi(x)$ is a solution of the ODE:

$$\prod_{\nu=0}^{N}(x - \alpha_\nu)\frac{d^2\psi}{dx^2} + \frac{1}{2}\sum_{\nu=0}^{N}\prod_{\lambda\neq\nu}(x - \alpha_\lambda)\frac{d\psi}{dx} + C(x)\psi = 0, \tag{2}$$

where, $C(x)$ is a polynomial of order $N-1$ depending linearly on the joint eigenvalues $(\lambda_0, \dots, \lambda_{N-1}) \in \text{Spec}\,(P_0, \dots, P_{N-1})$. The different species [WW] of harmonics are indexed by $m = 0, \dots, N+1$. Although for simplicity, we consider here the case where $m = 0$, our main result (Theorem 1.1) can be proved for the other cases corresponding to $m = 1, \dots, N+1$ in a similar fashion. When $m = 0$, the solutions $\psi(x)$ are called generalized Lamé polynomials.

Consider

$$\mathcal{E}(k) := \{\phi_1^{(k)}, \dots, \phi_{j(k)}^{(k)}\},$$

the set of Lamé polynomials of degree k. By the standard theory of spherical harmonics [WW] and the fact that the corresponding Lamé harmonics form a Hilbert basis, we know that $j(k) = \sigma(N, k) := \frac{(N+k-1)!}{k!(N-1)!}$. Let $\theta_{i,1}^{(k)} \leq \cdots \leq \theta_{i,k}^{(k)}$ denote the (real) zeroes of the polynomial, $\phi_i^{(k)}$, where $i = 1, \dots, \sigma(N, k)$.

In our main result (Theorem 1.1), we compute the asymptotic weak limit for the level spacings distribution averaged over the set, $\mathcal{E}(k)$, of k^{th} order Lamé polynomials. More precisely, consider

$$d\rho_{LS}^{AV}(x; N, k, \alpha) := \frac{1}{\sigma(N,k)}\sum_{l=1}^{\sigma(N,k)}\frac{1}{k-1}\sum_{j=1}^{k-1}\delta\left(x - k(\theta_{l,j+1}^{(k)}(\alpha) - \theta_{l,j}^{(k)}(\alpha))\right), \tag{3}$$

where $\alpha \in \Lambda^N$ and

$$\Lambda^N := \{(\alpha_0, \dots, \alpha_N) \in [0, 1]^{N+1}; \alpha_0 < \alpha_1 < \cdots < \alpha_{N-1} < \alpha_N\}. \tag{4}$$

We henceforth put the normalized Lebesgue measure $đ\alpha := (N+1)!d\alpha$ on Λ^N, so that meas $(\Lambda^N) = 1$. In order to state our first result, we will also need to introduce the integrated, averaged level-spacings distribution:

$$d\mu_{LS}(x; N, k) := \int_{\Lambda^N} d\rho_{LS}^{AV}(x; N, k, \alpha)\, đ\alpha. \tag{5}$$

Theorem 1.1. (i) *Fix* $0 < \epsilon < 1$ *and assume that* $k \sim N^{1-\epsilon}$ *as* $N \to \infty$. *Then,*

$$w - \lim_{N\to\infty} d\mu_{LS}(x; N, k) = e^{-x} dx.$$

(ii) *Suppose that* $k(N)$ *satisfies the hypotheses of part* (i). *Then, for any* $0 < \delta < \epsilon$ *there exist a measurable subset* $J^N \subset \Lambda^N$ *with meas* $(J^N) \geq 1 - N^{-\delta}$, *such that for any* $\alpha \in J^N$,

$$w - \lim_{N\to\infty} d\rho_{LS}^{AV}(x; N, k, \alpha) = e^{-x} dx.$$

In both (i) *and* (ii), *the weak-limit is taken in the dual space to* $C_0^0([a, b])$, *where* $0 \leq a < b < \infty$.

Remarks. (i) In recent work, Bleher, Shiffman and Zelditch [BSZ 1,2] have determined the asymptotics of various measures associated with the distribution of zeroes of eigensections of Toeplitz operators. The Lamé ensemble together with its complex analogues (the Gaudin spin chains) can be described in the Toeplitz framework [K]. However, in [BSZ 1,2] the averaging is carried out over a much larger ensemble: namely, *all* suitably normalized bases of Toeplitz eigensections. Most such bases are not quantum completely integrable and consequently, the situation considered in this paper is quite different from that in [BSZ 1,2]. The main point here is that we are really averaging over a comparatively small ensemble indexed by the parameters $\alpha \in \Lambda^N$ and the elements of which are all quantum completely integrable.

(ii) There are two asymptotic parameters that enter into our analysis here: k, the degree of the joint eigenfunctions, and N, the dimension of the base space, which in this case is just $\mathbb{S}^N$. So, Theorem 1.1 above is really a hybrid asymptotic result about the zeroes of the joint eigenfunctions of the P_j's on spheres of increasing dimension where we assume that the number of zeroes, k, satisfies $k(N) \sim N^{1-\epsilon}$ as $N \to \infty$. This asymptotic regime can be thought of as a kind of thermodynamic limit. It would also be of interest to determine what happens in other asymptotic ranges where the number of zeroes is permitted to grow at faster rates as $N \to \infty$. In particular, one would like to know what happens in the purely semiclassical regime, where N is *fixed* and $k \to \infty$. We hope to address these points in future work.

(iii) As the referee has pointed out, it would be of considerable interest to determine how the actual zeroes of the Lamé harmonics are distributed in the sense of a Riemann measure on $\mathbb{S}^N$ itself. A natural starting point would be to look at the density of states measures (see [ShZ, NV]). In light of our results in this paper, the zeroes should, at least on average, behave like random variables in the asymptotic regime where $k(N) \sim N^{1-\epsilon}$. Consequently, we believe that the density of states should on average tend to uniform measure on $\mathbb{S}^N$, but at present we do not know how to prove this. We plan to address this question for the Lamé harmonics as well as the more general complex XXX Gaudin spin chains in an upcoming paper.

2. The Lamé Differential Equation

We now give a brief introduction to the Lamé equation following the classical presentation in Whittaker and Watson [WW], where this equation is introduced via the theory of ellipsoidal harmonics. In his treatise on heat conduction in an ellipsoidal body, G. Lamé was led to consider the class of homogeneous, harmonic polynomials on $\mathbb{R}^{N+1}$ that vanish on a family of confocal quadrics. There is an analogous construction of spherical harmonics which we will now describe. Pick a set $\{\alpha_0, \dots, \alpha_N\}$ of positive real constants, all distinct, and ordered in increasing order. Define, for some real parameter θ, the diagonal matrix $A_\theta = \operatorname{diag}\big((\theta - \alpha_0)^{-1}, \dots, (\theta - \alpha_N)^{-1}\big)$. The problem then reduces to finding, for any positive integer k and multi-index $\beta = (\beta_0, \dots, \beta_N) \in \{0, 1\}^{N+1}$, k real numbers $\theta_1, \dots, \theta_k$ for which the Niven's functions

$$f_\beta(X) = X^\beta \prod_{j=1}^{k} (A_{\theta_j} X, X), \qquad X \in \mathbb{R}^{N+1}, \tag{6}$$

are solutions of Laplace's equation $\Delta(f_\beta) = 0$. The restrictions of the f_β's to $\mathbb{S}^N$ yield an important class of spherical harmonics: the *generalized Lamé harmonics*. Clearly, the functions $f_\beta(X)$ vanish on a family of confocal cones. Moreover, after substitution of the ansatz into Laplace's equation, a straightforward computation shows that the relevant θ_j are obtained as solutions of the equations

$$\begin{aligned} \sum_{\nu=0}^{N} \frac{1}{\theta_j(\alpha; l) - \alpha_\nu} + \sum_{\nu=0}^{N} \frac{2\beta_\nu}{\theta_j(\alpha; l) - \alpha_\nu} & \\ + \sum_{i \neq j} \frac{4}{\theta_j(\alpha; l) - \theta_i(\alpha; l)} &= 0 \quad \text{for} \quad j = 1, \dots, k. \end{aligned} \tag{7}$$

