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ESTIMATION OF A FUNCTION OBSERVED WITH A STATIONARY ERROR

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A process $y(t)$ is assumed to be observed for $t \in [-T, T]$,

$$y(t) = s(t) + x(t) \qquad (t \in [-T, T]),$$

where s is an unknown function (to be estimated) and x is stationary noise. The accuracy of the least-squares estimator $\mathbf{s}^*$ is compared with the accuracy of the best linear unbiased estimator $\mathbf{s}^\star$. Bibliography: 5 titles.

1. Observations with stationary noise

Let $x(t)$ be a real-valued process stationary in the broad sense with zero mean ($\mathbf{E}\, x(t) = 0$) and spectral density f, i.e., x is a function

$$x : R^1 \to L^2(d\mathbf{P})$$

such that

$$\mathbf{E}\, x(t) \cdot \overline{x(s)} = \int_{-\infty}^{\infty} \exp\{i(t-s)u\}\, f(u)\, du,$$

where $L^2(d\mathbf{P})$ is the L^2-space generated by a probability measure $\mathbf{P}$.

We denote by $x[\varphi]$ "the average value" of $x(t)$,

$$x[\varphi] = \int_{-\infty}^{\infty} x(t)\, \varphi(t)\, dt = (\varphi,\, x),$$

where $(\cdot, \cdot)$ is the inner product in an L^2-space, induced by the Lebesgue measure on R^1. Thus, we have

$$\mathbf{E}\, x[\varphi] \cdot \overline{x[\psi]} = \int_{-\infty}^{\infty} \hat{\varphi}(u) \cdot \overline{\hat{\psi}(u)}\, f(u)\, du = \left(\hat{\varphi},\, \hat{\psi}\right)_f,$$

where $\hat{\varphi}$ is the Fourier transform of the function φ,

$$\hat{\varphi}(v) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} \varphi(u) \exp\{ivu\}\, du;$$

$(\cdot, \cdot)_f, \|\cdot\|_f$ are the inner product and norm in the L^2-space generated by the measure $f(u)du$.

For some applications and theoretical problems we need to consider the case where the functional $x(t)$, "the value of a random function x at a point t," cannot be well defined. Hence, we need to introduce a generalized process when we know only such average values $x[\varphi]$.

Let $x[\cdot]$ be a generalized process stationary in the broad sense with spectral density f, i.e., (see [1]) x is an operator : $D \to L^2(d\mathbf{P})$ from the space D of infinitely differentiable functions with compact support to the L^2 space generated by the probability measure $\mathbf{P}$ such that

$$\mathbf{E}\, x[\varphi] \cdot \overline{x[\psi]} = \int_{-\infty}^{\infty} \hat{\varphi(u)} \cdot \overline{\hat{\psi}(u)}\, f(u) du = \left(\hat{\varphi}, \hat{\psi}\right)_f \quad (\varphi, \psi \in D). \tag{1}$$

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We assume that the process x is real-valued, i.e., the random variables $x[\varphi]$ are real for any real-valued function $\varphi \in D$. We assume that

$$\mathbf{E}x[\varphi] = 0, \quad \varphi \in D. \tag{2}$$

We can define the Hilbert structure generated by the norm $||\cdot||_x$, $||\varphi||_x = ||\hat{\varphi}||_f$, on the linear set D, and the process x can be extended to the linear set $D(x)$ of all locally integrable functions $\varphi \in L^2_{\text{loc}}$ such that $\hat{\varphi} \in L^2_f$ (which is contained in the closure of D in the above Hilbert space).

Further we assume that the process x is defined on $D(x)$ and denote

$$D_T = D_T(x) = \{\varphi : \varphi \in D(x), \operatorname{supp}\varphi \subset [-T, T]\}.$$

Assume that we observe a process $y[\varphi]$ for $\varphi \in D_T$,

$$y[\varphi] = s[\varphi] + x[\varphi] \qquad (\varphi \in D_T), \tag{3}$$

where $s[\cdot]$ is an unknown functional: $D_T \to \mathbf{C}^1$ (to be estimated) and $x[\cdot]$ is stationary noise.

