Semigroup Forum Vol. 63 (2001) 449–456

DOI: 10.1007/s002330010074

RESEARCH ARTICLE

Ideals, Idempotents and Right Cancelable Elements in the Uniform Compactification

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Communicated by John Pym

Introduction

In this note, we discuss the number of left ideals, idempotents and right cancelable elements in the uniform compactification of a topological group. These are related to covering numbers defined for the group.

G will denote a topological group with identity e. We assume that G has the right uniform structure in which a basis for the vicinities is provided by the sets of the form $\{(x,y) \in G \times G\colon\ xy^{-1} \in U\}$, where U denotes a neighbourhood of e in G. $C_u(G)$ will denote the set of real-valued bounded uniformly continuous functions defined on G, and uG will denote the uniform compactification of G. G can be topologically embedded in uG, and so we shall regard G as a subspace of uG. Furthermore, a semigroup operation can be defined on uG which extends that defined on G. This has the property that, for every $x \in uG$, the map $\rho_x\colon\ y \mapsto yx$ is a continuous map from uG to itself. Furthermore, the semigroup operation as a map from $uG \times uG$ to uG, is jointly continuous at every point of $G \times uG$. A real-valued bounded continuous function defined on G has a continuous extension to uG if and only if it is uniformly continuous. If $f \in C_u(G)$, we shall use $\overline{f}$ to denote the continuous extension of f to uG.

The reader is referred to [1] or [4] for proofs of these statements. We observe that uG is often denoted by G^{LUC} or $\mathcal{LC}(G)$ in the literature.

We shall use G_d to denote G with the discrete topology, and $\beta G_d \sim uG_d$ to denote its Stone-Čech compactification. There is a continuous surjective homomorphism $\pi\colon\ \beta G_d \to uG$. This has the property that, for every $x, y \in \beta G_d$, $\pi(x) = \pi(y)$ if and only if $f^\beta(x) = f^\beta(y)$ for every $f \in C_u(G)$, where $f^\beta\colon\ \beta G_d \to \mathbf{R}$ denotes the continuous extension of f.

If U is a neighbourhood of e in G, we define the cardinal $\kappa_U(G)$ to be the smallest number of sets of the form Us $(s \in G)$ required to cover G. We put

$$\kappa(G) = \sup\{\kappa_U(G)\colon\ U \text{ is a neighbourhood of } e \text{ in } G\}.$$

We show that, under certain conditions, uG has $2^{2^{\kappa(G)}}$ minimal left ideals and that $uG \setminus G$ has $2^{2^{\kappa(G)}}$ elements which are right cancelable in uG. These conditions are always satisfied if G is locally compact. We show that the number of left invariant means on $C_u(G)$ is at least equal to the number of minimal left ideals in uG, assuming that G is left amenable. We also give a result about the number of idempotents in uG in terms of another covering number.

Our results about the number of disjoint left ideals in uG and the number of left invariant means on uG are not new in the case in which G is locally compact. They were proved in [6] and in [5]. However, we believe that our results are new for many topological groups which are not locally compact.

M. Filali has previously obtained a result about right cancelable elements in uG. It was shown in [2] that $uG \setminus G$ contains a dense open set of right cancelable elements in the special case in which G is the product of $\mathbf{R}^n$ and a topological group which contains a compact open normal subgroup. M. Filali and J. S. Pym have recently proved that $uG \setminus G$ contains a dense set of right cancelable elements if G is any locally compact topological group [3].

We wish to thank J. S. Pym for very helpful comments. In particular, Example 1.7 is due to him.

1. Disjoint left ideals, idempotents and right cancelable points in uG

Lemma 1.1. *Let $X, Y \subseteq G$. If there is a neighbourhood U of e in G such that $x' \notin Ux$ whenever x and x' are distinct points of X, then $\pi\colon \beta G_d \to uG$ is injective on $\mathrm{cl}_{\beta G_d}(X)$. If $Y \subseteq G$ and $VX \cap Y = \emptyset$ for some neighbourhood V of e in G, then $\mathrm{cl}_{uG}(X) \cap \mathrm{cl}_{uG}(Y) = \emptyset$.*

Proof. Let $p, q \in \mathrm{cl}_{\beta G_d}(X)$, with $p \neq q$. We can choose $A, B \subseteq X$ such that $A \in p$, $B \in q$ and $A \cap B = \emptyset$. Since $UA \cap B = \emptyset$, there is a function $f \in C_u(G)$ such that $f = 0$ on A and $f = 1$ on B (see [4], ex. 21.5.3). Thus $f^\beta(p) = 0$ and $f^\beta(q) = 1$, and so $\pi(p) \neq \pi(q)$.