In the literature, these equations are commonly referred to as the "Bethe Ansatz" equations. Consequently, if we denote the solutions of (7) by $\theta_1, \dots, \theta_k$, it is not hard to see that the functions

$$\psi(x) = \prod_{\nu}^{N} (x - \alpha_\nu)^{\beta_\nu/2} \prod_{j=1}^{k} (x - \theta_j), \qquad \beta_\nu \in \{0, 1\} \tag{8}$$

satisfy the second order differential equation given by

$$\prod_{\nu=0}^{N} (x - \alpha_\nu) \frac{d^2\psi}{dx^2} + \frac{1}{2} \sum_{\nu=0}^{N} \prod_{\mu \neq \nu} (x - \alpha_\mu) \frac{d\psi}{dx} + C(x)\psi = 0, \tag{9}$$

where $C(x)$ is a polynomial of degree $N - 1$ that can be computed explicitly. This is exactly Eq. (2) of the introduction, and is known as the *generalized Lamé differential equation*. In the special case where the multi-index $\beta = 0$, it follows that the k^{th} degree polynomial $\phi(x) = \prod_{j=1}^{k} (x - \theta_j)$ is a solution of the Lamé equation. Following the terminology adopted previously, we refer to these as *Lamé polynomials*. Restricting our attention to the N-sphere $\mathbb{S}^N$, we can use elliptic-spherical [HW, T1, T2] coordinates $u = (u_1, \dots, u_N) \in (\alpha_0, \alpha_1) \times \cdots \times (\alpha_{N-1}, \alpha_N)$ to rewrite Niven's function in the form

$$f_\beta(u_1, \dots, u_N) = c \prod_{\nu=0}^{N} \prod_{j=1}^{N} (u_j - \alpha_\nu)^{\beta_\nu/2} \phi(u_j), \tag{10}$$

where c is some constant depending only on $\alpha_0, \dots, \alpha_N$ and $\phi(u) = \prod_{j=1}^{k} (u - \theta_j)$. The functions $\psi(u_1, \dots, u_N)$ in the introduction are simply linear combinations of the

f_β's and consequently, they are solutions of the eigenvalue problem for the Laplace's operator on $\mathbb{S}^N$ written in terms of elliptic-spherical coordinates; that is

$$\sum_{j=1}^{N} \frac{4}{\prod_{i\neq j}(u_j - u_i)} \left[\sqrt{U(u_j)}\frac{\partial}{\partial u_j}\left(\sqrt{U(u_j)}\frac{\partial \psi}{\partial u_j}\right)\right] = -\lambda_0 \psi,$$

where $U(x) = \prod_{\nu=0}^{N}(x-\alpha_\nu)$. The separated equations have the form

$$\prod_{\nu=0}^{N}(x-\alpha_\nu)\frac{d^2\psi}{dx^2} + \frac{1}{2}\sum_{\nu}^{N}\prod_{\mu\neq\nu}(x-\alpha_\mu)\frac{d\psi}{dx} + \frac{1}{4}\left(\sum_{j=0}^{N-1}\lambda_{N-j-1}x^j\right)\psi = 0, \tag{11}$$

where the separation constants $\lambda_1, \ldots, \lambda_{N-1}$ are the joint eigenvalues of the partial differential operators $P_0, \ldots, P_{N-1}$ defined in (1). Equation (11) is exactly the Lamé differential equation (2) considered in the introduction.

3. The Heine–Stieltjes Theorem

One of the key steps in the proof of Theorem 1.1 is based on a result originally obtained by M. Heine [H]. Shortly afterwards, Stieltjes [S] improved Heine's result in the special case of differential equations of Lamé's type, the case that we consider here. For a complete proof, we refer the reader to Szegö [Sz].

Theorem 3.1 (Heine–Stieltjes**).** *Let $A(x)$ be the polynomial of degree $N+1$ given by*

$$A(x) = (x-\alpha_0)\cdot(x-\alpha_1)\cdots(x-\alpha_N),$$

where $0 < \alpha_0 < \alpha_1 < \cdots < \alpha_N$ and $B(x)$ is a polynomial of degree N satisfying the condition

$$\frac{B(x)}{A(x)} = \frac{\rho_0}{x-\alpha_0} + \cdots + \frac{\rho_N}{x-\alpha_N},$$

for given numbers $\rho_\nu > 0$, $\nu = 0, \ldots, N$. Then, there are exactly $\sigma(N,k) = \frac{(N+k-1)!}{k!\,(N-1)!}$ polynomials $C(x)$ of degree $N-1$ for which the differential equation

$$A(x)\frac{d^2\phi}{dx^2} + 2B(x)\frac{d\phi}{dx} + C(x)\phi = 0 \tag{12}$$

has a polynomial solution of degree $k > 0$. In addition, for each of the $\sigma(N,k)$ solutions, $\phi(x)$, the zeroes are simple and uniquely determined by their distribution in the intervals $(\alpha_0, \alpha_1), \ldots, (\alpha_{N-1}, \alpha_N)$.

Note that the Lamé equation (11) is a particular case of the differential equation (12) appearing in the statement of the theorem: Indeed, in the Lamé case, we have that $\rho_\nu = 1/4$ for $\nu = 0, \ldots, N$. Taking into account the Heine–Stieltjes result, we denote the zeroes of $\phi(x)$ by $\theta_1(\alpha; l) \leq \cdots \leq \theta_k(\alpha; l)$, where $\alpha := (\alpha_0, \ldots, \alpha_N)$, whereas $l = (l_1, \ldots, l_k)$; $1 \leq l_1 \leq \cdots \leq l_k \leq N$ denotes the configuration of the zeroes. By this we mean that $\theta_1(\alpha; l)$ is the smallest zero lying in the interval $(\alpha_{l_1-1}, \alpha_{l_1})$, the next zero $\theta_2(\alpha; l)$ is contained in the interval $(\alpha_{l_2-1}, \alpha_{l_2})$ and so on.