2. Regular functionals

Let $\mathbf{H}$ be a Hilbert space of functions defined on $\mathbf{R}^1$ with inner product $\langle\cdot,\cdot\rangle$ and norm $||\cdot||$, and let $\mathbf{L}$ be a subspace of the space $\mathbf{H}$. We assume that $D_T \subset \mathbf{H}$ and $\overline{D_T} = \mathbf{H}$, where $\overline{D_T}$ is the closure of D_T in $\mathbf{H}$.

A functional $g : D_T \to \mathbf{C}^1$ is called regular if there exists an element $\mathbf{g} \in \mathbf{H}$ such that

$$g[\varphi] = \langle\varphi, \mathbf{g}\rangle \text{ for all } \varphi \in D_T.$$

We denote by $\boldsymbol{v}$ the operator that is defined on the linear subset of all regular functionals by the relation: $vg = \mathbf{g}$, where $\mathbf{g}$ is an element of $\mathbf{H}$ such that $g[\cdot] = \langle\cdot, \mathbf{g}\rangle$.

In the sequel, we assume that the unknown functional s is regular and denote $\mathbf{s} = vs$. Assume that the subspace $\mathbf{L}$ is known and $\dim(\mathbf{L}) = m < \infty$.

A random variable $x[\varphi], \varphi \in D_T$, is a function $\xi(\omega) = \xi_\varphi(\omega) = x[\varphi]$ from the space $L^2(d\mathbf{P})$. Conceivably, for some ω, the random functional $x[\cdot]$ is not regular. But the projection (in the sense that will be defined below) of x onto some finite-dimentional subspaces may be regular almost surely.

3. Projection onto a subspace

Let $\mathbf{L}$ and $\mathbf{L}^*$ be subspaces of the space $\mathbf{H}$ such that

$$\mathbf{L} \cap \mathbf{L}^* = \{\mathbf{0}\} \qquad \textit{textand} \qquad \mathbf{L} + \mathbf{L}^* = \mathbf{H}. \tag{4}$$

The projection of a functional $g : D_t \to \mathbf{C}^1$ onto the space $\mathbf{L}$ parallel to the subspace $\mathbf{L}^*$ is a functional $P(\mathbf{L}, \mathbf{L}^*)g$ (if it exists) such that

$$P(\mathbf{L}, \mathbf{L}^*)g\,[\varphi] = \begin{cases} g[\varphi] & \text{if} \quad \varphi \in \mathbf{L}_*, \\ \mathbf{0} & \text{if} \quad \varphi \in \mathbf{L}^\perp \end{cases} \tag{5}$$

(where we denote by $\mathbf{M}^\perp$ the orthogonal complement to the subspace $\mathbf{M}$ in the space $\mathbf{H}$, and we put $\mathbf{L}_* = \left(\mathbf{L}^*\right)^\perp$. Projection (5) is well defined if the functional g is defined on $\mathbf{L}_*$.

It is easy to see that if a functional g is defined on the subspace $\mathbf{L}_*$, then

$$P(\mathbf{L}, \mathbf{L}^*)g\,(\cdot) = \langle\cdot, \mathbf{h}\rangle \quad \text{with} \qquad \mathbf{h} = \sum_{j=1}^{m} \overline{g(\psi_j)} \cdot \varphi_j, \tag{6}$$

where $m = \dim(\mathbf{L})$, $\{\varphi_j, j = 1, ..., m\}$ and $\{\psi_j, j = 1, ..., m\}$ are orthogonal bases of the subspaces $\mathbf{L}$ and $\mathbf{L}_*$ such that $\langle \varphi_i, \psi_j \rangle = \delta_{ij}$.

In the case where $\mathbf{L}^* = \mathbf{L}^{\perp}$, the element $P(\mathbf{L})g = P(\mathbf{L}, \mathbf{L}^{\perp})g$ is called the orthogonal projection of the functional g onto the subspace $\mathbf{L}$.

