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Identification of Two Cracks in a Simply Supported Beam From Minimal Frequency Measurements

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Abstract: This paper presents a diagnostic technique for the identification of two cracks of equal severity in a simply supported beam under flexural vibrations. The crack is simulated by a rotational spring connecting the two adjacent segments of the beam. The analysis is based on an explicit expression of the frequency sensitivity to damage, and the damaged system is considered as a perturbation of the virgin system. By measuring the changes of the first three natural frequencies, it is possible to study the inverse problem—identification of crack location and severity. The inverse problem is ill-posed; namely, even by leaving symmetrical positions aside, cracks with different severity in two sets of different locations can produce identical changes in the first three natural frequencies. Numerical results show that if the natural frequencies used as data in identification are affected by errors relatively small with respect to the frequency-induced changes, then damage identification leads to satisfactory results.

Key Words: Damage detection, bending vibrating beams, inverse problems, cracks

1. INTRODUCTION

In this paper, we seek to detect two open cracks of equal severity in a simply supported beam from a minimum number of frequency measurements.

Despite the very extensive literature on damage identification (see Ruotolo, 1997; Salawu, 1997 for recent and complete state-of-the-art), most of the previous work dealing with cracked beams considers the damage sizing and location just if a single crack is present. Only recently, researchers have turned their attention to damage assessment in multicracked beams. The direct problem was considered in Ostachowicz and Krawczuk (1991) and in Ruotolo and Shifrin (1999). Ostachowicz and Krawczuk (1991) studied the effect of two open cracks upon the natural frequencies of the flexural vibration in a cantilever beam. Ruotolo and Shifrin (1999) presented an efficient technique for solving the eigenvalue problem related to the free bending vibration of a multicracked beam. The inverse problem of damage identification in multicracked beams has been considered in Liang, Hu, and Choy (1992); Ruotolo and Surace (1997); and Vestroni and Capecchi (1996). Liang, Hu, and Choy (1992) developed a diagnostic technique based on frequency sensitivity to localized damage for detecting and assessing multiple cracks in beam structures. Ruotolo and Surace

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(1997) formulated the inverse diagnostic problem in optimization terms and used a solution procedure employing genetic algorithms to identify two cracks in a cantilever beam. With the aim of reducing the indeterminacy of the diagnostic problem, Vestroni and Capecchi (1996) presented a damage identification procedure of variational type that is based on the a priori information that the damage is located in only a few sections of the beam.

The present paper deals with the identification of two small open cracks of equal severity in a simply supported uniform beam from the knowledge of the damage-induced shifts in first lower bending frequencies. As in Freund and Herrmann (1976), every crack is simulated by an equivalent massless rotational spring, of stiffness K, connecting the two segments of the beam adjacent to the damaged cross section. Assuming that the undamaged system is completely known, only three parameters need to be determined, namely, the stiffness K of the spring and the abscissas s_1, s_2 of the cracked cross sections. Therefore, we considered as a minimal set of data the first three natural frequencies of the beam. By using this set of data, the diagnostic problem is generally ill-posed, namely, even by leaving symmetrical solutions aside, cracks in different locations and of different severity can still produce identical changes in the first three natural frequencies. In spite of this ill-posedeness, the effects of the nonuniqueness of the solution are not so dramatic, because it is found that the stiffness K and the location variables $x = \cos 2\pi s_1/L, y = \cos 2\pi s_2/L$ (where L is the length of the beam) are solutions of polynomial equations of second degree. Therefore, in all cases, closed form expressions are deduced for x, y, and K in terms of the data. Our analysis is based on an explicit expression of the frequency sensitivity to damage derived by Morassi (1993), and our results are an extension of part of those obtained by Narkis (1994) and Morassi (1999) for the single crack identification problem. Some of the results are also valid for cracked rods in axial vibration.

Dynamic tests performed on simulated cracked beams supported the proposed method for the solution of the diagnostic problem in practical situations. Numerical results show that if the natural frequencies used as data in identification are affected by (model or measurement) errors relatively small with respect to the variations of the frequencies induced by the damage, then damage identification leads to satisfactory results.