Although we will not explicitly use the following result in this paper, it is of independent interest and so we include it here for future reference:

Lemma 3.2. *For any given configuration* $l = (l_1, \dots, l_k)$*, the zeroes* $\theta_1(\alpha; l), \dots, \theta_k(\alpha; l)$ *are differentiable functions of* $\alpha \in \Lambda^N$*.*

Proof. The proof is based on the argument given in [Sh]. Differentiating the Bethe Ansatz equations (7) with respect to the θ variables, we form the Jacobian matrix $B = (b_{ij})$ given by

$$b_{ij} = \begin{cases} -\sum_{\nu=0}^{N} \frac{\rho_\nu}{(\theta_j-\alpha_\nu)^2} - \sum_{m\neq i} \frac{1}{(\theta_j-\theta_m)^2} & \text{if } i = j \\ \frac{1}{(\theta_j-\theta_i)^2} & \text{if } i \neq j. \end{cases}$$

By a standard result in matrix theory (Gerŝgorin's Theorem) it follows that all the eigenvalues of B are strictly negative since if λ is an eigenvalue of B, then for some $j \in \{1, 2, \dots, k\}$,

$$\lambda \leq b_{jj} + \sum_{i\neq j} |b_{ij}| = -\sum_{\nu=0}^{N} \frac{\rho_\nu}{(\theta_j - \alpha_\nu)^2} < 0.$$

Therefore, the determinant of B is nonzero, so we can apply the implicit function theorem to conclude the proof. □

4. Level Spacings Distribution

Consider the Lamé system consisting of $N + 1$ particles located at $\alpha_0, \dots, \alpha_N$ with $0 < \alpha_0 < \cdots < \alpha_N < 1$ together with the k zeroes $\theta_1(\alpha; l) \leq \cdots \leq \theta_k(\alpha; l)$ of the polynomial solution $\phi(x)$ of the differential equation (12). Recall that, we use the notation $\theta_j(\alpha; l)$ to designate the j^{th} zero in configuration $l = (l_1, \dots, l_k)$ with respect to parameters $\alpha = (\alpha_0, \dots, \alpha_N)$. As a consequence of Heine–Stieltjes Theorem, we have that:

$$\begin{aligned} d\rho_{LS}^{AV}(x; N, k, \alpha) = \frac{1}{\sigma(N, k)} \sum_{1\leq l_1\leq\cdots\leq l_k\leq N} \frac{1}{k-1} \\ \cdot \sum_{j=1}^{k-1} \delta\left(x - k\left(\theta_{j+1}(\alpha; l) - \theta_j(\alpha; l)\right)\right), \end{aligned} \tag{13}$$

and so,

$$\begin{aligned} d\mu_{LS}(x; N, k) &:= \int_{\Lambda^N} d\rho_{LS}^{AV}(x; N, k, \alpha)\, đ\alpha \\ &= \frac{1}{\sigma(N, k)} \sum_{1\leq l_1\leq\cdots\leq l_k\leq N} \frac{1}{k-1} \sum_{j=1}^{k-1} \int_{\Lambda^N} \delta(x - k(\theta_{j+1}(\alpha; l) - \theta_j(\alpha; l)))\, đ\alpha. \end{aligned} \tag{14}$$

4.1. Proof of part (i) of Theorem 1.1. The proof is somewhat long and computational, so we divide it into several steps. For notational simplicity, we assume here that $[a, b] = [0, 1]$ and $\phi \in C_0^1([0, 1])$. The argument for more general non-negative intervals $[a, b]$ follows in the same way.

Step 1. We first show that $d\mu_{LS}(x; N, k)$ can be asymptotically approximated by fairly simple integrals that do not depend on the explicit formulas for the zeroes $\theta_1(\alpha; l), \ldots, \theta_k(\alpha; l)$. More precisely, we claim that:

$$d\mu_{LS}(x; N, k)(\phi) = \frac{1}{\sigma(N,k)} \sum_{1\le l_1\le\cdots\le l_k\le N} \frac{1}{k-1} \sum_{j=1}^{k-1} \int_{\Lambda^N} \phi(k(\alpha_{l_{j+1}} - \alpha_{l_j}))\, đ\alpha + \mathcal{O}\left(\frac{k}{N}\right). \quad (15)$$

To obtain the estimate in (15), we make a first-order Taylor expansion in (14) around $k(\alpha_{l_{j+1}} - \alpha_{l_j})$ to get:

$$d\mu_{LS}(x; N, k)(\phi) = \frac{1}{\sigma(N,k)} \sum_{1\le l_1\le\cdots\le l_k\le N} \frac{1}{k-1} \sum_{j=1}^{k-1} \int_{\Lambda^N} \phi(k(\alpha_{l_{j+1}} - \alpha_{l_j}))\, đ\alpha + E_1(N,k,\phi), \quad (16)$$

where the error term $E_1(N, k, \phi)$ is given by,

$$E_1(N,k,\phi) = \frac{1}{\sigma(N,k)} \sum_{1\le l_1\le\cdots\le l_k\le N} \frac{k}{k-1} \cdot \sum_{j=1}^{k-1} \int_{\Lambda^N} \phi'(k\xi_j(\alpha)) \left[\left(\theta_{j+1}(\alpha; l) - \alpha_{l_{j+1}}\right) - \left(\theta_j(\alpha; l) - \alpha_{l_j}\right)\right]\, đ\alpha,$$

with $\xi_j(\alpha) \in (0, 1)$. We need to show that $E_1(N, k, \phi) = \mathcal{O}\left(\frac{k}{N}\right)$.

First, we start with a simple calculus lemma:

Lemma 4.1. *For any* $0 \le i \le j \le N$ *and multi-indices* $\beta = (\beta_1, \beta_2) \in \mathbb{N}^2 \setminus \{(0,0)\}$, *we have*

$$\int_{\Lambda^N} \alpha_i^{\beta_1} \alpha_j^{\beta_2}\, đ\alpha = \frac{\prod_{l=1}^{\beta_1}(i+l) \prod_{l=1}^{\beta_2}(\beta_1 + j + l)}{\prod_{l=1}^{|\beta|}(N+1+l)},$$

where we define products of the form $\prod_{l=1}^{0}$ *to be equal to* 1 *and* $|\beta| := \beta_1 + \beta_2$.

Proof of Lemma 4.1. A direct computation with iterated integrals gives

$$
\begin{aligned}
\int_{\Lambda^N} \alpha_i^{\beta_1}\alpha_j^{\beta_2}\, đ\alpha &= \int_{0<\alpha_0<\cdots<\alpha_N<1} \alpha_i^{\beta_1}\alpha_j^{\beta_2}\, đ\alpha \\
&= (N+1)! \int_0^1 \int_0^{\alpha_N} \cdots \int_0^{\alpha_1} \alpha_j^{\beta_2}\alpha_i^{\beta_1}\, d\alpha_0 \cdots d\alpha_N \\
&= \frac{(N+1)!}{i!} \int_0^1 \int_0^{\alpha_N} \cdots \int_0^{\alpha_{i+1}} \alpha_j^{\beta_2}\alpha_i^{\beta_1+i}\, d\alpha_i \cdots d\alpha_N \\
&= \frac{(N+1)!}{i!(\beta_1+i+1)\ldots(\beta_1+j)} \int_0^1 \int_0^{\alpha_N} \cdots \int_0^{\alpha_{j+1}} \alpha_j^{\beta_2+\beta_1+j}\, d\alpha_j \cdots d\alpha_N \\
&= \frac{(N+1)!}{i!\,(\beta_1+i+1)\ldots(\beta_1+j)\,(\beta_1+\beta_2+j+1)\ldots(\beta_1+\beta_2+N+1)} \\
&= \frac{\prod_{l=1}^{\beta_1}(i+l)\ \prod_{l=1}^{\beta_2}(\beta_1+j+l)}{\prod_{l=1}^{|\beta|}(N+1+l)}. \qquad \square
\end{aligned}
$$

As a consequence of Lemma 4.1, we see that the integrals of consecutive monomials over the truncated positive Weyl chamber Λ^N are asymptotically equal as $N \to \infty$. Combined with the Heine–Stieltjes result, this fact leads to the following simple corollary of Lemma 4.1:

Corollary 4.2. *For any configuration $l = (l_1, \ldots, l_k)$ and integer j satisfying $1 \le j \le k$, we have that*

$$\int_{\Lambda^N} \left|\theta_j(\alpha; l) - \alpha_{l_j}\right|\, đ\alpha = \mathcal{O}(N^{-1}) \tag{17}$$

uniformly in k.

