COLLOCATION AND ITERATED DEFECT CORRECTION

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1. INTRODUCTION

In this paper an equivalence between solutions of collocation methods and fixed points of Iterated Defect Correction (IDeC) methods is proved. Therefore the IDeC-methods can be regarded as efficient schemes for solving collocation equations. Attention is restricted to the application of the IDeC to ordinary differential equations (initial value problems and two point boundary value problems). Extension to other types of operator equations (e.g. partial differential equations, integral equations,...) is straightforward.

In Section 2 special variants of collocation methods, which are of importance in connection with the IDeC are discussed. The basic ideas behind the IDeC are presented in Section 3. The equivalence between collocation schemes and the fixed points of the IDeC-methods is established in Section 4.

2. COLLOCATION METHODS

2.1. Collocation methods for two point boundary value problems

We consider problems of the form

(2.1a) $$y' = f(t,y), \quad t \in [a,b]$$

(2.1b) $$g(y(a),y(b)) = 0$$

where y,f and g are vector-valued functions of dimension n with f and g sufficiently smooth. A number of papers about collocation methods applied to (2.1) have appeared recently in the literature on the numerical solution of BVPs for ODEs (e.g. de Boor, Swartz [3], Russel, Shampine [9], Weiss [11]). From the class of collocation schemes, we consider the following special type (cf. Weiss [11]):

The collocation solution is a continuous piecewise polynomial which satisfies (2.1a) at given (collocation) points.

We now introduce the notation to be used below. The grid is given by

$$(2.2)\qquad \begin{aligned} &a = t_0 < \ldots < t_I = b \\ &H_i := t_{i+1} - t_i, \quad i = 0(1)I-1. \end{aligned}$$

We consider the space of continuous piecewise polynomial functions $P(t)$ [vector-valued of dimension n] defined by

$$(2.3)\qquad \begin{aligned} &P(t) := P_i(t), \quad t \in [t_i, t_{i+1}], \quad i = 0(1)I-1 \\ &P_i(t_{i+1}) = P_{i+1}(t_{i+1}), \quad i = 0(1)I-2 \end{aligned}$$

where *all* polynomials P_i are of *degree m*. On (2.2) we construct the subgrid

$$(2.4)\qquad t_{i,k} := t_i + \xi_k H_i, \quad i = 0(1)I-1, \quad k = 1(1)m$$

with collocation nodes

$$(2.5)\qquad 0 \leq \xi_1 < \ldots < \xi_m \leq 1$$

(important special cases satisfying (2.5) are the Gauss-Legendre, the Lobatto and the Radau points). The collocation equations become

$$(2.6)\qquad P_i'(t_{i,k}) = f(t_{i,k}, P_i(t_{i,k})), \quad i = 0(1)I-1, \quad k = 1(1)m.$$

If $\xi_1 = 0$ or $\xi_m = 1$ in (2.5), then $P_i'(t_{i,1})$ or $P_i'(t_{i,m})$ is interpreted as the right derivative or the left derivative, respectively. If $\xi_1 = 0$ *and* $\xi_m = 1$, then two collocation equations (2.6) hold at every gridpoint $t_i = t_{i,1} = t_{i-1,m}$. Together with the boundary condition

$$(2.7)\qquad g(P_0(a), P_{I-1}(b)) = 0$$

and the continuity conditions

$$(2.8)\qquad P_i(t_{i+1}) = P_{i+1}(t_{i+1}), \quad i = 0(1)I-2,$$

the collocation conditions yield $n \cdot I \cdot (m+1)$ equations for the $n \cdot I \cdot (m+1)$ unknown coefficients of P.

2.2. Collocation methods for initial value problems

The method of Section 2.1 can be interpreted as a method for solving IVPs, if in (2.1) the boundary condition is replaced by

$$(2.9)\qquad g(y(a), y(b)) = y(a) - y_a = 0.$$

In this situation, it is possible to solve the equations (2.6) block by block (where one block contains the equations for the interval $[t_i, t_{i+1}]$).

Collocation methods can only be justified as efficient computational strategies for IVPs when the equations are stiff. For such equations these methods have the advantage of good stability properties combined with high order accuracy (cf. e.g. Wright [12], Axelsson [1], Ehle [4], Chipman [2]).