There is a function $g \in C_u(G)$ such that $g = 0$ on X and $g = 1$ on Y. Since $\overline{g} = 0$ on $\mathrm{cl}_{uG}(X)$ and $\overline{g} = 1$ on $\mathrm{cl}_{uG}(Y)$, $\mathrm{cl}_{uG}(X) \cap \mathrm{cl}_{uG}(Y) = \emptyset$. ∎

Theorem 1.2. *Let U be a symmetric neighbourhood of e in G. If $\kappa_U(G)$ is infinite, uG has at least $2^{2^{\kappa_U(G)}}$ points.*

Proof. Let $X \subseteq G$ be maximal subject to the condition that $x' \notin Ux$ whenever x and x' are distinct elements of X. Since $\bigcup_{x \in X} Ux$ covers G, $|X| \geq \kappa_U(G)$. So $|\mathrm{cl}_{\beta G}(X)| \geq 2^{2^{\kappa_U(G)}}$ ([4], Theorem 3.58). Our claim therefore follows from Lemma 1.1. ∎

Our next theorem concerns the number of disjoint left ideals and the number of right cancelable elements of uG.

Theorem 1.3. *Let $\kappa(G)$ be infinite.*

Suppose that there exists a neighbourhood U of e in G such that G cannot be covered by less than $\kappa(G)$ sets of the form sUt $(s, t \in G)$.

Let $\mathcal{A}$ denote the family of subsets $A \subseteq G$ which have the property that $G \setminus A$ can be covered by less than $\kappa(G)$ sets of the form sUt $(s,t \in G)$ and define $C := \bigcap_{A\in\mathcal{A}} \mathrm{cl}_{uG}A$. If $p \in C$ and N is a neighbourhood of p in uG, then there is a set $Q \subseteq N \cap (uG \setminus G)$ such that $|Q| = 2^{2^{\kappa(G)}}$, the left ideals of the form $(uG)q$ $(q \in Q)$ are pairwise disjoint and the elements of Q are right cancelable in uG.

Proof. Suppose N to be closed and consider a symmetric neighbourhood V of e in G such that $V^3 \subseteq U$. By hypothesis we can cover G with $\kappa(G)$ sets Vs_α $(\alpha < \kappa)$ where each $s_\alpha \in G$. We may also suppose that $s_0 = e$. Define then by induction a κ-sequence $(t_\alpha)_{\alpha<\kappa}$ in N in such a way that $t_\beta \notin s_\gamma^{-1}Us_\delta t_\alpha$ whenever $\alpha,\gamma,\delta < \beta$. Let K be the set of κ-uniform ultrafilters on $\{t_\alpha\colon \alpha < \kappa\}$. The cardinality of K is $2^{2^{\kappa(G)}}$ (see [4], Theorem 3.58). Since $t_\beta \notin Ut_\alpha$ whenever $\alpha,\beta < \kappa$ are distinct, it follows from Lemma 1.1 that $\pi\colon \beta G_d \to uG$ is injective on K. We put $Q := \pi(K)$ and consider two distinct elements $q_1, q_2 \in Q$ with $q_1 = \pi(x_1)$ and $q_2 = \pi(x_2)$, $(x_1, x_2 \in K)$. Let $X_1 \in x_1$ and $X_2 \in x_2$ be disjoint. We define sets $\tilde{X}_i := \{vs_\alpha t_\beta\colon v \in V, \alpha < \beta, t_\beta \in X_i\}$, $(i = 1,2)$. We note that, for every $x \in uG$, $xq_i \in \mathrm{cl}_{uG}\tilde{X}_i$, because $xq_i = \lim_{vs_\alpha\to x}\lim_{t_\beta\to q_i} vs_\alpha t_\beta$. Using the property which defines $(t_\alpha)_{\alpha<\kappa}$ and the fact that $V^3 \subseteq U$, we have $V\tilde{X}_1 \cap \tilde{X}_2 = \emptyset$. So, by Lemma 1.1, $\mathrm{cl}_{uG}(\tilde{X}_1) \cap \mathrm{cl}_{uG}(\tilde{X}_2) = \emptyset$ and hence $(uG)q_1 \cap (uG)q_2 = \emptyset$, as required. Notice also that $Q \subseteq uG \setminus G$, because $(uG)a = uG$ if $a \in G$.