Proof of Corollary 4.2. As a consequence of the Heine–Stieltjes theorem, we know that given a configuration l, the j^{th} zero necessarily lies in the interval $(\alpha_{l_j-1}, \alpha_{l_j})$; that is,

$$\alpha_{l_j-1} \le \theta_j(\alpha; l) \le \alpha_{l_j}.$$

On the other hand, by Lemma 4.1, $\int_{\Lambda^N} \alpha_j\, đ\alpha = \frac{j+1}{N+2}$. Thus,

$$
\begin{aligned}
\int_{\Lambda^N} \left|\theta_j(\alpha; l) - \alpha_{l_j}\right|\, đ\alpha &\le \int_{\Lambda^N} \left(\alpha_{l_j} - \alpha_{l_j-1}\right)\, đ\alpha \\
&= \frac{l_j+1}{N+2} - \frac{l_j}{N+2} \\
&= \mathcal{O}(N^{-1}). \qquad \square
\end{aligned}
$$

Given Corollary 4.2, we can now estimate the error term, $E_1(N, k, \phi)$, in (11) as follows:

$$E_1(N,k,\phi) \le \frac{1}{\sigma(N,k)} \sum_{1\le l_1\le\cdots\le l_k\le N} \frac{k}{k-1} \cdot \sum_{j=1}^{k-1} \left(\int_{\Lambda^N} \left|\theta_{j+1}(\alpha;l) - \alpha_{l_{j+1}}\right| \, đ\alpha + \int_{\Lambda^N} \left|\theta_j(\alpha;l) - \alpha_{l_j}\right| \, đ\alpha \right) \|\phi\|_{C^1} = \mathcal{O}\left(\frac{k}{N}\right).$$

This proves the identity in (15) and so Step 1 is complete.

Step 2. The next step involves computing the first term on the RHS of (15) explicitly. We claim that:

$$\frac{1}{\sigma(N,k)} \sum_{1\le l_1\le\cdots\le l_k\le N} \frac{1}{k-1} \sum_{j=1}^{k-1} \int_{\Lambda^N} \phi\left(k\left(\alpha_{l_{j+1}} - \alpha_{l_j}\right)\right) \, đ\alpha = \frac{k}{N+k-1}\,\phi(0) + \frac{N+1}{\sigma(N,k)} \sum_{m=0}^{N-2} \sigma(N-m-1, k-1) \int_0^1 \phi(kx)\mathrm{binom}(N,m;x)\,dx \tag{18}$$

where $\mathrm{binom}(N, m; x) := \frac{N!}{m!(N-m)!} x^m (1-x)^{N-m}$ for $x \in [0, 1]$. In order to prove the identity in (18), we start with a simple lemma which involves a successive application of the Fubini theorem.

Lemma 4.3. *For any integers i, j with $0 \le i < j \le N$, we have that*

$$\int_{\Lambda^N} \phi\left(k\left(\alpha_j - \alpha_i\right)\right) \, đ\alpha = (N+1) \int_0^1 \phi(kx)\mathrm{binom}(N, j-i-1; x)\,dx. \tag{19}$$

Proof of Lemma 4.3. Given the definition of Λ^N, it is clear that

$$\int_{\Lambda^N} \phi\left(k\left(\alpha_j - \alpha_i\right)\right) \, đ\alpha = (N+1)! \int_0^1 \int_0^{\alpha_N} \cdots \int_0^{\alpha_1} \phi\left(k\left(\alpha_j - \alpha_i\right)\right) \, d\alpha_0 \cdots d\alpha_N.$$

By repeated application of Fubini's theorem, we can ensure that the iterated integrals with respect to α_i and α_j are carried out last. More precisely, we apply Fubini's theorem on the double integral with respect to α_j and α_{j+1} to reverse the order of integration. We then repeat the same procedure for the double integral with respect to α_j and α_{j+2} and so on, until we bring the last integration with respect to α_j. This gives

$$\int_{\Lambda_N} d\alpha = \int_0^1 \int_{\alpha_j}^1 \int_{\alpha_j}^{\alpha_N} \cdots \int_{\alpha_j}^{\alpha_{j+2}} \int_0^{\alpha_j} \cdots \int_0^{\alpha_1} d\alpha_0 \ldots d\alpha_{j-1} d\alpha_{j+1} \ldots d\alpha_{N-1} d\alpha_N d\alpha_j.$$

We proceed in a similar manner for α_i to finally obtain:

$$\int_{\Lambda^N} d\alpha = \int_0^1 \int_{\alpha_j}^1 \cdots \int_{\alpha_j}^{\alpha_{j+2}} \int_0^{\alpha_j} \int_{\alpha_i}^{\alpha_j} \cdots \int_{\alpha_i}^{\alpha_{i+2}} \int_0^{\alpha_i} \cdots \int_0^{\alpha_1} d\alpha_0 \ldots \\ \ldots d\alpha_{i-1} d\alpha_{i+1} \ldots d\alpha_{j-1} d\alpha_i d\alpha_{j+1} \ldots d\alpha_N d\alpha_j$$

$$= \int_0^1 \int_0^{\alpha_j} \int_{\alpha_j}^1 \cdots \int_{\alpha_j}^{\alpha_{j+2}} \int_{\alpha_i}^{\alpha_j} \cdots \int_{\alpha_i}^{\alpha_{i+2}} \int_0^{\alpha_i} \cdots \int_0^{\alpha_1} d\alpha_0 \ldots \widehat{d\alpha_i} \ldots \\ \ldots \widehat{d\alpha_j} \ldots d\alpha_N d\alpha_i d\alpha_j,$$

where $\widehat{d\alpha_i}$, $\widehat{d\alpha_j}$ means that these variables are omitted in the product measure $d\alpha_0 \cdots d\alpha_N$. We then carry out the iterated integration over the first $N-2$ variables $\alpha_0 < \alpha_1 < \cdots < \alpha_{i-1} < \alpha_{i+1} < \cdots < \alpha_{j-1} < \alpha_{j+1} < \cdots < \alpha_N$ to get

$$\int_{\Lambda^N} \bar{d}\alpha = (N+1)! \int_0^1 \int_0^{\alpha_j} \frac{\alpha_i^i}{i!} \frac{(\alpha_j - \alpha_i)^{j-i-1}}{(j-i-1)!} \frac{(1-\alpha_j)^{N-j}}{(N-j)!} \, d\alpha_i d\alpha_j.$$

Finally, we make the change of variables $x = \alpha_j - \alpha_i$, $y = \alpha_j$ and integrate by parts i times with respect to y. It follows that

$$\int_{\Lambda^N} \phi\left(k\left(\alpha_j - \alpha_i\right)\right) \bar{d}\alpha = (N+1)! \int_0^1 \phi(kx) \frac{x^{j-i-1}}{(j-i-1)!} \frac{(1-x)^{N-(j-i-1)}}{(N-(j-i-1))!} \, dx \\ = (N+1) \int_0^1 \phi(kx) \mathrm{binom}(N, j-i-1; x) dx.$$

This completes the proof of Lemma 4.3. □

To complete Step 2, we need to compute the asymptotic averages (18) of the integrals $\int_{\Lambda^N} \phi(k(\alpha_{l_{j+1}} - \alpha_{l_j})) \, \bar{d}\alpha$. First, we start with a simple combinatorial lemma. In order to state the lemma, it is useful to introduce some notation at this point: We denote by $S_j(m)$ the set of all configurations $l = (l_1, \ldots, l_k)$ for which $l_{j+1} - l_j = m$. As the following lemma shows, the cardinality of $S_j(m)$ is independant of j.

