THE USE OF THE "TRIDENT" MODEL IN THE ANALYSIS OF PLASTIC ZONES NEAR CRACK TIPS AND CORNER POINTS

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UDC 539.375

The paper deals with calculation of a plastic zone near a crack tip in a homogeneous elastoplastic solid and near a corner point of the boundary of this solid. The calculations are conducted for a solid subject to plane strain and within the framework of models with plastic strips. It is shown that in comparison with the widely used model with two straight slip-lines, the process of plastic deformation is described by the "trident" model more accurately. The results of calculations of the plastic zone by the "trident" model that correspond to different stages of the development of plastic deformation are given for a crack of normal separation in a quasibrittle material.

Problems arising in studying the fracture of elastoplastic [2–6, 10, 11], viscoelastic [12], and inhomogeneous [13, 15] materials call for the development of more perfect models of cracks representing the actual pattern of the fracture process.

A great number of papers devoted to the calculation of plastic zones near crack tips under plane deformation have been published in recent years. In these studies, the plastic zones are modeled by two narrow rectilinear plastic strips coming from the crack tip and representing slip lines [6, 8, 10]. On a slip line, only the tangential displacement may have a discontinuity, while the tangential stress is equal to the shear yield point τ_s.

The basis for such a simulation is the results of the experimental investigation [5]. According to this study, at the initial stage of plastic deformation, two plastic strips appear near a tip of a crack of normal separation. The strips are tilted to the line of crack continuation at an angle of approximately 72° [10]. In [10], it is shown that the results of calculation of the initial plastic zone near a tip of a crack of normal separation within the framework of the mentioned model comply, in some sense, with those of numerical calculation of a "fuzzy" plastic zone having the shape of plastic "ears" [14]. Therefore, the plastic zone corresponding to this model can be considered as some approximation of a "fuzzy" plastic zone.

However, as follows from the investigations of the plastic zone near the crack tips using an electron microscope and the x-ray diffraction method [2, 3], at all stages of the fracture process, there exists a frontal prefracture zone including the initial one along with strongly developed "butterfly"-shaped side plastic zones. The linear dimension of this zone at the continuation of the crack is much less than the maximum linear dimensions of the side elastic zones. With an increase in the load, the plastic prefracture zone turns to a destruction zone distinguished by the maximum level of plastic strains and the presence of pores and microcracks. As the load increases, the plastic zones grow and the angle of inclination of side plastic zones changes and achieves 50° at subsequent stages of the fracture process.

The existence of the third plastic zone (along with the two side ones) that develops from a crack tip can be proved not only in an experimental but also in a theoretical way. To this end, it is sufficient to examine the behavior of the stresses near a crack tip in the symmetric problem [10] of the theory of elasticity for a plane whose point is the origin of a semiinfinite crack and two finite slip lines. This study was carried out in [1]. Its results show that after the occurrence of side plastic slip lines coming from the tip of a crack of normal separation, the crack tip remains a concentrator of stresses with a power singularity, though weaker than the crack tip in an elastic body. The degree of singularity of the stresses depends on the angle of inclination of the slip line to the line of the crack continuation. If the angle is equal to 72°, then the degree of singularity

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 1063-7095/00/3603-0372$25.00

is approximately equal to 0.20049. The presence of the mentioned stress concentration implies that once two side plastic zones coming from a crack tip appear, the third plastic zone may be expected to develop from it.

The symmetric problem of the theory of elasticity is solved in [1] for a plane whose point is the origin of a semiinfinite crack, two slip lines of finite length, and the Dugdale line of considerably smaller length, on which only normal shear rupture is possible, and the direct stress is equal to the tension yield point σ_s. The examination of the stress behavior near a crack tip in the given problem shows that once the third plastic line coming from a tip of a normal-separation crack appears, the crack tip is not a stress concentrator any more. Therefore, except for the three plastic zones developing from the crack tip, new plastic zones may not be expected.

These experimental and theoretical results suggest that, as compared to a model with two slip lines, the "trident" model more accurately describes the process of plastic deformation near a crack tip. According to this model, the set of three narrow rectilinear plastic strips (two slip lines and the Dugdale line) coming from a crack tip models a plastic zone near the crack tip. If, at the initial stage, the plastic strains near the crack tip are localized in thin layers of the material — three plastic strips coming from the tip — then, at the subsequent stages of plastic-strain development, the plastic zone corresponding to the "trident" model should be considered as some approximation of the actual "fuzzy" plastic zone observed in the above-mentioned experiments [2, 3] for a crack of normal separation.

For hardenable materials, it is desirable to introduce the parameter σ_B (the ultimate strength of a material) instead of σ_s, since, as follows from [2, 3], the strains in the prefracture zone achieve an extremely high level (up to 50%) at the stages of continuation and growth of the crack, and, hence, the stresses in this zone considerably exceed the yield point of the material.