2. FREQUENCY SENSITIVITY TO DAMAGE

The physical model, which will be mainly investigated in this paper, is a simply supported uniform Euler-Bernoulli beam with two cracks of equal severity located at cross sections of abscissa s_1 and s_2. We assume that $0 < s_1 < s_2 < L$, where L is the length of the beam. Assuming that cracks remain always open during the flexural vibration, every crack is represented by inserting a massless rotational spring, as in Freund and Herrmann (1976). As it is well-known, the stiffness K of the spring can be related in a precise way to the geometry of the damage, as suggested, for example, by Dimarogonas and Paipetis (1983). Denoting by E the Young's modulus of the material and by γ the volume mass-density, the mth eigenpair $(w_m(s), \nu_{dm} \equiv \omega_{dm}^2), m = 1, 2, \ldots,$ of the bending vibrations of the cracked beam satisfies the following boundary value problem:

$$EI\frac{d^4 w_m(s)}{ds^4} = \nu_{dm}\gamma A w_m(s) \quad \text{for } s \in (0, s_1) \cup (s_1, s_2) \cup (s_2, L), \tag{1}$$

$$w_m = 0 = \frac{d^2 w_m}{ds^2} \quad \text{at } s = 0 \text{ and } s = L, \tag{2}$$

where the jump conditions

$$[w_m(s)] = \left[\frac{d^2 w_m(s)}{ds^2}\right] = \left[\frac{d^3 w_m(s)}{ds^3}\right] = 0, \tag{3}$$

$$EI\frac{d^2 w_m(s)}{ds^2} = K\left[\frac{dw_m(s)}{ds}\right], \tag{4}$$

hold at the cross sections of abscissa $s = s_1$ and $s = s_2$ where cracks occur. In the equations above, I and A represent the moment of inertia and the area of the cross section of the beam, respectively. In equations (3) through (4), $[\phi(s)] \equiv (\phi(s^+) - \phi(s^-))$ denotes the jump of the function ϕ at s. The undamaged system corresponds to $K \to \infty$ or $\varepsilon \equiv 1/K \to 0$.

If cracks are small, namely, ε is small enough, then we may find the first-order variation of the natural frequencies with ε as shown in Morassi (1993) or in Ruotolo (1997) (Section 5.2, equation (5.7)). By taking

$$\nu_{dm} = \nu_m + \varepsilon\left(\Delta\nu_m\right), \tag{5}$$

we find that the first variation of the mth eigenvalue is given by

$$\delta\nu_m \equiv \varepsilon\left(\Delta\nu_m\right) = -\frac{M_m^2(s_1)}{K} - \frac{M_m^2(s_2)}{K}, \tag{6}$$

where the normalizing condition $\int_0^L \gamma A w_m^2(s)\, ds = 1$ has been taken into account. Note that the change in a natural frequency produced by a single crack may be expressed as the product of two terms, the first of which is proportional to the severity and the second of which depends only on the location of the damage. In particular, this second term is the square of the bending moment

$$M_m(s) \equiv -EI\frac{d^2 w_m(s)}{ds^2} \tag{7}$$

in the mth mode shape of the undamaged beam evaluated at the cracked cross section. We will see in the next section that the explicit expression (6) for the damage sensitivity of natural frequencies plays a crucial role in our analysis. Finally, we observe that the assumption of small damages confines the range of application of the method to cracked configurations that are a perturbation of the undamaged one. However, this is not a severe limitation because in most practical situations it is crucial to be in position to identify the damage right as it arises.

3. THEORETICAL RESULTS

We can now pose the problem of identifying the positions s_1, s_2 of the cracks and their severity K from the knowledge of the changes in the lower natural frequencies of the beam. Since only three parameters need to be determined, it is reasonable to investigate to what extent the measurement of the first three natural frequencies can be useful for identifying the damage. The system is symmetrical with respect to $s = L/2$, and therefore a crack located at any one of a set of symmetrically placed points will produce identical changes in natural frequencies. It follows that, without affecting the character of generality of the analysis, we can assume that the two cracks are located on the interval $(0, L/2)$ corresponding to the left half of the beam, for example, $0 < s_1 < s_2 \leq L/2$. Let us denote by C_m the quantity

$$C_m = -\frac{\delta \nu_m}{Bm^4}, \tag{8}$$

where $m \geq 1$ is an integer and B is the constant

$$B = \left(EI \sqrt{\frac{2}{\gamma AL}} \left(\frac{\pi}{L}\right)^2 \right)^2 . \tag{9}$$

The eigenpairs of the simply supported uniform beam in bending vibrations are

$$\nu_m = \frac{EI}{\gamma A} \left(\frac{m\pi}{L}\right)^4 , \; w_m(s) = \sqrt{\frac{2}{\gamma AL}} \sin\left(m\pi \frac{s}{L}\right), \tag{10}$$

$m = 1, 2, \ldots$.