3. ITERATED DEFECT CORRECTION

The Iterated Defect Correction (IDeC) consists essentially in an iterative improvement of a given numerical solution (obtained from some finite difference method). In this section we describe methods derived from this concept for which an equivalence with collocation methods will be established below.

3.1. IDeC-methods for two point boundary value problems

In order to obtain an initial approximation to (2.1) by some finite difference method, we introduce a grid which is a refinement of grid (2.2):

$$\text{(3.1)} \qquad \left.\begin{array}{l} t_i = s_{i,o} < \ldots < s_{i,K} = t_{i+1} \\ h_{i,k} := s_{i,k+1} - s_{i,k}\,, \quad k = 0(1)K-1 \end{array}\right\} \quad i = 0(1)I-1$$

<u>Note</u>: The points t_i have two different names in the notation of (3.1):

$$\text{(3.2)} \qquad t_i = s_{i,o} = s_{i-1,K}\,, \quad i = 1(1)I-1$$

On the grid (3.1), we next consider three well-known finite difference methods.

For $i = 0(1)I-1, \quad k = 0(1)K-1$:

$$\text{(3.3)} \qquad \begin{cases} (\eta_{i,k+1} - \eta_{i,k})/h_{i,k} = f(s_{i,k} + h_{i,k}/2, (\eta_{i,k} + \eta_{i,k+1})/2) \\ g(\eta_{o,o}, \eta_{I-1,K}) = 0 \end{cases}$$

$$\text{(3.4)} \qquad \begin{cases} (\eta_{i,k+1} - \eta_{i,k})/h_{i,k} = f(s_{i,k}, \eta_{i,k}) \\ g(\eta_{o,o}, \eta_{I-1,K}) = 0 \end{cases}$$

$$(3.5)\qquad \begin{cases} (\eta_{i,k+1}-\eta_{i,k})/h_{i,k} = f(s_{i,k+1},\eta_{i,k+1}) \\ g(\eta_{o,o},\eta_{I-1,K}) = 0. \end{cases}$$

Because of (3.2), we require (for all three methods) that

$$(3.6)\qquad \eta_{i,o} = \eta_{i-1,K}\ , \quad i = 1(1)I-1$$

which ensures that the number of equations and the number of unknowns are indentical.

These methods are usually defined on grids with only one index. However, the more involved notation of (3.1) will turn out to be essential for the analysis in the next section.

The solution of a given finite difference scheme ((3.3), (3.4) or (3.5)) is denoted by

$$\eta^o := \left(\eta^o_{o,o},\ldots,\eta^o_{o,K},\eta^o_{1,o},\eta^o_{1,1},\ldots,\eta^o_{I-1,K}\right).$$

Interpolation of η^o by a piecewise polynomial function P^o

$$(3.7)\qquad \left.\begin{array}{ll} P^o(t) := P^o_i(t), & t\in[t_i,t_{i+1}] \\ P^o_i(s_{i,k}) = \eta^o_{i,k}\ , & k = 0(1)K \end{array}\right\}\ i = 0(1)I-1$$

yields the defect

$$(3.8)\qquad \begin{array}{l} d^o_i(t) := (P^o_i)'(t) - f(t,P^o_i(t)), \\ \qquad\qquad t\in[t_i,t_{i+1}], \quad i = 0(1)I-1. \end{array}$$

By adding the defect d^o_i to the righthand side of the original problem (2.1), we obtain a new BVP of a slightly more general type:

From the set of continuous piecewise functions

$$(3.9)\qquad \begin{array}{l} \{y|y(t) := y_i(t),\ t\in[t_i,t_{i+1}], \quad i = 0(1)I-1; \\ \qquad y_i(t_{i+1}) = y_{i+1}(t_{i+1}), \quad i = 0(1)I-2\} \end{array}$$

(where sufficiently high derivatives of the functions y_i *exist), we determine that function which satisfies the following relations*

$$(3.10)\qquad \begin{array}{l} y_i' = f(t,y_i) + d^o_i(t), \quad t\in[t_i,t_{i+1}], \quad i = 0(1)I-1 \\ g(y_o(a),y_{I-1}(b)) = 0 \end{array}$$