Lemma 4.4. *For each $m = 0, \ldots, N-1$ and each $j = 1, \ldots, k$, the number of k-tuples $(l_1, \ldots, l_k)$ with $1 \le l_1 \le \cdots \le l_k \le N$ and satisfying $l_{j+1} - l_j = m$ is given by*

$$\sigma(N-m, k-1) = \frac{(N-m+k-2)!}{(k-1)! \cdot (N-m-1)!}.$$

Proof of Lemma 4.4. By identifying the j^{th} and the $j+1^{\text{st}}$ zeroes, we are reduced to the problem of distributing $k-1$ zeroes amongst the remaining $N-m$ slots. Clearly, this number is given by $\sigma(N-m, k-1)$. □

As a consequence of Lemma 4.4,

$$\frac{1}{\sigma(N,k)} \sum_{1 \le l_1 \le \cdots \le l_k \le N} \frac{1}{k-1} \sum_{j=1}^{k-1} \int_{\Lambda^N} \phi\left(k\left(\alpha_{l_{j+1}} - \alpha_{l_j}\right)\right) \bar{d}\alpha \\ = \frac{1}{(k-1)} \frac{1}{\sigma(N,k)} \sum_{m=0}^{N-1} \sum_{j=1}^{k-1} \sum_{l \in S_j(m)} \int_{\Lambda^N} \phi\left(k\left(\alpha_{l_{j+1}} - \alpha_{l_j}\right)\right) \bar{d}\alpha.$$

In order to apply Lemma 4.3, we need to treat the two cases where $m = 0$ and $m > 0$ separately. Since by Lemma 4.4, we know that $S_j(0) = \sigma(N, k-1)$, it is clear that

$$\frac{1}{\sigma(N,k)} \sum_{1\le l_1\le\cdots\le l_k\le N} \frac{1}{k-1} \sum_{j=1}^{k-1} \int_{\Lambda^N} \phi\left(k\left(\alpha_{l_{j+1}} - \alpha_{l_j}\right)\right)\, đ\alpha \tag{20}$$
$$= \frac{\sigma(N,k-1)}{\sigma(N,k)}\phi(0) + \frac{1}{(k-1)}\frac{1}{\sigma(N,k)} \sum_{m=1}^{N-1}\sum_{j=1}^{k-1}\sum_{l\in S_j(m)} \int_{\Lambda^N} \phi\left(k\left(\alpha_{l_{j+1}} - \alpha_{l_j}\right)\right)\, đ\alpha.$$

On one hand, we can apply Lemma 4.3 to the RHS of (20) to get

$$\frac{\sigma(N,k-1)}{\sigma(N,k)}\phi(0)$$
$$+ \frac{1}{(k-1)}\frac{1}{\sigma(N,k)} \sum_{m=1}^{N-1}\sum_{j=1}^{k-1}\sum_{l\in S_j(m)} \int_{\Lambda^N} \phi\left(k\left(\alpha_{l_{j+1}} - \alpha_{l_j}\right)\right)\, đ\alpha = \frac{k}{N+k-1}\phi(0)$$
$$+ \frac{N+1}{(k-1)\sigma(N,k)} \sum_{m=1}^{N-1}\sum_{j=1}^{k-1}\sum_{l\in S_j(m)} \int_0^1 \phi(kx)\mathrm{binom}(N, l_{j+1} - l_j - 1; x)\, dx. \tag{21}$$

On the other hand, an application of Lemma 4.4 allow us to remove the summations over j and l in (21), so that

$$\frac{N+1}{(k-1)\sigma(N,k)} \sum_{m=1}^{N-1}\sum_{j=1}^{k-1}\sum_{l\in S_j(m)} \int_0^1 \phi(kx)\mathrm{binom}(N, l_{j+1} - l_j - 1; x)\, dx$$
$$= \frac{N+1}{\sigma(N,k)} \sum_{m=0}^{N-2} \sigma(N-m-1, k-1) \int_0^1 \phi(kx)\mathrm{binom}(N, m; x) dx. \tag{22}$$

The identity in (18) follows from (21) and (22) and so, Step 2 is complete.

Step 3. Summing up, as a result of Steps 1 and 2 we have shown that:

$$d\mu_{LS}(x; N, K)(\phi) =$$
$$\frac{N+1}{\sigma(N,k)} \sum_{m=0}^{N-2} \sigma(N-m-1, k-1) \int_0^1 \phi(kx)\mathrm{binom}(N, m; x)\, dx + \mathcal{O}\left(\frac{k}{N}\right). \tag{23}$$

Our next task is to further simplify the expression on the RHS of (23) by appealing to the theory of Bernstein approximations [D]. First, we need to estimate the quotient $\frac{\sigma(N-m-1,k-1)}{\sigma(N,k)}$ appearing on the RHS of (23). For this, it is convenient to consider two cases:

Case 1. $(m << (N/k)^{1+\beta})$; $0 < \beta < 1$. Under this assumption,

$$\begin{aligned}\frac{\sigma(N-m-1,k-1)}{\sigma(N,k)} &= \frac{k}{N+k-1}\prod_{j=0}^{k-2}\left(1-\frac{m+1}{N+j}\right)\\ &= \frac{k}{N+k-1}\exp\left[\sum_{j=0}^{k-2}\log\left(1-\frac{m+1}{N+j}\right)\right]\\ &= \frac{k}{N+k-1}\exp\left[\sum_{j=0}^{k-2}\log\left(1-\frac{m+1}{N}\cdot\frac{1}{1+\frac{j}{N}}\right)\right]\\ &= \frac{k}{N+k-1}\exp\left[-\sum_{j=0}^{k-2}\frac{m+1}{N}\cdot\frac{1}{1+\frac{j}{N}}+\mathcal{O}\left(\frac{km^2}{N^2}\right)\right].\end{aligned} \tag{24}$$

After some further simplification involving Taylor expansions, we get that

$$\begin{aligned}&\frac{\sigma(N-m-1,k-1)}{\sigma(N,k)}\\ &= \frac{k}{N+k-1}\exp\left[-\sum_{j=0}^{k-2}\frac{m+1}{N}\left(1+\mathcal{O}\left(\frac{j}{N}\right)\right)+\mathcal{O}\left(\frac{km^2}{N^2}\right)\right]\\ &= \frac{k}{N+k-1}\exp\left[-\frac{(k-2)(m+1)}{N}+\mathcal{O}\left(\frac{mk^2}{N^2}\right)+\mathcal{O}\left(\frac{km^2}{N^2}\right)\right].\\ &= \frac{k}{N+k-1}\exp\left(\frac{-mk}{N}\right)\left(1+\mathcal{O}\left(\frac{mk^2}{N^2}\right)+\mathcal{O}\left(\frac{km^2}{N^2}\right)\right).\end{aligned}$$

Finally, using the fact that $x^pe^{-x} = \mathcal{O}_p(1);$ for all $x \geq 0$, we get

$$\frac{\sigma(N-m-1,k-1)}{\sigma(N,k)} = \frac{k}{N+k-1}\exp\left(\frac{-mk}{N}\right)+\mathcal{O}\left(\frac{k^2}{N^2}\right)+\mathcal{O}\left(\frac{1}{N}\right). \tag{25}$$