It remains to show that all $q \in Q$ are right cancelable in uG. Let $x, y \in uG$ be distinct and suppose, by contradiction, that $xq = yq$ with $q \in Q$. Since $x \neq y$ there exists $h \in C_u(G)$ such that $\bar{h}(x) = 0$ and $\bar{h}(y) = 1$. Since h is uniformly continuous there exists a neighbourhood W of e in G (that we may suppose contained in V) such that $|h(s) - h(t)| < \frac{1}{3}$ whenever $s,t \in G$ are such that $st^{-1} \in W$. Let $A := \{vs_\alpha t_\beta\colon v \in V, \alpha < \beta, h(vs_\alpha) < \frac{1}{3}\}$ and $B := \{vs_\alpha t_\beta\colon v \in V, \alpha < \beta, h(vs_\alpha) > \frac{2}{3}\}$. Since, as before, $xq \in \mathrm{cl}_{uG}(A)$ and $yq \in \mathrm{cl}_{uG}(B)$, we have $WA \cap B \neq \emptyset$. So $wvs_\alpha t_\beta = v's_\gamma t_\delta$ for some $w \in W$, $v, v' \in V$, $\alpha, \beta < \kappa$, where $\alpha < \beta$, $\gamma < \delta$, $h(vs_\alpha) < \frac{1}{3}$ and $h(v's_\gamma) > \frac{2}{3}$. Using once more the property of $(t_\alpha)_{\alpha<\kappa}$ we get $\beta = \delta$ and hence $wvs_\alpha = v's_\gamma$. So $|h(vs_\alpha) - h(v's_\gamma)| < \frac{1}{3}$, which is a contradiction. ∎

When G is locally compact and non-compact, $\kappa(G)$ is the cardinality of the smallest number of compact subsets required to cover G. Therefore hypothesis † of Theorem 1.3 holds for any compact neighbourhood U of e. So the conclusions of Theorem 1.3 hold for all locally compact, non-compact groups. In particular, if G is locally compact and σ-compact, there is a dense subset of $uG \setminus G$ whose elements are all right cancelable in G.

Hypothesis † also holds for many other groups. For example, suppose that G is a SIN-group. (This means that e has a basis of neighbourhoods U for which $tUt^{-1} = U$ for every $t \in G$.) If $\kappa(G) = \omega$ and if G is not totally bounded, then hypothesis † of Theorem 1.3 holds. It is not hard to prove, in this case, that we again have a dense subset of $uG \setminus G$ whose elements are all right cancelable in uG. In particular, these statements hold for all separable topological vector spaces.

We note that † also holds for any SIN-group G for which $\kappa(G) = \kappa_U(G)$ for some neighbourhood U of e in G.

In some cases, † is necessary, as well as sufficient, for the existence of $2^{2^{\kappa(G)}}$ minimal left ideals. For example, if G is a SIN-group and $\kappa(G) = \omega$, then † fails to hold if and only if G is totally bounded. In this case, uG is the completion of G, a compact topological group, which has precisely one left ideal. In the following theorem, we see that † is necessary for the existence of $2^{\mathfrak{c}}$ disjoint left ideals in uG if G is a metrisable topological group with $\kappa(G) = \omega$.

Theorem 1.4. *Let G be a metrisable group. Suppose that, for every neighbourhood of e, G is covered by a finite number of sets of the form xUy $(x, y \in G)$. Then uG cannot have more than $\mathfrak{c}$ disjoint left ideals.*

Proof. Let (U_n) be a basis of neighbourhoods of e. For every $n \in \mathbf{N}$ let F_n be a finite subset of $G \times G$ such that $G = \bigcup_{(x,y)\in F_n} xU_n y$. Let $\Phi := \prod_{n\in\mathbf{N}} F_n$. Then $|\Phi| \leq \mathfrak{c}$. Let $p \in uG$. For every $n \in \mathbf{N}$, $G = \bigcup_{(x,y)\in F_n} xU_n y$, and so there exists $(x, y) \in F_n$ such that $p \in \mathrm{cl}_{uG}(xU_n y)$. We can choose $f \in \Phi$ such that $f(n) = (x_n, y_n)$ implies that $p \in \mathrm{cl}_{uG}(x_n U_n y_n)$. We shall prove that if p_1 and p_2 correspond to the same function f, then the principal left ideals corresponding to p_1 and p_2 intersect.