The *exact* solution of this "piecewise" BVP is P^o (cf. (3.8)). Despite our knowledge of the exact solution, we solve the new BVP (3.10) in the same way as the original BVP (2.1), i.e. the same finite difference scheme [(3.3), (3.4) or (3.5)] which was used to obtain η^o, is now applied to (3.10). This yields

$$\pi^o = \left(\pi^o_{o,o},\ldots,\pi^o_{o,K},\ \pi^o_{1,o},\ \pi^o_{1,1},\ldots,\pi^o_{I-1,K}\right).$$

We can now use the *known* global discretization errors $\pi^o_{i,k} - P^o_i(s_{i,k})$ of (3.10) as estimates for the unknown global discretization errors $\eta^o_{i,k} - y(s_{i,k})$ of (2.1). The original idea of estimating the global discretization error in this way is due to Zadunaisky [13].

If we replace the unknown error term in the identity

$$(3.11)\qquad y(s_{i,k}) = \eta^o_{i,k} - \left(\eta^o_{i,k} - y(s_{i,k})\right).$$

by our estimate, we obtain the following formula for the improvement of our first solution η^o:

$$(3.12)\qquad \eta^1_{i,k} := \eta^o_{i,k} - \left(\pi^o_{i,k} - P^o_i(s_{i,k})\right)$$

The whole procedure may be used iteratively,

$$(3.13)\qquad \eta^{j+1}_{i,k} := \eta^o_{i,k} - \left(\pi^j_{i,k} - P^j_i(s_{i,k})\right),\qquad j = 1,2,\ldots$$

where P^j denotes the polynomial which interpolates η^j (analog to (3.7)).

The above iterative strategy is called the *Iterated Defect Correction (IDeC)*, and the different methods which can be constructed using this concept are called *IDeC-methods*. More details about the IDeC and IDeC-methods are available (see, for example Stetter [8] or Frank, Ueberhuber [6]).

The IDeC-methods described above use estimates of the *global* discretization error. We will now discuss other IDeC-methods which use estimates of the *local* discretization errors. As in the "global case", we start with η^o (solution of (3.3), (3.4) or (3.5)), interpolate η^o by P^o and construct the new BVP (3.10) the exact solution of which is P^o. Therefore, the exact local discretization error associated with problem (3.10) can be evaluated: e.g., for the box-scheme (3.3), we obtain

$$(3.14)\quad \begin{aligned} l^{o}_{i,k} &:= \left(P^{o}_{i}(s_{i,k+1}) - P^{o}_{i}(s_{i,k})\right)/h_{i,k} - \\ &\quad - f\,(s_{i,k}+h_{i,k}/2, [P^{o}_{i}(s_{i,k}) + P^{o}_{i}(s_{i,k+1})]/2) - \\ &\quad - \left(P^{o}_{i}\right)'(s_{i,k}+h_{i,k}/2) + \\ &\quad + f\,(s_{i,k}+h_{i,k}/2,\ P^{o}_{i}(s_{i,k}+h_{i,k}/2)) = \\ &= -\left(P^{o}_{i}\right)'(s_{i,k}+h_{i,k}/2) + f(s_{i,k}+h_{i,k}/2,\ P^{o}_{i}(s_{i,k}+h_{i,k}/2)) = \\ &= -\,d^{o}_{i}(s_{i,k}+h_{i,k}/2) \end{aligned}$$

as an estimate for the unknown local discretization error of (2.1)

$$(3.15)\quad \begin{aligned} l_{i,k} &:= (y(s_{i,k+1}) - y(s_{i,k}))/h_{i,k} - \\ &\quad - f\,(s_{i,k}+h_{i,k}/2,\ [y(s_{i,k}) + y(s_{i,k+1})]/2). \end{aligned}$$

To obtain the improved approximation η^1, it is necessary to solve

$$(3.16)\quad \begin{aligned} &\left(\eta^{1}_{i,k+1} - \eta^{1}_{i,k}\right)/h_{i,k} - f(s_{i,k}+h_{i,k}/2, [\eta^{1}_{i,k} + \eta^{1}_{i,k+1}]/2) = \\ &\qquad = l^{o}_{i,k} = -\,d^{o}_{i}(s_{i,k}+h_{i,k}/2) \\ &g\left(\eta^{1}_{o,o},\ \eta^{1}_{I-1,K}\right) = 0,\quad \eta^{1}_{i,K} = \eta^{1}_{i+1,o}. \end{aligned}$$