The initial plastic zone near the tip of a crack of normal separation is calculated in [1] within the framework of the "trident" model. However, for a detailed study of the development of plastic strains near the tip of a normal-separation crack, it is necessary to have the values of the length l of the slip lines, the length d of the Dugdale line, and the opening δ of the crack at its tip, which correspond to subsequent stages of this process.

The mentioned quantities l, d, and δ are evaluated below for different values of the angle α between the slip line and the line of continuation of a crack of normal separation in a quasibrittle material.

According to [1], we obtain the following formulas for determination of the lengths of plastic strips:

$$l = L(\alpha)\frac{K_I^2}{\tau_s^2}, \quad L = \frac{\pi}{32}\left[\frac{G_1^+(-1)}{G_1^+(-1/2)}\right]^2 \sin^2\alpha\cos^2\frac{\alpha}{2},$$

$$d = D^*\frac{K_I^2}{\tau_s^2}, \quad D^* = D(\alpha)\left(\frac{\tau_s}{\sigma_s}\right)^{-1/\lambda}, \quad D = \frac{\pi}{32}D_0,$$

$$D_0 = \left(\frac{S^*}{2\lambda+1}\right)^{-1/\lambda}\left[\frac{G_1^+(-1)}{G_1^+(-1/2)}\right]^2\left[\frac{G_1^+(-\lambda-1)G_2^+(-1)}{G_1^+(-1)G_2^+(-\lambda-1)}\right]^{-1/\lambda}\sin^2\alpha\cos^2\frac{\alpha}{2},$$

$$s^* = \frac{4(\lambda+2)\sin(\lambda+1)(\pi-\alpha)\sin\lambda\pi\sin\alpha}{\lambda s},$$

$$\begin{aligned} s = {} & -2\pi\sin 2\lambda\pi + [2(\pi-\alpha)\cos 2\alpha - \sin 2\alpha] \\ & \times\sin 2(\lambda+1)(\pi-\alpha) + 4(\alpha\cos\alpha + \sin\alpha)(\lambda+1) \\ & \times\cos 2(\lambda+1)\alpha\sin\alpha - [(2\alpha+\sin 2\alpha)\lambda(\lambda+2) \\ & - (2\alpha\cos 2\alpha + \sin 2\alpha)(\lambda+1)^2]\sin 2(\lambda+1)\alpha \\ & - 4(\pi-\alpha)(\lambda+1)\cos 2(\lambda+1)(\pi-\alpha)\sin\alpha\cos\alpha + 4(\lambda+1)\sin^2\alpha, \end{aligned}$$

$$G_j^+(p) = \exp\left[\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\ln G_j(z)}{z-p}dz\right], \quad \operatorname{Re} p < 0, \quad j = 1,2,$$

$$G_1(z) = \frac{\Delta_0(z)}{2\sin^2 z\pi},$$

$$\Delta_0(z) = (\sin 2z\alpha + z\sin 2\alpha)[\sin 2z(\pi-\alpha) - z\sin 2\alpha]$$

$$+ 2(\cos 2z\alpha - \cos 2\alpha)[\sin^2 z(\pi-\alpha) - z^2\sin^2\alpha],$$

$$G_2(z) = \frac{\Delta(z)\cos z\pi}{\Delta_0(z)\sin z\pi},$$

$$\Delta(z) = 2\{(\sin 2z\alpha + z\sin 2\alpha)[\sin^2 z(\pi-\alpha) - z^2\sin^2\alpha]$$

$$+ [\sin 2z(\pi-\alpha) - z\sin 2\alpha](\sin^2 z\alpha - z^2\sin^2\alpha)\}.$$

Here, K_I is the factor of intensity of the stresses at the tip of the crack and λ is the solution (unique on the interval]–1; 0[) of the equation

$$[\sin 2(\lambda+1)\alpha + (\lambda+1)\sin 2\alpha]$$

$$\times[\sin 2(\lambda+1)(\pi-\alpha) - (\lambda+1)\sin 2\alpha] + 2[\cos 2(\lambda+1)\alpha - \cos 2\alpha]$$

$$\times[\sin^2(\lambda+1)(\pi-\alpha) - (\lambda+1)^2\sin^2\alpha] = 0.$$

Assume that the material satisfies the Tresca condition of plasticity. In this case, $\sigma_s = 2\tau_s$. The values of $L, D^*, \Delta = \delta/[(1-\nu^2)K_I^2 E^{-1}\tau_s^{-1}]$, where E is Young's modulus and ν is Poisson's ratio, are presented below for some values of α ($\alpha°$ is the value of α in degrees):

$\alpha°$	48	51	54	60	66	69	72
L	0.0452	0.0472	0.0507	0.0548	0.0573	0.0578	0.0583
D^*	0.0006	0.0008	0.0009	0.0012	0.0016	0.0017	0.0019
Δ^*	0.1527	0.1719	0.1913	0.2302	0.2690	0.2884	0.3078

Some errata are contained in [1]. The exponent in the expression for R in (3.6) should be equal to 1/2 (in [1] it is equal to 1/220), and the quantity C_2 in the following formula should be equal to 4.92523 (in [1], it is equal to 0.08452).