Inserting the expression of $w_m(s)$, for $m = 1, 2, 3$, into equation (6), we obtain the following system of three nonlinear equations:

$$\frac{1}{K}\left(\sin^2 \pi\frac{s_1}{L} + \sin^2 \pi\frac{s_2}{L}\right) = C_1 \tag{11a}$$
$$\frac{1}{K}\left(\sin^2 2\pi\frac{s_1}{L} + \sin^2 2\pi\frac{s_2}{L}\right) = C_2 \tag{11b}$$
$$\frac{1}{K}\left(\sin^2 3\pi\frac{s_1}{L} + \sin^2 3\pi\frac{s_2}{L}\right) = C_3, \tag{11c}$$

to be solved with respect to (s_1, s_2, K). Note that $C_m > 0, m = 1, 2, 3$, since $0 < s_1 < s_2 \leq L/2$. By using standard trigonometric identities, we can rewrite the system (11a) through (11c) in the following equivalent form:

$$x + y = 2 - 2KC_1 \tag{12a}$$
$$x^2 + y^2 = 2 - KC_2 \tag{12b}$$
$$x^3 + y^3 = 2 - \frac{K}{2}(3C_1 + C_3), \tag{12c}$$

where

$$x \equiv \cos 2\pi\frac{s_1}{L} \in [-1, 1), \quad y \equiv \cos 2\pi\frac{s_2}{L} \in [-1, 1). \tag{13}$$

Since $0 < s_1 < s_2 \leq L/2$, functions $x = x(s_1), y = y(s_2)$ are one-to-one correspondences and $x \neq y$. By using the algebraic identities $x^3 + y^3 = (x+y)(x^2+y^2-xy)$ and $2xy = (x+y)^2 - (x^2+y^2)$ in equation (12c), and by expressing $(x+y)$ and (x^2+y^2) by means of equations (12a) through (12b), we can deduce the following polynomial equation in the unknown K:

$$K\left[4C_1^3K^2 + 3C_1(C_2 - 4C_1)K + \left(\frac{15}{2}C_1 - 3C_2 + \frac{C_3}{2}\right)\right] = 0. \tag{14}$$

By neglecting the trivial solution $K = 0$ (which clearly does not correspond to the assumption of small damage), we now prove that the polynomial of second degree enclosed in square brackets in equation (14) always has two real positive solutions K_1, K_2 for every set of data (C_1, C_2, C_3). Recalling that $C_1 > 0$, we start showing that the coefficient of K is negative and the coefficient of the term of order zero is positive. In fact, from equations (12a) through (12c), we have

$$C_2 - 4C_1 = -\frac{1}{K}\left((x-1)^2 + (y-1)^2\right) < 0 \tag{15}$$

for $x \in [-1, 1)$ and $y \in [-1, 1)$. Moreover, a simple computation shows that

$$\frac{15}{2}C_1 - 3C_2 + \frac{C_3}{2} = \frac{1}{K}\left((1-x)^3 + (1-y)^3\right) > 0 \tag{16}$$

for $x \in [-1, 1)$ and $y \in [-1, 1)$. Finally, we show that the discriminant Δ of the polynomial in the variable K is nonnegative:[1]

$$\Delta \equiv \left(3C_1C_2 - 12C_1^2\right)^2 - 16C_1^3\left(\frac{15}{2}C_1 - 3C_2 + \frac{C_3}{2}\right) \geq 0. \tag{17}$$

In fact, by using the expressions (12a) through (12c) for C_1, C_2, C_3, we have the following:

$$\Delta \equiv C_1^2\left(24C_1^2 - 24C_1C_2 + 9C_2^2 - 8C_1C_3\right) \equiv \frac{(2-x-y)^2}{4K^2}f(x,y), \tag{18}$$

where

$$\begin{aligned} f(x,y) &= x^4 + y^4 + 4x^3 + 4y^3 + 18x^2y^2 - 8xy^3 - 8x^3y \\ &- 12xy^2 - 12x^2y + 24xy - 8x - 8y + 4. \end{aligned} \tag{19}$$