This procedure may again be used iteratively, yielding η^2, η^3,... . The error estimate used in obtaining η^2 is

$$(3.17)\quad \begin{aligned} l^{1}_{i,k} &= \left(P^{1}_{i}(s_{i,k+1}) - P^{1}_{i}(s_{i,k})\right)/h_{i,k} - \\ &\quad - f\,(s_{i,k}+h_{i,k}/2,\ [P^{1}_{i}(s_{i,k}) + P^{1}_{i}(s_{i,k+1})]/2) - \\ &\quad - \left(P^{1}_{i}\right)'(s_{i,k}+h_{i,k}/2) + \\ &\quad + f\,(s_{i,k}+h_{i,k}/2,\ P^{1}_{i}(s_{i,k}+h_{i,k}/2)) = \\ &= \left(\eta^{1}_{i,k+1} - \eta^{1}_{i,k}\right)/h_{i,k} - \\ &\quad - f\,(s_{i,k}+h_{i,k}/2,\ [\eta^{1}_{i,k} + \eta^{1}_{i,k+1}]/2) - \\ &\quad - \left(P^{1}_{i}\right)'(s_{i,k}+h_{i,k}/2) + \\ &\quad + f\,(s_{i,k}+h_{i,k}/2,\ P^{1}_{i}(s_{i,k}+h_{i,k}/2)) = \\ &= -\,d^{o}_{i}(s_{i,k}+h_{i,k}/2) - d^{1}_{i}(s_{i,k}+h_{i,k}/2). \end{aligned}$$

The general formula for the error estimate is

(3.18) $$l^j_{i,k} = - d^o_i(s_{i,k} + h_{i,k}/2) - \dots - d^j_i(s_{i,k} + h_{i,k}/2).$$

Note This method is a special case of the difference correction of Fox and Pereyra (cf. e.g. Pereyra [8]). Another "local version" of the IDeC which is more similar to Pereyra's approach has been discussed by Frank, Hertling, Ueberhuber [7].

We now establish an equivalence result for both the above variants.

THEOREM 3.1. *Consider the following general formulation for a linear BVP*

(3.19) $$\begin{aligned} y' &= Ay \\ By(a) + Cy(b) &= e. \end{aligned}$$

Then the approximations $\eta^1, \eta^2, \dots$ *given by both the above mentioned variants of the IDeC are identical.*

PROOF: For the linear equations (3.19), the schemes (3.3), (3.4) and (3.5) [together with (3.6)] may be written as

(3.20) $$D\eta^o = \begin{pmatrix} 0 \\ e \end{pmatrix}.$$

For example, in the case of the box-scheme, (3.20) becomes

(3.21) $$\begin{aligned} \left(\eta^o_{i,k+1} - \eta^o_{i,k}\right)/h_{i,k} - (1/2)A\left(\eta^o_{i,k} + \eta^o_{i,k+1}\right) &= 0 \\ \eta^o_{i,K} - \eta^o_{i+1,o} &= 0 \\ B\eta^o_{o,o} + C\eta^o_{I-1,K} &= e. \end{aligned}$$

We use induction. For $j = 1$, we obtain

(i) "local variant":

$$D\eta^1 = \begin{pmatrix} 0 \\ e \end{pmatrix} - \begin{pmatrix} d^o \\ 0 \end{pmatrix} \qquad \text{(cf. (3.14), (3.16))}$$

i.e.

$$\eta^1 = D^{-1}\begin{pmatrix} 0 \\ e \end{pmatrix} - D^{-1}\begin{pmatrix} d^o \\ 0 \end{pmatrix}$$

Note 1 For example, in the case of the box-scheme, d^o becomes

$$d^o := \left(d^o_o(s_{o,o} + h_{o,o}/2), \dots, d^o_{I-1}(s_{I-1,K-1} + h_{I-1,K-1}/2)\right).$$

Note 2 If (3.19) has a unique solution, then it is well known that (3.20) has also a unique solution for sufficiently fine grids, i.e. D^{-1} is defined.