If a model with two slip lines is used to calculate the initial plastic zone near the tip of a crack of normal separation, then, according to [10], the opening of the crack at its tip is equal to $0.2222(1-\nu^2)K_I E^{-1}\tau_s^{-1}$. When the "trident" model is used, the result corresponding to $\alpha° = 72°$ in the above table shows that the opening of the crack at its tip is equal to $0.3078(1-\nu^2)K_I E^{-1}\tau_s^{-1}$. Thus, in the presence of the third plastic strip at the tip of a crack of normal separation, the opening of the crack at this tip will be greater than in the case of existence of only two plastic strips coming from it. Moreover, the calculation results presented above show that the greater the angle between the side plastic strips, the greater the lengths of these strips and the third one.

The initial plastic zone near a corner point of the boundary of a body under the condition of a symmetric problem was calculated in [4] within the framework of a model with two slip lines. A formula for determination of the length of the slip lines was derived, and the direction of their development was established. It was shown that the slip line forms an angle γ with the stress-free boundary of the body. This angle increases with increase in the angle 2β between the lines of the boundary. Some values of γ are as follows:

β°	95	105	115	125	135	145	155	165	175
γ°	47	52	58	64	70	77	83	92	101

It is possible to show that after the appearance of plastic slip lines developing from a corner point, this point remains a stress concentrator. To this end, it is sufficient to take advantage of the general provisions on the behavior of the stresses near corner points of elastic bodies [7, 9]. Following the mentioned provisions, it is necessary to consider the respective homogeneous problem of the theory of elasticity for a wedge and two semiinfinite straight slip lines coming from its vertex. The secular equation of this problem has the form

$$2\,\Delta_1\,(\sin^2 p\gamma - p^2\sin^2\gamma) + \Delta_2\,(\sin 2p\gamma + p\sin 2\gamma) = 0,$$

$$\Delta_1 = \cos 2p\,(\beta-\gamma) - \cos 2\,(\beta-\gamma),$$

$$\Delta_2 = \sin 2p\,(\beta-\gamma) + p\sin 2\,(\beta-\gamma),$$

$$p = -\lambda - 1\,.$$

An analysis of this equation shows that it has a unique root λ_1 on the interval $]-1;\,0[$. The values of λ_1 for the β given above and the γ corresponding to them are presented below:

β°	95	105	115	125	135	145	155	160	165	170	175
$-\lambda_1^\circ$	$61\cdot10^{-6}$	$63\cdot10^{-6}$	$64\cdot10^{-6}$	$65\cdot10^{-6}$	$67\cdot10^{-6}$	$69\cdot10^{-6}$	$339\cdot10^{-6}$	0.000611	0.0392	0.1077	0.1587

According to the general provisions on the behavior of stresses near corner points of elastic bodies, the stresses behave as r^{λ_1} as $r \to 0$ in the problem of the theory of elasticity for slip lines coming from a corner point of the boundary of a body (r, θ is a polar coordinate system with its pole at the corner point). Hence, the corner point remains a stress concentrator with a power singularity. The degree of singularity of stresses increases with increase in the angle between the boundary lines.

Thus, if the angle between the boundary lines of a body exceeds 320°, then in calculating the initial plastic zone near the corresponding corner point near which the boundary is free of stresses, it is expedient to use the "trident" model. If this angle is less than 320°, then, since the singularity of the stresses is very weak, which is seen from the last table, it is expedient to use a model with two slip lines.

When the legs of the corner are rigidly fastened and hinged (the tangential stress and the normal displacement are equal to zero), the corresponding secular equations are as follows:

$$\Delta_1\Big[\,\kappa_1 - 2\,(\kappa\sin^2 p\gamma + p^2\sin^2\gamma)\,\Big] - \Delta_2(\kappa\sin 2p\gamma - p\sin 2\gamma) = 0,$$

$$\Delta_1(\sin 2p\gamma + p\sin 2\gamma) + \Delta_2(\cos 2p\gamma - \cos 2\gamma) = 0,$$

$$\kappa_1 = (\kappa+1)^2/2, \quad \kappa = 3 - 4\nu\,.$$

Each of these equations has a unique root on the interval $]-1;\,0[$. Some values of the angle γ and the root λ_2 of the secular equation corresponding to the corner point with rigidly fastened legs of the corner are as follows ($\nu = 0.25$):

β °	95	100	105	110	115	120	125	130
γ °	45	50	55	59	64	68	73	77
$-\lambda_2 \cdot 10^3$	327	361	394	415	444	463	487	503

The data in this table show that, in calculating the initial plastic zone near a corner point of the type being considered, it is expedient to use the "trident" model.

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