We have $x_n^{-1} p_1 y_n^{-1} \in \mathrm{cl}_{uG} U_n$ for every $n \in \mathbf{N}$. So $x_n^{-1} p_1 y_n^{-1} \longrightarrow e$ in uG. Let (z, y) be a limit point of (x_n^{-1}, y_n) in $uG \times uG$ and let $(x_{n_\iota}^{-1}, y_{n_\iota})$ be a net in $G \times G$ converging to (z, y) in uG. The semigroup operation of uG is jointly continuous at every point of $G \times uG$ (see [4], Theorem 21.44). So $(x_{n_\iota}^{-1} p_1 y_{n_\iota}^{-1}) y_{n_\iota} = x_{n_\iota}^{-1} p_1$ converges both to y and zp_1. Thus $zp_1 = y$. We also have $zp_2 = y$, and so $(uG)p_1 \cap (uG)p_2 \neq \emptyset$. ∎

Theorem 1.5. *Suppose that $\kappa_U(G) < \kappa(G)$ for every neighbourhood U of e in G and that there is a basis of cardinality at most $\kappa(G)$ for the neighbourhoods of e in G. Then, assuming the generalised continuum hypothesis, uG contains at most $2^{\kappa(G)}$ points.*

Proof. Let $(U_\alpha)_{\alpha<\kappa(G)}$ be a basis for the neighbourhoods of e in G. For each $\alpha < \kappa(G)$, we can choose $X_\alpha \subseteq G$ such that $G = U_\alpha X_\alpha$ and $|X_\alpha| = \kappa_{U_\alpha}(G)$. For each $p \in uG$ and each $\alpha < \kappa(G)$, let $\phi(\alpha, p) := \{Y \in \mathcal{P}(X_\alpha)\colon p \in \mathrm{cl}_{uG}(U_\alpha Y)\}$. We observe that $|\phi(\alpha, p)| \leq 2^{2^{\kappa_{U_\alpha}(G)}}$. Hence, assuming the generalised continuum hypothesis, $|\phi(\alpha, p)| < \kappa(G)$.

We shall show that, if $p, q \in u(G)$ have the property that $\varphi(\alpha, p) = \varphi(\alpha, q)$ for every $\alpha < \kappa(G)$, then $p = q$.

To see this, assume that $p \neq q$. We can choose $f \in C_U(G)$ such that $\overline{f}(p) = 0$ and $\overline{f}(q) = 1$. We can then choose $\alpha < \kappa(G)$ such that $|f(x) - f(y)| < \frac{1}{8}$ whenever $x, y \in G$ satisfy $xy^{-1} \in U_\alpha$. Then, if $Y := \{x \in X_\alpha\colon f(x) < \frac{1}{4}\}$, we have $Y \in \phi(p, \alpha)$ and hence $Y \in \phi(q, \alpha)$. However, $f < \frac{3}{8}$ on $U_\alpha Y$ and so $q \notin \mathrm{cl}_{uG}(U_\alpha Y)$—a contradiction.

Thus we have shown that we have an injective mapping from uG into $\kappa(G)^{\kappa(G)}$, defined by $p \mapsto (\phi(p,\alpha))_{\alpha<\kappa(G)}$. Our claim now follows from the fact that $\kappa(G)^{\kappa(G)} = 2^{\kappa(G)}$. ∎

Theorem 1.6. *Let U be a symmetric neighbourhood of e in G and let $\lambda_U(G)$ denote the least number of sets of the form sUt $(s,t \in G)$ required to cover G. If $\lambda_U(G)$ is infinite, uG has at least $2^{2^{\lambda_U(G)}}$ idempotents.*

Proof. Put $\lambda = \lambda_U(G)$. We can inductively choose a λ-sequence $(x_\alpha)_{\alpha<\lambda}$ in G such that $x_0 = e$ and, whenever F and F' are finite subsets of $[0,\beta)$ arranged in increasing order, $x_\beta \notin (\prod_{\alpha\in F} x_\alpha)^{-1} U (\prod_{\alpha\in F'} x_\alpha)$.