(ii) "global variant":

$$D\pi^o = \begin{pmatrix} 0 \\ e \end{pmatrix} + \begin{pmatrix} d^o \\ 0 \end{pmatrix} \qquad \text{(cf. (3.10))}$$

$$\pi^o = D^{-1}\begin{pmatrix}0\\e\end{pmatrix} + D^{-1}\begin{pmatrix}d^o\\0\end{pmatrix}$$

$$\eta^1 = \eta^o - (\pi^o - \eta^o) = \qquad \text{(cf. (3.12))}$$

$$= 2\eta^o - \pi^o =$$

$$= 2D^{-1}\begin{pmatrix}0\\e\end{pmatrix} - \left[D^{-1}\begin{pmatrix}0\\e\end{pmatrix} + D^{-1}\begin{pmatrix}d^o\\0\end{pmatrix}\right] =$$

$$= D^{-1}\begin{pmatrix}0\\e\end{pmatrix} - D^{-1}\begin{pmatrix}d^o\\0\end{pmatrix}$$

which establishes the identity for the case $j = 1$. Let us now assume, for $j = r$, that

$$\eta^r = D^{-1}\left[\begin{pmatrix}0\\e\end{pmatrix} - \begin{pmatrix}d^o\\0\end{pmatrix} - \dots - \begin{pmatrix}d^{r-1}\\0\end{pmatrix}\right]$$

is valid for both variants of the IDeC. Then for $j = r+1$, we obtain

(i) <u>"local variant"</u>:

$$\eta^{r+1} = D^{-1}\left[\begin{pmatrix}0\\e\end{pmatrix} - \begin{pmatrix}d^o\\0\end{pmatrix} - \dots - \begin{pmatrix}d^r\\0\end{pmatrix}\right] \qquad \text{(cf. (3.18))}.$$

(ii) <u>"global variant"</u>:

$$D\pi^r = \begin{pmatrix}0\\e\end{pmatrix} + \begin{pmatrix}d^r\\0\end{pmatrix}$$

$$\pi^r = D^{-1}\begin{pmatrix}0\\e\end{pmatrix} + D^{-1}\begin{pmatrix}d^r\\0\end{pmatrix}$$

$$\eta^{r+1} = \eta^o - (\pi^r - \eta^r) =$$

$$= D^{-1}\begin{pmatrix}0\\e\end{pmatrix} - D^{-1}\begin{pmatrix}0\\e\end{pmatrix} - D^{-1}\begin{pmatrix}d^r\\0\end{pmatrix} +$$

$$+ D^{-1}\begin{pmatrix}0\\e\end{pmatrix} - D^{-1}\begin{pmatrix}d^o\\0\end{pmatrix} - \dots - D^{-1}\begin{pmatrix}d^{r-1}\\0\end{pmatrix} =$$

$$= D^{-1}\left[\begin{pmatrix}0\\e\end{pmatrix} - \begin{pmatrix}d^o\\0\end{pmatrix} - \dots - \begin{pmatrix}d^r\\0\end{pmatrix}\right]$$

which proves the assertion. ■

<u>3.2. IDeC-methods for initial value problems</u>

If, in Section 3.1, the special boundary condition

(3.22) $$g(y(a), y(b)) = y(a) - y_a = 0$$

is used, certain IDeC-methods for IVPs are immediately defined. The schemes (3.3), (3.4) and (3.5) are now the implicit midpoint rule, the explicit Euler method and the implicit Euler method. Other IDeC-methods for IVPs are obtained when more general RK-methods are used. This is discussed in Frank, Ueberhuber [5] where the following asymptotic result is proved for equidistant grids ($H_i \equiv H = (b-a)/I$, $h_{i,k} \equiv h = H/K$):

THEOREM 3.2. *If a RK-scheme of order* $p(\leq K)$ *is used, and if f satisfies suitable smoothness conditions, then*

$$\eta^j_{i,k} - y(s_{i,k}) = O\left(h^{\min(p(j+1),K)}\right) \quad \text{for } h \to 0. \tag{3.23}$$ [1]

Note We interpret "$h \to 0$" in the sense of "$I \to \infty$ and K fixed".

We have introduced the IDeC as an iterative scheme, but up to now we have not dicussed how to terminate the process. According to (3.23), a reasonable termination criterion is given by the maximum achievable order K, which is reached for η^J if $K/p \in (J,J+1]$. In section 4, other termination criteria will be discussed.