Case 2. $(m >> (N/k)^{1+\beta})$. In this case, we can choose $0 < \beta < \frac{1-\epsilon}{\epsilon}$ so that with appropriate constants $C_1, C_2 > 0$,

$$\begin{aligned}\frac{\sigma(N-m-1,k-1)}{\sigma(N,k)} &= \frac{k}{N+k-1}\prod_{j=0}^{k-2}\left(1-\frac{m+1}{N+j}\right)\\ &\leq \frac{k}{N+k-1}\prod_{j=0}^{k-2}\left(1-C_1\frac{N^\beta}{k^{1+\beta}}\right)\\ &= \mathcal{O}\left(e^{-C_2(N/k)^\beta}\right).\end{aligned} \tag{26}$$

Substituting the estimates (25) and (26) into (23) and using the fact that

$$\sum_{m=0}^{N}\mathrm{binom}(N,m;x) = 1 \quad \text{and} \quad \int_0^1 \phi(kx)dx = \mathcal{O}(k^{-1})$$

gives:

$$
\begin{aligned}
d\mu_{LS}(x; N, K)(\phi) &= \frac{k(N+1)}{N+k-1} \sum_{m=0}^{N-2} e^{-\frac{mk}{N}} \\
&\quad \cdot \int_0^1 \phi(kx)\mathrm{binom}(N, m; x)\, dx + \mathcal{O}\left(\frac{k}{N}\right) + \mathcal{O}\left(\frac{1}{k}\right) \\
&= \frac{k(N+1)}{N+k-1} \sum_{m=0}^{N} e^{\frac{-mk}{N}} \\
&\quad \cdot \int_0^1 \phi(kx)\mathrm{binom}(N, m; x)\, dx + \mathcal{O}\left(\frac{k}{N}\right) + \mathcal{O}\left(\frac{1}{k}\right) \qquad (27)
\end{aligned}
$$

since the terms for $m = N - 1$ and $m = N$ are bounded by $1/N$. Recall that for a function $f(x)$ defined on $[0, 1]$, the N^{th} degree Bernstein polynomial of $f(x)$ is defined to be [D]:

$$B_N(f; x) = \sum_{m=0}^{N} f\left(\frac{m}{N}\right) \mathrm{binom}(N, m; x).$$

It is easy to see that in the special case where $\exp_{-k}(x) := e^{-kx}$, there is a concise closed-form expression for $B_N(\exp_{-k}; x)$; indeed,

$$B_N(\exp_{-k}; x) = \left(xe^{-\frac{k}{N}} + (1-x)\right)^N. \qquad (28)$$

From the identity in (28) we easily derive the following:

Lemma 4.5. *For $x \geq 0$, we have that*

$$B_N(\exp_{-k}; x) = e^{-kx} + \mathcal{O}\left(\frac{k}{N}\right). \qquad (29)$$

Proof of Lemma 4.5. Expand $e^{-\frac{k}{N}}$ in a second-order Taylor series and use the identity (28) directly to get

$$B_N(\exp_{-k}; x) = \left[1 + x\left(e^{-\frac{k}{N}} - 1\right)\right]^N = \left[1 - \frac{kx}{N}\left(1 + \mathcal{O}\left(\frac{k}{N}\right)\right)\right]^N.$$

From the inequality

$$0 \leq e^{-x} - \left(1 - \frac{x}{N}\right)^N \leq \frac{x^2 e^{-x}}{N},$$

and the fact that $x^p e^{-x} = \mathcal{O}_p(1)$ for all $x \geq 0$, it follows that

$$B_N(e^{-kx}; x) = e^{-kx}\left(1 + \mathcal{O}\left(\frac{k^2 x}{N}\right)\right) + \mathcal{O}(N^{-1}) = e^{-kx} + \mathcal{O}\left(\frac{k}{N}\right).$$

This completes the proof of the lemma. □

Substituting (29) into (27), we finally obtain

$$\begin{aligned} d\mu_{LS}(x; N, K)(\phi) =& \frac{k(N+1)}{N+k-1} \int_0^1 \phi(kx) B_N(\exp_{-k}; x)\, dx + \mathcal{O}\left(\frac{k}{N}\right) + \mathcal{O}\left(\frac{1}{k}\right) \\ &= \frac{k(N+1)}{N+k-1} \int_0^1 \phi(kx) e^{-kx}\, dx + \mathcal{O}\left(\frac{k}{N}\right) + \mathcal{O}\left(\frac{1}{k}\right) \\ &= \int_0^1 \phi(x) e^{-x}\, dx + \mathcal{O}\left(\frac{k}{N}\right) + \mathcal{O}\left(\frac{1}{k}\right). \end{aligned} \tag{30}$$

By noting that $C_0^1([0, 1])$ is dense in $C_0^0([0, 1])$, this completes the proof of Theorem 1.1 (i). □

4.2. Proof of part (ii) of Theorem 1.1. For convenience, we henceforth denote by $\sum_{l,l'}$ the double sum over the indices $1 \leq l_1 \leq \cdots \leq l_k \leq N$ and $1 \leq l'_1 \leq \cdots \leq l'_k \leq N$. We also define $\phi_k(x) := \phi(kx)$. First, we claim that

$$\begin{aligned} &\int_{\Lambda^N} \left| d\rho_{LS}^{AV}(x; N, k, \alpha)(\phi) \right|^2 \, d\!\!\!{}^-\alpha \\ &= \frac{1}{\sigma^2(N, k)} \sum_{l,l'} \frac{1}{(k-1)^2} \sum_{i,j=1}^{k-1} \int_{\Lambda^N} \phi_k(\alpha_{l_{j+1}} - \alpha_{l_j}) \phi_k(\alpha_{l'_{i+1}} - \alpha_{l'_i})\, d\!\!\!{}^-\alpha + \mathcal{O}\left(\frac{k}{N}\right). \end{aligned} \tag{31}$$

To obtain (31), we essentially repeat the argument of Step 1 in Sect. 4.1. That is, we expand each of the functions $\phi_k(\theta_{j+1}(\alpha; l) - \theta_j(\alpha; l))$ and $\phi_k(\theta_{i+1}(\alpha; l') - \theta_i(\alpha; l'))$ in a first-order Taylor series around the points $(\alpha_{l_{j+1}} - \alpha_{l_j})$ and $(\alpha_{l'_{i+1}} - \alpha_{l'_i})$ respectively. First, we claim that the terms involving the derivative of ϕ_k are all $\mathcal{O}\left(\frac{k}{N}\right)$. Indeed, the Heine–Stieltjes Theorem and Lemma 4.1 imply that

$$\begin{aligned} \int_{\Lambda^N} \left|\theta_j(\alpha, l) - \alpha_{l_j}\right| \left|\theta_i(\alpha, l') - \alpha_{l'_i}\right| \, d\!\!\!{}^-\alpha &\leq \int_{\Lambda^N} \left(\alpha_{l_j+1} - \alpha_{l_j}\right) (\alpha_{l'_i+1} - \alpha_{l'_i})\, d\!\!\!{}^-\alpha \\ &= \frac{(l'_i+2)(l_j+3) - (l'_i+1)(l_j+3) - (l'_i+2)(l_j+2) + (l'_i+1)(l_j+2)}{(N+2)(N+3)} \\ &= \frac{1}{(N+2)(N+3)}. \end{aligned}$$

Thus, it follows that

$$\int_{\Lambda^N} \left|\theta_j(\alpha, l) - \alpha_{l_j}\right| \left|\theta_i(\alpha, l') - \alpha_{l_{i'}}\right| \, d\!\!\!{}^-\alpha = \mathcal{O}(N^{-2}), \tag{32}$$

uniformly for all $l'_i, l_j \in \{1, \ldots, N\}$. Consequently, the terms involving the derivative of ϕ are all $\mathcal{O}\left(\frac{k}{N}\right)$ as desired. In the integral (32) we have assumed without loss of generality that $l'_i < l_j$. The other cases $l'_i = l_j$ and $l'_i > l_j$ can be treated in a similar fashion. We next prove an L^2 estimate for $d\rho_{LS}^{AV}(x; N, k, \alpha)(\phi)$ and then derive as an immediate corollary an estimate for the variance of $d\rho_{LS}^{AV}(x; N, k, \alpha)(\phi)$. In order to simplify the writing in the next proposition, we introduce the following notation for the multinomial coefficient:

$$\mathrm{multi}(N-1, m, m'; x, y) := \frac{N!}{m!m'!(N-m-m')!} x^m y^{m'} (1-x-y)^{N-m-m'},$$

where m, m' are positive integers satisfying $m + m' \leq N$ and $x, y \in [0, 1]$.