Let $X = \{x_\alpha\colon \alpha < \beta\}$ and let $FP(X)$ denote the set of all finite products of the form $x_{\alpha_1}x_{\alpha_2}\cdots x_{\alpha_m}$ with $\alpha_1 < \alpha_2 < \cdots < \alpha_m < \lambda$. If $\beta < \lambda$, let $FP_\beta(X)$ denote the set of products of this form with $\alpha_1 > \beta$. Let $C = \bigcap_{\beta<\lambda} \mathrm{cl}_{\beta G_d}(FP_\beta(X))$.

We observe that C is a subsemigroup of βG_d (see [4], Theorem 4.20). We shall show that $Cp \cap Cq = \emptyset$ whenever p and q are distinct uniform ultrafilters on X. To see this, choose disjoint subsets P and Q of X with $P \in p$ and $Q \in q$. Put $Y = \{x_{\alpha_1}x_{\alpha_2}\cdots x_{\alpha_m}\colon \alpha_1 < \alpha_2 < \cdots < \alpha_m, \alpha_m \in P\}$ and $Z = \{x_{\alpha_1}x_{\alpha_2}\cdots x_{\alpha_m}\colon \alpha_1 < \alpha_2 < \cdots < \alpha_m, \alpha_m \in Q\}$. Then $Y \cap Z = \emptyset$ and so $\mathrm{cl}_{\beta G_d}(Y) \cap \mathrm{cl}_{\beta G_d}(Z) = \emptyset$. However, $Cp \subseteq \mathrm{cl}_{\beta G_d}(Y)$ and $Cq \subseteq \mathrm{cl}_{\beta G_d}(Z)$.

Since there are 2^{2^λ} uniform ultrafilters on X, there are 2^{2^λ} disjoint left ideals in C. Each of these contains an idempotent (see [4], Corollary 2.6).

We claim that $\pi\colon \beta G_d \to uG$ is injective on $\mathrm{cl}_{\beta G_d}(FP(X))$. To see this, we shall apply Lemma 1.1 and show that, if y and z are distinct elements of $FP(X)$, then $z \notin Uy$. We suppose the contrary. Let $y = x_{\alpha_1}x_{\alpha_2}\cdots x_{\alpha_m}$ and $z = x_{\beta_1}x_{\beta_2}\cdots x_{\beta_n}$ with $\alpha_1 < \alpha_2 < \cdots < \alpha_m < \lambda$ and $\beta_1 < \beta_2 < \cdots < \beta_n < \lambda$. We assume that $m+n$ has been chosen to be as small as possible subject to the condition that $z \in Uy$. Our choice of $\{x_\alpha\colon \alpha < \lambda\}$ implies that $\alpha_m = \beta_n$ and hence that $x_{\alpha_1}x_{\alpha_2}\cdots x_{\alpha_{m-1}} \in Ux_{\beta_1}x_{\beta_2}\cdots x_{\beta_{n-1}}$, contradicting the minimality of $m+n$.

Since π is injective on $\mathrm{cl}_{\beta G_d}(FP(X))$, uG has at least 2^{2^λ} idempotents. ∎

For each neighbourhood U of e in G, let $\lambda_U(G)$ denote the smallest number of sets of the form xUy $(x,y \in G)$ required to cover G, and let $\lambda(G) = \sup\{\lambda_U(G)\colon U \text{ a neighbourhood of } e \text{ in } G\}$. We remark that it follows from Theorem 1.6 that uG has at least $\lambda(G)$ idempotents, unless $\lambda_U(G)$ is finite for every neighbourhood U of e in G.

We are indebted to J. S. Pym for the following example of a group G for which $\lambda(G) < \kappa(G)$. This example also allows us to show that † is not, in general, necessary for the conclusions of Theorem 1.3 to hold.

Example 1.7. Let $\eta > 1$ be any cardinal. We shall define a group G for which $\kappa(G) = \max(\mathfrak{c}, \eta)$ and $\lambda(G) = \omega$.

Let F be a group with identity 1_F and $|F| = \eta$, and let $F_i = F$ for each

$i \in (0,1)$. We put $H := \bigoplus_{i\in(0,1)} F_i$, with the topology defined by choosing as a base of neighbourhoods of the identity 1_H the sets of the form $U_\epsilon = \bigoplus_{i\in(0,1)} V_i$, where $\epsilon \in (0,1)$, $V_i = F$ if $i < \epsilon$ and $V_i = \{1_F\}$ if $i \geq \epsilon$.