For BVPs, each IDeC-method consists in computing successively each of the iterates $\eta^1, \eta^2, \ldots$ for the whole interval [a,b]. For IVPs, it is of course possible to proceed in a blockwise manner, as is indicated by the following:

$$\left[\begin{array}{ccc} \eta^o_{o,o}, & \cdots, & \eta^o_{o,K} \\ \eta^1_{o,o}, & \cdots, & \eta^1_{o,K} \\ \vdots & & \vdots \\ \eta^J_{o,o}, & \cdots, & \eta^J_{o,K} \end{array}\right.$$
$$\left[\begin{array}{ccc} \eta^o_{1,o}, & \cdots, & \eta^o_{1,K} \\ \vdots & & \vdots \\ \eta^J_{1,o}, & \cdots, & \eta^J_{1,K} \end{array}\right.$$
$$\left[\begin{array}{ccc} \vdots & & \vdots \end{array}\right.$$

The use of such a strategy yields an economy in storage. In Frank, Ueberhuber [6], a more detailed discussion of this procedure may be found.

Just as for BVPs, there exist two possibilities (using either estimates of the *global* or estimates of the *local* discretization error) to construct IDeC-methods for IVPs. Theorem 3.1 may immediately be applied to IVPs, which means that both variants yield identical results for the linear problem.

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1) K is the degree of the interpolating polynomials P^j_i (cf. (3.7)).

4. RELATIONS BETWEEN COLLOCATION AND IDEC

When IDeC-methods are applied to certain problems, the convergence of the iterates $\eta^1, \eta^2, \ldots$ to a fixed point η^* may be observed. In this section, we will show that those IDeC-methods based on the schemes (3.3), (3.4) or (3.5) (cf. Section 3) have fixed points that coincide with the solutions of collocation schemes (discussed in Section 2).

4.1. Boundary value problems

4.1.1. IDeC-methods based on the box-scheme

We start our discussion by establishing a relationship between the IDeC based on the box-scheme (3.3) and an appropriate collocation scheme. For the IDeC-methods, we assume a grid of the form (3.1) with the same proportional spacing of the subgrid-points $s_{i,k}$ in $[t_i, t_{i+1}]$, i.e. for $k = 1(1)K$

(4.1) $$(s_{o,k} - s_{o,k-1})/H_o = \ldots = (s_{I-1,k} - s_{I-1,k-1})/H_{I-1}.$$

The related collocation is defined on the following grid:

(4.2) $$t_{i,k} := (s_{i,k-1} + s_{i,k})/2, \qquad i = 0(1)I-1, \quad k = 1(1)m$$

with

$$m = K.$$

Therefore the corresponding collocation nodes satisfy

(4.3) $$0 < \xi_1 < \ldots < \xi_m < 1.$$

Note A straightforward generalization would consist in dropping the relation (4.1), resulting in a different collocation scheme on every interval $[t_i, t_{i+1}]$.

THEOREM 4.1. *Consider an IDeC-method which uses the box-scheme and a* global *discretization error estimate. η^* is a fixed point of this IDeC-method iff P^*, defined by*

(4.4) $$P_i^*(s_{i,k}) = \eta_{i,k}^*, \quad i = 0(1)I-1, \quad k = 0(1)m$$

is the solution of the corresponding collocation scheme, i.e.

(4.5) $$d_i^*(t_{i,k}) := (P_i^*)'(t_{i,k}) - f(t_{i,k}, P_i^*(t_{i,k})) = 0, \qquad i = 0(1)I-1, \quad k = 1(1)m$$

PROOF: By definition, η^* is a fixed point, iff one step of the IDeC-method applied to η^* leads again to η^*, i.e.

$$(4.6) \qquad \eta^*_{i,k} = \eta^o_{i,k} - (\pi^*_{i,k} - \eta^*_{i,k}), \qquad i = 0(1)I-1, \quad k = 0(1)m.$$

Fixed points are therefore characterized by

$$(4.7) \qquad \pi^* = \eta^o.$$

a) Assume P^* is the solution of the (corresponding) collocation scheme, i.e. (4.5) is satisfied. The equations defining η^o are

$$(4.8) \qquad \begin{aligned} &(\eta^o_{i,k+1} - \eta^o_{i,k})/h_{i,k} = f(s_{i,k} + h_{i,k}/2, (\eta^o_{i,k} + \eta^o_{i,k+1})/2) \\ &\eta^o_{i,o} = \eta^o_{i-1,m} \\ &g(\eta^o_{o,o}, \eta^o_{I-1,m}) = 0. \end{aligned}$$