We order $f(x,y)$ with respect to the variable x:

$$\begin{aligned} f(x,y) &= x^4 + 2x^3(2-4y) + x^2\left(18y^2 - 12y\right) \\ &+ 2x\left(-4y^3 - 6y^2 + 12y - 4\right) + y^4 + 4y^3 - 8y + 4. \end{aligned} \tag{20}$$

Observing that

$$
\begin{aligned}
y^4+4y^3-8y+4 &= \left(y^2+2y-2\right)^2, &\text{(21a)}\\
-4y^3-6y^2+12y-4 &= \left(y^2+2y-2\right)(2-4y), &\text{(21b)}\\
18y^2-12y &= (2-4y)^2+2\left(y^2+2y-2\right), &\text{(21c)}
\end{aligned}
$$

we can rewrite $f(x,y)$ as follows:

$$
\begin{aligned}
f(x,y) &= x^4+x^2(2-4y)^2+\left(y^2+2y-2\right)^2\\
&+ 2x^3(2-4y)+2x^2\left(y^2+2y-2\right)+2x\left(y^2+2y-2\right)(2-4y), \quad (22)
\end{aligned}
$$

that is,

$$
f(x,y)=\left(x^2+x(2-4y)+\left(y^2+2y-2\right)\right)^2, \quad \textbf{(23)}
$$

which clearly is a nonnegative quantity.

Taking into account conditions (15), (16), and (17), we can conclude that there exist two real positive (possibly equal) roots of the polynomial in the variable K, that is, there exist two values of the stiffness K of the spring simulating the damage:

$$
K_{1,2}=\frac{-3\left(C_2-4C_1\right)\pm\left[9\left(C_2-4C_1\right)^2-16C_1\left(\frac{15}{2}C_1-3C_2+\frac{C_3}{2}\right)\right]^{1/2}}{8C_1^2}, \quad (24)
$$

where indexes 1 and 2 correspond to $+$ sign and $-$ sign, respectively.

By inserting the expression (24) of K into equations (12a) through (12b), we can localize the damage. Note that the role of variables x and y in system (12a) through (12c) is completely interchangeable, namely, if (K,x,y) is a solution of the diagnostic problem, then (K,y,x) is also a solution. Then, it is enough to determine the position variable x. By using equations (12a) through (12b), we can deduce the following polynomial equation of second degree in the variable x:

$$
g(x)\equiv 2x^2-2x\left(2-2KC_1\right)+\left(2-2KC_1\right)^2-\left(2-KC_2\right)=0. \quad \textbf{(25)}
$$

The polynomial in equation (25) has two distinct real roots; in fact, the discriminant Δ is strictly positive:

$$
\Delta=\left(2-2KC_1\right)^2-2\left[\left(2-2KC_1\right)^2-\left(2-KC_2\right)\right]=(x-y)^2>0, \quad (26)
$$

because $x\neq y$. To prove that two roots x_1,x_2 of equation (25) belong to the interval $[-1,1)$, and then they correspond to physically reasonable damage locations (see definition (13)), it is enough to verify (i) that the value of the polynomial $g(x)$ of equation (25) evaluated at

$x = -1$ and $x = 1^-$ is nonnegative and positive, respectively, and (ii) that $g(x)$ has minimum at $x_{\min} \in [-1, 1)$. Recalling equations (12a) through (12c), we have the following:

$$g(-1) = 2 + 2(2 - 2KC_1) + (2 - 2KC_1)^2 - (2 - KC_2) = 2(1 + x)(1 + y) \geq 0, \quad (27)$$

and

$$g(1^-) = 2(1 - x)(1 - y) > 0, \quad (28)$$

for every $x \in [-1, 1)$ and $y \in [-1, 1)$. Then, the condition (i) is satisfied. A direct computation shows that $g(x)$ has minimum at $x_{\min} = 1 - KC_1$. Then, using equation (12a) and recalling that $x \in [-1, 1)$ and $y \in [-1, 1)$, it turns out that $x_{\min} \in [-1, 1)$ and (ii) is proved. Hence, we can evaluate

$$x_{1,2} = \frac{(2 - 2KC_1) \pm [-4K^2C_1^2 + 8KC_1 - 2KC_2]^{1/2}}{2}, \quad (29)$$

where indexes 1 and 2 represent + sign and − sign before the square root, respectively.