Lemma 4.6. (i) *For any* $x, y \in [0, 1]$,

$$\sum_{m'=0}^{N-2} \sum_{m=0}^{N-2-m'} e^{-\frac{km}{N}} e^{-\frac{km'}{N}} \mathrm{multi}(N-1, m', m; x, y) = e^{-kx-ky} + \mathcal{O}\left(\frac{k}{N}\right). \quad (33)$$

(ii) *Also, for* $0 \leq x \leq y \leq \frac{1}{k}$,

$$\sum_{m=0}^{N-2} \sum_{m'=0}^{m} e^{-\frac{km}{N}} e^{-\frac{km'}{N}} \mathrm{multi}(N-1, m', N-m; x, 1-y) = e^{-kx-ky} + \mathcal{O}\left(\frac{k}{N}\right). \quad (34)$$

Proof of Lemma 4.6. (i) As in the proof of the first part of the theorem, modulo $\mathcal{O}(N^{-1})$ errors, we can replace $N-2$ by $N-1$ in the upper limit of both summations. Define $\exp_{-k}(x) := \exp(-kx)$. Then, as a consequence of Lemma 4.5, we have that

$$\begin{aligned}
&\sum_{m'=0}^{N-2} \sum_{m=0}^{N-2-m'} e^{-\frac{km}{N}} e^{-\frac{km'}{N}} \mathrm{multi}(N-1, m', m; x, y) \\
&\quad = \sum_{m'=0}^{N-1} \sum_{m=0}^{N-1-m'} \mathrm{multi}(N-1, m', m; xe^{-\frac{k}{N}}, ye^{-\frac{k}{N}}) + \mathcal{O}(N^{-1}) \\
&\quad = \left(1 - (x+y) + (x+y)e^{-\frac{k}{N}}\right)^{N-1} + \mathcal{O}(N^{-1}) \\
&\quad = B_N(\exp_{-k}; x+y) + \mathcal{O}\left(\frac{k}{N}\right) \\
&\quad = \exp(-kx-ky) + \mathcal{O}\left(\frac{k}{N}\right).
\end{aligned}$$

(ii) Once again, we can replace $N-2$ by $N-1$ in the upper limit of the first sum. We make successive applications of the binomial theorem to get:

$$\begin{aligned}
&\sum_{m=0}^{N-2}\sum_{m'=0}^{m} e^{-\frac{km}{N}} e^{-\frac{km'}{N}} \mathrm{multi}(N-1, m', N-m; x, 1-y) \\
&\quad = (N-1)! \sum_{m=0}^{N-1} e^{-\frac{km}{N}} \frac{\left(x-y+ye^{-\frac{k}{N}}\right)^{m-1}}{(m-1)!} \frac{(1-x)^{N-m}}{(N-m)!} + \mathcal{O}(N^{-1}) \\
&\quad = e^{-\frac{k}{N}} \left(1-(x+ye^{-\frac{k}{N}}) + (x+ye^{-\frac{k}{N}})e^{-\frac{k}{N}}\right)^{N-1} + \mathcal{O}(N^{-1}) \\
&\quad = e^{-\frac{k}{N}} B_{N-1}(\exp_{-k}; x+e^{-\frac{k}{N}}y) + \mathcal{O}(N^{-1}) \\
&\quad = e^{-kx-ky} + \mathcal{O}\left(\frac{k}{N}\right). \qquad \square
\end{aligned}$$

We now use the combinatorial identities in Lemma 4.6 to estimate the variance of the averaged level-spacings measures:

Proposition 4.7. *For any $\phi \in C_0^1([0,1])$, we have that*

$$\int_{\Lambda^N} \left| d\rho_{LS}^{AV}(x; N, k, \alpha)(\phi) \right|^2 \, d\!\!\!{}^-\alpha = \left(\int_0^1 e^{-x}\phi(x)\,dx \right)^2 + \mathcal{O}\left(\frac{k}{N}\right) + \mathcal{O}\left(\frac{1}{k}\right). \tag{35}$$

Proof of Proposition 4.7. As a consequence of the estimate in (31), it suffices to show that

$$\begin{aligned}
&\frac{1}{\sigma^2(N,k)} \sum_{l,l'} \frac{1}{(k-1)^2} \sum_{i,j=1}^{k-1} \int_{\Lambda^N} \phi_k(\alpha_{l_{j+1}} - \alpha_{l_j}) \phi_k(\alpha_{l'_{i+1}} - \alpha_{l'_i}) \, d\!\!\!{}^-\alpha \\
&\qquad\qquad = \left(\int_0^1 e^{-x}\phi(x)dx \right)^2 + \mathcal{O}\left(\frac{k}{N}\right) + \mathcal{O}\left(\frac{1}{k}\right).
\end{aligned}$$

In order to show this, we need to distinguish three different cases corresponding to the various relative configurations of $\alpha_{l_i'}, \alpha_{l_{i+1}'}, \alpha_{l_j}$ and $\alpha_{l_{j+1}}$:

Case 1. $\alpha_{l_i'} < \alpha_{l_{i+1}'} < \alpha_{l_j} < \alpha_{l_{j+1}}$ *(or equivalently,* $\alpha_{l_j} < \alpha_{l_{j+1}} < \alpha_{l_i'} < \alpha_{l_{i+1}'}$ *).*

The argument is essentially the same as in Lemma 4.3. The only difference is that instead of getting a simple integral, we obtain a double integral at the end of the iterated integration. More precisely, just as in Step 2, we make repeated applications of Fubini's Theorem to ensure that the last four iterated integrals are with respect to $\alpha_{l_i'}, \alpha_{l_{i+1}'}, \alpha_{l_j}$ and $\alpha_{l_{j+1}}$ variables. We then integrate by parts in the first $N-4$ integrals with respect to the remaining α's. As before, we make the change of variables $x = \alpha_{l_{i+1}'} - \alpha_{l_i'}$ and $y = \alpha_{l_{j+1}} - \alpha_{l_j}$ and then integrate by parts with respect to $\alpha_{l_{i+1}'}$ and $\alpha_{l_{j+1}}$. The end result is that:

$$\begin{aligned}
&\int_{\Lambda^N} \phi_k(\alpha_{l_{j+1}} - \alpha_{l_j})\phi_k(\alpha_{l'_{i+1}} - \alpha_{l'_i})\, d\!\!\!{}^-\alpha \\
&\quad = (N+1)N \int_0^1 \int_0^{1-y} \phi_k(x)\phi_k(y) \\
&\qquad \cdot \text{multi}(N-1, l'_{i+1} - l'_i - 1, l_{j+1} - l_j - 1; x, y)\, dxdy \\
&\quad = (N+1)N \int_0^1 \int_0^1 \phi_k(x)\phi_k(y) \\
&\qquad \cdot \text{multi}(N-1, l'_{i+1} - l'_i - 1, l_{j+1} - l_j - 1; x, y)\, dxdy + \mathcal{O}(k^{-1}),
\end{aligned}$$

since $0 \le y \le 1/k$ in supp $\phi_k(y)$.