π^* is defined by

$$(4.9) \qquad \begin{aligned} &(\pi^*_{i,k+1} - \pi^*_{i,k})/h_{i,k} = f(s_{i,k} + h_{i,k}/2, (\pi^*_{i,k} + \pi^*_{i,k+1})/2) + \\ &\qquad\qquad + d^*_i(s_{i,k} + h_{i,k}/2) \\ &\pi^*_{i,o} = \pi^*_{i-1,m} \\ &g(\pi^*_{o,o}, \pi^*_{I-1,m}) = 0. \end{aligned}$$

Since $t_{i,k} = s_{i,k-1} + h_{i,k-1}/2$ [cf. (4.2)] and (4.5) is satisfied, the equations (4.8) and (4.9) are identical, i.e. (4.7) is satisfied.

b) Let η^* be a fixed point, i.e. $\eta^o = \pi^*$. Subtraction of (4.8) from (4.9) leads immediately to the desired result (4.5). ■

<u>THEOREM 4.2</u>. *Consider an IDeC-method which uses the box-scheme and a local discretization error estimate. η^* is a fixed point of this IDeC-method iff*

$$(4.10) \qquad d^*_i(t_{i,k}) = 0, \qquad i = 0(1)I-1, \qquad k = 1(1)m.$$

<u>PROOF</u>: Let us consider one IDeC-step starting from η^*. The estimate of the local discretization error is

$$(4.11) \qquad \begin{aligned} l^*_{i,k} = &(\eta^*_{i,k+1} - \eta^*_{i,k})/h_{i,k} - f(s_{i,k} + h_{i,k}/2, (\eta^*_{i,k} + \eta^*_{i,k+1})/2) - \\ &- d^*_i(s_{i,k} + h_{i,k}/2). \end{aligned}$$

The equations for the next iterate are

$$(4.12) \qquad (\eta^{**}_{i,k+1} - \eta^{**}_{i,k})/h_{i,k} - f(s_{i,k} + h_{i,k}/2, (\eta^{**}_{i,k} + \eta^{**}_{i,k+1})/2) = l^*_{i,k}.$$

a) Suppose (4.10) is satisfied. Then (4.11) and (4.12) imply that $\eta^* = \eta^{**}$, i.e. η^* is a fixed point.

b) Suppose $\eta^* = \eta^{**}$, then (4.11) and (4.12) imply (4.10). ■

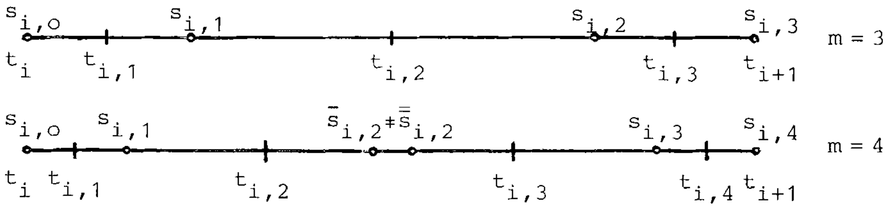
REMARK 1: According to Theorem 3.1 the "local" and "global" variants of the IDeC yield identical results, when applied to *linear* problems. This is of course not true for *nonlinear* problems, but Theorem 4.1 and Theorem 4.2 show that in the nonlinear case both variants have the same collocation solution as fixed point.

REMARK 2: From (4.2), it follows that we can construct for any "$s_{i,k}$-grid" of the IDeC-methods a corresponding "$t_{i,k}$-grid" of the collocation methods. Unfortunately the reverse is not true, e.g. for equidistant collocation nodes

$$\xi_k = k/(m+1), \quad k = 1(1)m,$$

there exists no corresponding "$s_{i,k}$-grid" for the box-scheme, which satisfies (4.2). Gauss-Legendre points with m even do not have a corresponding "$s_{i,k}$-grid" either, but for m odd an "$s_{i,k}$-grid" satisfying (4.2) may be found (see Fig. 1).

Figure 1



4.1.2. IDeC-methods based on the scheme (3.4)

The relation (4.2) becomes

$$(4.13) \qquad t_{i,k} = s_{i,k-1}, \quad i = 0(1)I-1, \quad k = 1(1)m.$$

The corresponding collocation nodes satisfy

$$(4.14) \qquad 0 = \xi_1 < \ldots < \xi_m < 1.$$

Theorems corresponding to Theorem 4.1 and Theorem 4.2 will now hold for the IDeC-methods based on the scheme (3.4).