Finally, the complete set of solutions of the system (12a) through (12c) with reference to cracks located on the left half of the beam is given by

$$(K_1, s_1(K_1), s_2(K_1)), \quad (K_2, s_1(K_2), s_2(K_2)). \quad (30)$$

In fact, once the value of the stiffness of the rotational spring simulating the crack is determined (via expression (24)), let's say K, we can evaluate the possible damage locations x_1, x_2 of one crack via expression (29). Given one x-unknown, let's say x_1, we can evaluate the corresponding value of the y-unknown, let's say y_1, via equation (12a). It turns out that

$$\begin{aligned} y_1 &= (2 - 2KC_1) - x_1 \\ &= (2 - 2KC_1) - \frac{(2 - 2KC_1) + [-4K^2C_1^2 + 8KC_1 - 2KC_2]^{1/2}}{2} = x_2, \end{aligned} \quad (31)$$

and, similarly, $y_2 = x_1$. Then, for one fixed value of the stiffness K, there is a pair of solutions of the inverse problem:

$$(K, x_1(K), y_1 = x_2(K)), \quad (K, x_2(K), y_2 = x_1(K)). \quad (32)$$

Since the relations $x \equiv \cos 2\pi \frac{s}{L}$ and $y \equiv \cos 2\pi \frac{s}{L}$ are one-to-one correspondences for $0 < s \leq L/2$, by solving with respect to the s-variable, we obtain

$$\begin{aligned} &(K, s_1(K) \equiv s(x_1(K)), \; s_2(K) \equiv s(x_2(K))), \\ &(K, s_2(K) \equiv s(x_2(K)), \; s_1(K) \equiv s(x_1(K))). \end{aligned} \quad (33)$$

These two damage configurations clearly coincide, and then there exists only one solution of the inverse problem corresponding to a given value of K, for example, $(K, s_1(K), s_2(K))$. That is, we have shown that two cracks of same severity K_1 (evaluated via expression (24)—with plus sign) located at the cross sections of abscissa $s_1(K_1), s_2(K_1)$ (evaluated via expressions (29) and (13)) produce changes in the first three natural frequencies identical to those induced by two cracks of the same severity K_2 (expression (24)—with minus sign) located at the cross sections of abscissa $s_1(K_2), s_2(K_2)$ (expressions (29) and (13)).

In conclusion of the section, it should be noted that the present method can be adapted to identify two small cracks in an axially vibrating beam with free ends. To show this, suffice to observe that the mth eigenpair, $(u_m(s), \mu_m \equiv \omega_m^2), m = 0, 1, 2, \dots,$ of a free-free uniform beam in axial vibration are

$$\mu_m = \frac{E}{\gamma}\left(\frac{m\pi}{L}\right)^2, \quad u_m(x) = \sqrt{\frac{2}{\gamma AL}}\cos m\pi\frac{s}{L}, \tag{34}$$

and then repeat the same procedure used for the bending vibration case to formulate the inverse problem. According to Freund and Herrmann (1976), here a crack is represented by the insertion of a massless translational spring, of stiffness K, at the damaged cross section. If cracks are small and of equal severity, namely, K is large enough, on proceeding as in Morassi (1993) and with the above notation, the first-order variation of the mth eigenvalue with $1/K$ is given by

$$\delta\mu_m = -\frac{N_m^2(s_1)}{K} - \frac{N_m^2(s_2)}{K}, \tag{35}$$

where $N_m(s) \equiv EA\frac{du_m(s)}{ds}$ is the axial force at the cross section of abscissa s in the mth (normalized) axial mode of the undamaged beam. In equation (35), s_1 and s_2 denote the abscissas of the damaged cross sections. Taking the expression (34) of vibrating modes into account and considering as data the variations of the first three natural frequencies (rigid mode is omitted, e.g., $m = 1, 2, 3$), we obtain a system of three nonlinear equations formally coincident with the previous system (12a) through (12c). By repeating now the same procedure used for the bending vibration case, we can solve the inverse problem.