We define Φ to be the group of functions $t \mapsto t^r$ defined on (0,1), where r denotes a positive rational number, with composition as the group operation. We denote the identity of Φ by 1_Φ. We give Φ the discrete topology. We define an action of Φ on H by putting $\phi(x_i) = (x_{\phi(i)})$, where $\phi \in \Phi$ and $(x_i) \in H$.

We take G to be the semidirect product of H and Φ. So $G = H \times \Phi$ as a topological space, with the group operation of G given by $(x,\phi)(y,\psi) = (x\phi(y), \phi\psi)$.

Let $\widetilde{H} = H \times \{1_\Phi\}$ and $\widetilde{U}_\epsilon = U_\epsilon \times \{1_\Phi\}$. Note that $\widetilde{H}$ is an open subsemigroup of G and that $\{\widetilde{U}_\epsilon \colon \epsilon \in (0,1)\}$ is a base for the neighbourhoods of $e = (1_H, 1_\Phi)$ in G.

A set of the form $\widetilde{U}_\epsilon(x,\phi)$ meets $\widetilde{H}$ if and only if $\phi = 1_H$. Since $\kappa_{\widetilde{U}_\epsilon}(\widetilde{H}) = \kappa_{U_\epsilon}(H) = \max(\mathfrak{c}, \eta)$, we have $\kappa(G) = \max(\mathfrak{c}, \eta)$.

On the other hand, we have $(1_H,\phi)\widetilde{U}_\epsilon(1_H,\phi^{-1}) = U_{\phi(\epsilon)} \times \{1_\Phi\}$ for every $\phi \in \Phi$. Let $\phi_n = t^{\frac{1}{n}}$. Since $\phi_n \nearrow 1$, $\bigcup_{n\in\mathbf{N}}(1_H,\phi_n)\widetilde{U}_\epsilon(1_H,\phi_n^{-1}) = \widetilde{H}$. Now $G/\widetilde{H}$ is countable and so $\lambda(G) = \omega$.

We now claim that, with a suitable choice of η, G nevertheless satisfies the conclusions of Theorem 1.3. Our proof is similar to that of Theorem 1.3.

We choose $\eta > \mathfrak{c}$, with the property that $\eta = \sup\{\eta_n \colon n \in \mathbf{N}\}$ for some increasing sequence (η_n) of distinct cardinals. We enumerate $\widetilde{H}$ as $(s_\alpha)_{\alpha<\eta}$, with $s_0 = e$, and Φ as $(\phi_n)_{n\in\mathbf{N}}$, with $\phi_0 = 1_\Phi$. We observe that every element of G can be expressed uniquely in the form $(1_H,\phi_n)s_\alpha$ for some $n \in \mathbf{N}$ and some $\alpha < \eta$.

We inductively choose $(t_\beta)_{\beta<\eta}$ in $\widetilde{H}$, with $t_0 = e$ and

$$t_\delta \notin s_\beta^{-1}(1_H,\phi_n)^{-1}\widetilde{U}_{\frac{1}{2}}(1_H,\phi_n)s_\alpha t_\gamma = s_\beta^{-1}\widetilde{U}_{\phi_n^{-1}(\frac{1}{2})}s_\alpha t_\gamma,$$

whenever $\alpha, \beta, \gamma, \eta_n < \delta$. This is possible, because, for a given $\delta < \eta$, $\{n \in \mathbf{N} \colon \eta_n < \delta\}$ is finite.

As in the proof of Theorem 1.3, we choose K to be the set of η-uniform ultrafilters on $\{t_\beta \colon \beta < \eta\}$, and we put $Q = \pi(K)$. We note that $|K| = 2^{2^\eta}$ and hence that $|Q| = 2^{2^\eta}$, because π is injective on K, by Lemma 1.1.

We claim that, if q_1 and q_2 are distinct elements of Q, then $(uG)q_1 \cap (uG)q_2 = \emptyset$. To see this, we choose x_1 and x_2 in K with $\pi(x_1) = q_1$ and $\pi(x_2) = q_2$. We then choose disjoint subsets X_1 and X_2 of $\{t_\beta \colon \beta < \eta\}$ with $X_1 \in x_1$ and $X_2 \in x_2$. For $i \in \{1,2\}$, we put

$$\widetilde{X}_i := \{(1_H,\phi_n)s_\alpha t_\beta \colon \eta_n < \beta, \alpha < \beta, t_\beta \in X_i\}.$$

We note that an equation of the form $u(1_H,\phi_n)s_\alpha t_\beta = (1_H,\phi_{n'})s_{\alpha'}t_{\beta'}$, with $u, s_\alpha, s_{\alpha'}, t_\beta, t_{\beta'} \in \widetilde{H}$, can only hold if $n = n'$. It follows that $\widetilde{U}_{\frac{1}{2}}\widetilde{X}_1 \cap \widetilde{X}_2 = \emptyset$.