The "grid-restrictions" formulated in Remark 2 for the IDeC-methods based on the box-scheme do not apply in the present situation. There is a one-to one correspondence between "$t_{i,k}$-grids" and the "$s_{i,k}$-grids" (cf. (4.13)).

4.1.3. IDeC-methods based on the scheme (3.5)

Remarks analogous to those made for the scheme (3.4) apply to the scheme (3.5) with

$$t_{i,k} = s_{i,k}, \quad i = 0(1)I-1, \quad k = 1(1)m$$

and

$$0 < \xi_1 < \ldots < \xi_m = 1.$$

4.2. Initial value problems

All the results of Section 4.1 hold for IVPs, if the boundary condition

$$(4.16) \qquad g(y(a),y(b)) = y(a) - y_a = 0$$

is used. Collocation schemes for IVPs are only competitive for stiff systems of ODEs. For such problems methods with good stability properties are needed. Collocation schemes based on Gauss-Legendre points are known to be A-stable (cf. Wright [12]), and therefore, the IDeC based on the implicit midpoint rule (3.3)(with m odd) seems to be an appropriate scheme for solving stiff ODEs. Collocation schemes based on Radau points (with $\xi_m = 1$) are strongly A-stable (cf. Wright [12]) and therefore, the IDeC-methods based on the implicit Euler method (3.5) is perhaps an even more interesting scheme for solving stiff problems.

Up to now we have not examined whether the iterates *converge* to the fixed point. Consider the IDeC-methods based on the implicit Euler-method applied to stiff systems. This possibility has been investigated by Frank, Ueberhuber [6]. It is shown that, for equidistant nodes ($\xi_k = k/m$), very promising convergence results hold. Some of these results do not apply when the nodes are Radau points. As a consequence, it would appear that an IDeC-method based on the impl. Euler method on an *equidistant grid* is the preferred implementation.

If the IDeC is examined as a method for solving collocation equations, then J (the maximum number of IDeC-steps) is not determined by the asymptotic result (3.23). In this situation, the standard stopping criterion

$$\|\eta^j - \eta^{j-1}\| < \varepsilon$$

may be used.

4.3. Fixed points of IDeC-methods for other discretizations

In the previous sub-sections (4.1, 4.2), a relation between collocation schemes and the fixed point of the IDeC-methods based on methods (3.3), (3.4) and (3.5) was established. As a consequence, it is natural to examine whether the IDeC-methods based on arbitrary RK-methods always have "collocation fixed-points". In general, this question has to be answered in the negative, as a simple counter-example shows:

If the trapezoidal rule is used as the basic discretization method for the "global variant" of the IDeC, then the fixed point is characterized by

$$d_i^*(s_{i,k}) + d_i^*(s_{i,k+1}) = 0.$$

This fixed point is therefore only equivalent to a rather general weighted residual method, where instead of requiring the defects to vanish at any single grid point, a linear combination of the defects must vanish.

5. CONCLUSION

In this paper, iterative methods for solving collocation equations were introduced. Any step of the iterative process produces an approximation η^j which is usually more accurate than η^{j-1}. Compared with Newton's method for solving the collocation equations, the above strategy yields more information by which its implementation can be controlled (for example, step size control).

In Frank, Ueberhuber [6] the fact is discussed, that the effort necessary to perform the IDeC-steps is low compared with the effort necessary to solve the basic finite difference scheme ((3.3), (3.4) or (3.5)). Moreover, the structure of the equations (3.3), (3.4) and (3.5) is much simpler than the structure of the collocation equations. The IDeC-methods are therefore a more economical way for solving the collocation equations, than Newton's method. E.g. the application of the IDeC to stiff IVPs requires the solution of systems of non-linear equations of the same dimension n as the given problem, whereas the dimension of the collocation equations is $n \times m$, if a scheme with m collocation nodes is used.

A further advantage of the IDeC-methods for certain IVPs (with high stiffness) is the fast convergence of the approximations to the fixed point which corresponds to the solution of a collocation method. In some situations, an approximation to the fixed point which agrees with it to machine accuracy is obtained after *one* IDeC-step (see Frank, Ueberhuber [6]).

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