4. APPLICATIONS

In the preceding section, it was elucidated how to employ the measurement of the first three bending frequencies of a simply supported beam with two cracks of equal severity so as to assess the location as well as the magnitude of the damage. Aiming to account for the practical use of the results above within the analysis of real cases, the present section is devoted to outlining some applications of numerical character.

The inverse problem of damage detection is solved for different cases, using pseudo-experimental data, that is, the frequencies are obtained from the direct problem in undamaged conditions and in some damaged conditions defined by the three damage parameters K, s_1, s_2. The beam for the case study is a double T steel beam of the series IPE 300 with the following geometrical and mechanical characteristics: length $L = 5$ m, bending stiffness $EI = 1.721 \cdot 10^7$ Nm^2, and linear mass density $\rho \equiv \gamma A = 42.2$ kg/m.

Table 1. Frequencies for the undamaged beam and their values associated to the cases of damage (cases free of error). Values in Hz. $\Delta\% = (f_{undamaged} - f_{damaged})/f_{undamaged} \cdot 100$.

		Small Damage ($K = 100EI/L$)				Moderate Damage ($K = 25EI/L$)			
f	Undam.	Case C	$\Delta\%$	Case S	$\Delta\%$	Case C	$\Delta\%$	Case S	$\Delta\%$
f_1	40.13	39.64	1.22	39.64	1.22	38.25	4.68	38.25	4.68
f_2	160.51	157.84	1.66	158.58	1.20	151.14	5.84	153.55	4.34
f_3	361.16	359.42	0.48	356.70	1.23	354.82	1.76	344.03	4.74

Table 2. Results of damage identification (cases free of error). Determination of the spring stiffness K (Nm/rad, equation (24)) and of corresponding damage locations s_1, s_2 (m, equations (29) and (13)).

	Small Damage $K = 100EI/L = 3.443 \cdot 10^8 Nm/rad$		Moderate Damage $K = 25EI/L = 0.861 \cdot 10^8 Nm/rad$	
	Case C	Case S	Case C	Case S
Damage Parameters	$s_1 = 1.25$ $s_2 = 1.67$	$s_1 = 1.00$ $s_2 = 2.00$	$s_1 = 1.25$ $s_2 = 1.67$	$s_1 = 1.00$ $s_2 = 2.00$
$K_1\,(\cdot 10^8)$	3.572	3.561	0.990	0.978
$s_1\,(K_1)$	1.25	1.01	1.27	1.04
$s_2\,(K_1)$	1.70	2.02	1.77	2.06
$K_2\,(\cdot 10^8)$	2.020	2.825	0.567	0.758
$s_1\,(K_2)$	0.27	0.62	0.29	0.61
$s_2\,(K_2)$	1.56	1.88	1.62	1.91

Two main different cases of damage among several studied are presented: they are illustrative of the main features of the inverse problem and of the identification technique. The first case is characterized by "small" damage, that is, the value of the stiffness K is such that the variations of the first three frequencies are about 0.5% to 1.7% of the initial values for a different set of damage locations. For the other case, "moderate" damage corresponding to variations of the same frequencies about 1.7% to 6% is considered. In both cases, identification results are presented for a set of two damage locations: $s_1 = L/4, s_2 = L/3$ (close cracks, Case C) and $s_1 = L/5, s_2 = 2L/5$ (separate cracks, Case S).

The frequency values for the undamaged beam and their values associated with the cases of damage are shown in Table 1; the latter are obtained by solving in exact way the eigenvalue problem (1) through (4), as shown, for example, in Ostachowicz and Krawczuk (1991). The results of identification are presented in Table 2. It is possible to observe that in the absence of errors, the pair of two solutions (30) predicted by the theory for the mathematical problem contains (a satisfactory estimate of) the real solution of the damage problem. The deviations from the exact damage parameters, which are exclusively due to the perturbation assumption—see equations (5) and (6)—are negligible for damage locations and are of order 4% to 15% for damage severity. Discrepancies are smaller for less severe damages, and this behavior is expected because the inverse diagnostic problem is formulated on the assumption that the damaged system is a "small" perturbation of the virgin system.