So $\mathrm{cl}_{uG}\widetilde{X}_1 \cap \mathrm{cl}_{uG}\widetilde{X}_2 = \emptyset$, by Lemma 1.1. Now, if $i \in \{1,2\}$, $(uG)q_i \subseteq \mathrm{cl}_{uG}\widetilde{X}_i$, and so our claim follows.

It is also true that each element of Q is right cancelable in uG. We omit the proof, which is essentially the same as the proof of the corresponding statement in Theorem 1.3.

2. Consequences

The main consequence of Theorem 1.3 concerns the number of left invariant means on a left amenable group. In the following theorem we use the terminology of [1].

A mean on $C_u(G)$ is a real linear functional μ on $C_u(G)$ with the property that

$$\inf_{s\in G} f(s) \leq \mu(f) \leq \sup_{s\in G} f(s).$$

for every $f \in C_u(G)$. If ν is a mean on $C_u(G)$, we define $T_\nu\colon C_u(G) \to C_u(G)$ by $T_\nu(f)(s) = \nu(L_s f)$, where $L_s f \in C_u(G)$ is given by $L_s f(t) = f(st)$ $(\forall s,t \in G)$. If μ, ν are means on $C_u(G)$ a mean $\mu\nu$ on $C_u(G)$ is defined by $\mu\nu(f) = \mu(T_\nu(f))$.

A mean μ on uG is said to be multiplicative if $\mu(fg) = \mu(f)\mu(g)$ for every $f,g \in C_u(G)$. We note that uG can be identified with the set of multiplicative means on $C_u(G)$, endowed with the w^*-topology of $C_u^*(G)$. If μ is a multiplicative mean on $C_u(G)$ and $f \in C_u$, then $\overline{f}(\mu) = \mu(f)$.

A mean μ on $C_u(G)$ is said to be left invariant if $\mu(L_s f) = \mu(f)$ for every $f \in C_u(G)$ and every $s \in G$. $C_u(G)$ is said to be left amenable if a left invariant mean on $C_u(G)$ exists.

We shall use $M(C_u(G))$, $MM(C_u(G))$ and $LIM(C_u(G))$ respectively for the set of means, the set of multiplicative means and the set of left invariant means on $C_u(G)$. We shall use the fact that $MM(C_u(G))$ is the set of extreme points of $M(C_u(G))$. We refer the reader to [1], Chapter 2, for the proofs of these statements.

Theorem 2.1. *Let G be left amenable. Suppose that uG has η minimal left ideals, then there are at least η means in $LIM(C_u(G))$.*

Proof. Let $\mu \in LIM(C_u(G))$.

By Proposition 3.5, p. 81 of [1], $LIM(G)$ is a right ideal in $M(C_u(G))$. We shall prove that, if L and L' are disjoint closed left ideals in uG, then $\mu\nu \neq \mu\nu'$ if $\nu \in L$ and $\nu' \in L'$. Notice that we are regarding uG as $MM(C_u(G))$. Since $\mu \in M(C_u(G))$, μ is in the closed convex hull of $MM(C_u(G))$, by the Krein-Milman Theorem. Let $f \in C_u(G)$ satisfy $\overline{f}(L) = \{0\}$ and $\overline{f}(L') = \{1\}$, where $\overline{f}\colon uG \to \mathbf{R}$ denotes the continuous extension of f. If λ is a convex combination of elements in $MM(C_u(G))$, then $(\lambda\nu)(f) = 0$ and $(\lambda\nu')(f) = 1$, because $\lambda\nu$ is a convex combination of elements in L and $\lambda\nu'$ is a convex combination of elements in L'. So $(\mu\nu)(f) = 0$ and $(\mu\nu')(f) = 1$, and therefore $\mu\nu \neq \mu\nu'$. ■

Corollary 2.2. *If $C_u(G)$ is left amenable and G satisfies hypothesis † of Theorem 1.1, then* $|LIM(C_u(G))| \geq 2^{2^{\kappa(G)}}$.

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Received November 6, 2000
and in final form November 27, 2000
Online publication June 29, 2001