Case 2. $\alpha_{l'_i} < \alpha_{l_j} < \alpha_{l'_{i+1}} < \alpha_{l_{j+1}}$ *(or equivalently* $\alpha_{l_j} < \alpha_{l'_i} < \alpha_{l_{j+1}} < \alpha_{l'_{i+1}}$*).*

When compared with all possible relative configurations, the proportion of configurations satisfying the assumptions of Case 2 are asymptotically small. Indeed, the proportion of such relative configurations is $\mathcal{O}(k^{-1})$. One can see this as follows: Given $N+1$ positive real numbers $0 < \alpha_0 < \ldots < \alpha_N < 1$, we consider the following two subsets of k elements given by:

$$\alpha_{l_1} \le \cdots \le \alpha_{l_k} \text{ and } \alpha_{l'_1} \le \cdots \le \alpha_{l'_k}. \tag{36}$$

For each of the subsets above, there are $k-1$ pairs of the form $(\alpha_{l_j}, \alpha_{l_{j+1}})$ and $(\alpha_{l'_i}, \alpha_{l'_{i+1}})$. From (36) it follows that for any fixed pair $(\alpha_{l_j}, \alpha_{l_{j+1}})$, there is at most one pair $(\alpha_{l'_i}, \alpha_{l'_{i+1}})$ for which Case 2 is possible.

Case 3. $\alpha_{l_j} < \alpha_{l'_i} < \alpha_{l'_{i+1}} < \alpha_{l_{j+1}}$ *(or equivalently,* $\alpha_{l'_i} < \alpha_{l_j} < \alpha_{l_{j+1}} < \alpha_{l'_{i+1}}$*).*

This case can be dealt with in a similar fashion to Case 1. That is, we apply the Fubini Theorem repeatedly to ensure that the last four iterated integrals involve $\alpha_{l'_i}, \alpha_{l_j}, \alpha_{l_{j+1}}$ and $\alpha_{l'_{i+1}}$. Then, we integrate by parts with respect to the remaining α's. Finally, we make the change of variables $x = \alpha_{l'_{i+1}} - \alpha_{l'_i}$ and $y = \alpha_{l_{j+1}} - \alpha_{l_j}$ and integrate by parts again with respect to $\alpha_{l'_{i+1}}$ and $\alpha_{l_{j+1}}$ to get:

$$\begin{aligned}
&\int_{\Lambda^N} \phi_k(\alpha_{l_{j+1}} - \alpha_{l_j})\phi_k(\alpha_{l'_{i+1}} - \alpha_{l'_i})\, d\!\!\!{}^-\alpha \\
&\quad = (N+1)N \int_0^1 \int_x^1 \phi_k(x)\phi_k(y) \\
&\qquad \cdot \text{multi}(N-1, l'_{i+1} - l'_i - 1, N - l_{j+1} - l_j + 1; x, 1-y)\, dydx \\
&\quad = (N+1)N \int_0^1 \int_0^1 \phi_k(x)\phi_k(y) \\
&\qquad \cdot \text{multi}(N-1, l'_{i+1} - l'_i - 1, N - l_{j+1} - l_j + 1; x, 1-y)\, dydx + \mathcal{O}(k^{-1}).
\end{aligned}$$

As in the proof of part (i) of Theorem 1.1, we make the substitution $m = l_{j+1} - l_j - 1$ and $m' = l'_{i+1} - l'_i - 1$ in order to apply Lemma 4.6. From the estimate in (31) and the analysis of Cases 1-3 above, we deduce that

$$\begin{aligned}
&\int_{\Lambda^N} \left| d\rho_{LS}^{AV}(x; N, k, \alpha)(\phi) \right|^2 \, d\!\!\!{}^-\alpha \\
&\quad = \frac{k^2(N+1)N}{(N+k-1)^2} \Big[\sum_{m'=0}^{N-2} \sum_{m=0}^{N-2-m'} e^{-\frac{mk}{N}} e^{-\frac{m'k}{N}} \int_0^1 \int_0^1 \phi_k(x)\phi_k(y) \\
&\qquad \cdot \mathrm{multi}(N-1, m', m; x, y)\, dxdy \\
&\qquad + \sum_{m'=0}^{N-2} \sum_{m=0}^{m'} e^{-\frac{mk}{N}} e^{-\frac{m'k}{N}} \int_0^1 \int_0^1 \phi_k(x)\phi_k(y) \\
&\qquad \cdot \mathrm{multi}(N-1, m', N-m; x, 1-y)\, dydx \Big] \\
&\qquad\quad + \mathcal{O}\left(\frac{k}{N}\right) + \mathcal{O}\left(\frac{1}{k}\right).
\end{aligned}$$

By Lemma 4.6, we finally conclude that

$$\begin{aligned}
\int_{\Lambda^N} \left| d\rho_{LS}^{AV}(x; N, k, \alpha)(\phi) \right|^2 \, d\!\!\!{}^-\alpha &= \frac{k^2(N+1)N}{(N+k-1)^2} \left(\int_0^1 \phi_k(x) e^{-kx} dx \right)^2 \\
&\quad + \mathcal{O}\left(\frac{k}{N}\right) + \mathcal{O}\left(\frac{1}{k}\right) \\
&= \left(\int_0^1 \phi(x) e^{-x} dx \right)^2 + \mathcal{O}\left(\frac{k}{N}\right) + \mathcal{O}\left(\frac{1}{k}\right). \qquad \square
\end{aligned}$$

Theorem 1.1 (ii) is then an immediate consequence of the Chebyshev inequality and the following corollary of Proposition 4.7:

Corollary 4.8. *For any $\phi \in C_0^1([0,1])$, we have*

$$\int_{\Lambda^N} \left(d\rho_{LS}^{AV}(x; N, k, \alpha)(\phi) - \int_0^1 e^{-x}\phi(x)\, dx \right)^2 \, d\!\!\!{}^-\alpha = \mathcal{O}\left(\frac{k}{N}\right) + \mathcal{O}\left(\frac{1}{k}\right). \quad (37)$$

Prof of Corollary 4.8. The corollary follows directly from Proposition 4.7 and the estimate for the convergence of the integrated, averaged level-spacings measure in (30). □

Remark. We should point out that one can also quite easily determine the weak limit of the level-spacings measures before "unfolding the zeroes" (i.e. rescaling to unit mean level-spacing). Indeed, by carrying out exactly the same analysis as above, one can show that

$$w - \lim_{N\to\infty} \frac{1}{\sigma(N,k)} \sum_{l=1}^{\sigma(N,k)} \frac{1}{k-1} \sum_{j=1}^{k-1} \delta(x - (\theta_{l,j+1}^{(k)}(\alpha) - \theta_{l,j}^{(k)}(\alpha))) = \delta_0(x), \quad (38)$$

both in the mean, and pointwise for an asymptotically full measure of $\alpha \in \Lambda^N$. Indeed, the computation of the weak-limit in (38) turns out to be a simpler problem that the corresponding one after "unfolding", since the error terms are much easier to control.

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