We have developed the damage analysis in absence of errors so far, but, as it is well-known, the results of most identification techniques strictly depend on possible measurement

Table 3. Results of damage identification for Case C (small damage) with errors on the data. Determination of the spring stiffness K (Nm/rad, equation (24)) and of corresponding damage locations s_1, s_2 (m, equations (29) and (13)). $err\% f = (f_{error} - f_{exact}) / f_{exact} \cdot 100$. The symbol $(*)$ means imaginary solution.

	Case Free of Error	Exact values: $K = 3.443 \cdot 10^8\ Nm/rad,\ s_1 = 1.25m,\ s_2 = 1.67m$ Errors of the same sign				Errors with alternate sign			
$err\% f_1$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$err\% f_2$	0.00	–0.10	–0.20	–0.40	–0.50	0.10	0.20	0.40	0.50
$err\% f_3$	0.00	–0.20	–0.40	–0.80	–1.00	–0.20	–0.40	–0.80	–1.00
$K_1\ (\cdot 10^8)$	3.572	3.385	3.191	2.757	(*)	3.547	3.513	3.376	(*)
$s_1\ (K_1)$	1.25	1.16	1.07	0.88	(*)	1.17	1.10	0.94	(*)
$s_2\ (K_1)$	1.70	1.68	1.66	1.58	(*)	1.78	1.85	1.96	(*)
$K_2\ (\cdot 10^8)$	2.020	2.040	2.067	2.168	(*)	2.212	2.413	2.885	(*)
$s_1\ (K_2)$	0.27	0.34	0.40	0.53	(*)	0.38	0.47	0.68	(*)
$s_2\ (K_2)$	1.56	1.54	1.53	1.51	(*)	1.63	1.70	1.88	(*)

Table 4. Results of damage identification for Case C (moderate damage) with errors on the data. Determination of the spring stiffness K (Nm/rad, equation (24)) and of corresponding damage locations s_1, s_2 (m, equations (29) and (13)). $err\% f = (f_{error} - f_{exact}) / f_{exact} \cdot 100$.

	Case Free of Error	Exact values: $K = 0.861 \cdot 10^8\ Nm/rad,\ s_1 = 1.25m,\ s_2 = 1.67m$ Errors of the same sign				
$err\% f_1$	0.00	0.00	0.00	0.00	0.00	−0.50
$err\% f_2$	0.00	−0.10	−0.20	−0.40	−0.50	−1.50
$err\% f_3$	0.00	−0.20	−0.40	−0.80	−1.00	−3.00
$K_1\ (\cdot 10^8)$	0.990	0.976	0.963	0.936	0.922	0.743
$s_1\ (K_1)$	1.27	1.24	1.22	1.17	1.14	0.92
$s_2\ (K_1)$	1.77	1.77	1.76	1.76	1.75	1.74
$K_2\ (\cdot 10^8)$	0.567	0.570	0.572	0.578	0.581	0.597
$s_1\ (K_2)$	0.29	0.31	0.33	0.37	0.39	0.58
$s_2\ (K_2)$	1.62	1.61	1.61	1.61	1.60	1.59

and modeling errors. To take the effect of errors in the experimental data into account, we considered a series of cases in which the natural frequencies were corrupted by some noise. To give an example, Tables 3 and 4 refer to the previous Case C—with small and moderate damage—in presence of increasing errors of the same sign or of an alternate sign. As a general remark, it is possible to observe that if the natural frequencies used as data in identification are affected by errors relatively small with respect to the variations of the frequencies induced by the damage, then damage identification leads to satisfactory results. However, in the inverse problem solution, the noise in the data is usually amplified strongly and the estimates of the damage parameters seem to be rather sensitive to input errors.

5. CONCLUSIONS

In this paper, we have focused on detecting two cracks of equal severity from the knowledge of the damage-induced changes in the first three natural frequencies of a simply supported beam under bending vibration. The analysis is based on an explicit expression of the frequency sensitivity to damage and the damaged system is considered as a perturbation of the virgin system. It was found that, even by leaving symmetrical positions aside, cracks with different severity in two sets of different locations can produce identical changes in the first three natural frequencies. The theoretical results are confirmed by a comparison with numerical tests performed on cracked beams.

NOTE

1. This argument was pointed out to us by Michele Di Lena.

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