If the function g has regular representation $g(\cdot) = \langle \cdot, \mathbf{g} \rangle$, it is obvious that

$$P(\mathbf{L}, \mathbf{L}^*)g\,(\cdot) = \langle \cdot, \mathbf{P}(\mathbf{L}, \mathbf{L}^*)\mathbf{g} \rangle,$$

where $\mathbf{P}(\mathbf{L}, \mathbf{L}^*)$ is the corresponding projector in the space $\mathbf{H}$, that is,

$$\mathbf{P}(\mathbf{L}, \mathbf{L}^*)\mathbf{h} = \begin{cases} \mathbf{h} & \text{if} \quad \mathbf{h} \in \mathbf{L}, \\ \mathbf{0} & \text{if} \quad \mathbf{h} \in \mathbf{L}^*. \end{cases} \tag{7}$$

For brevity, we shall write $\mathbf{P}(\mathbf{L}, \mathbf{L}^*)g$ instead of $v\,P(\mathbf{L}, \mathbf{L}^*)g$. Thus, for example, by definition we have

$$\mathbf{P}(\mathbf{L}, \mathbf{L}^*)g = \mathbf{P}(\mathbf{L}, \mathbf{L}^*)\mathbf{g}, \quad \text{where} \quad \mathbf{g} = vg.$$

4. Linear estimators. Unbiased estimators

Consider the problem of linear estimation of an unknown function $\mathbf{s} \in \mathbf{H}$, based on the observations

$$y[\varphi] = s[\varphi] + x[\varphi] \qquad (\varphi \in D_T, D_T \subset \mathbf{H})$$

(for details, see [2, 3, 5]). Assume that $s[\varphi] = \langle \varphi, \mathbf{s} \rangle$, the unknown function $\mathbf{s}$ belongs to a finite-dimensional subspace $\mathbf{L}$ of the space $\mathbf{H}$, and the subspace $\mathbf{L}$ is known. We also assume that $x[\cdot]$ is a stationary generalized process with spectral density f.

Let $\{\varphi_j,\ j = 1, ..., m\}$ be a basis of $\mathbf{L}$ and

$$\mathbf{s}(\cdot) = \sum_{j=1}^{m} \theta_j \varphi_j(\cdot).$$

We consider linear estimators $\hat{\mathbf{s}}$ for the unknown function $\mathbf{s}$, based on the observations $y[\varphi]$, $\varphi \in D_T$, i.e., estimators such that they can be represented in the form

$$\hat{\mathbf{s}}(\cdot) = \sum_{j=1}^{m} \hat{\theta}_j \varphi_j(\cdot), \tag{8}$$

where $\hat{\theta}_j \in \mathbf{Y}_{-T}^{T}$ $(j = 1, ..., m)$. We denote

$$\mathbf{Y}_a^b = \overline{\mathrm{sp}}\{y[\varphi], \operatorname{supp}\varphi \in [a, b]\}.$$

It is clear that

$$\hat{\theta}_j = \overline{y[\psi_j]} = \overline{s[\psi_j]} + \overline{x[\psi_j]} \quad \text{for some} \quad \psi_j \in D_T \quad (j = 1, ..., m)$$

and

$$\overline{s[\psi_j]} = \langle \mathbf{s},\ \psi_j \rangle \quad (j = 1, ..., m).$$

A linear estimator $\hat{\mathbf{s}}$ is called an unbiased estimator if $\mathbf{E}\,\hat{\mathbf{s}} = \mathbf{s}$ for any $\mathbf{s} \in \mathbf{L}$. If an estimator (8) is unbiased, then, by the fact that $\mathbf{E}\,y[\varphi] = s[\varphi]$ (see (2)),

$$\sum_{j=1}^{m} \mathbf{E}\,\hat{\theta}_j \varphi_j(\cdot) = \sum_{j=1}^{m} \langle \mathbf{s},\ \psi_j \rangle \varphi_j(\cdot) = \mathbf{s}(\cdot)$$

for any $\mathbf{s} \in \mathbf{L}$. Thus, if an estimator $\hat{\mathbf{s}}$ is unbiased, then $\langle \varphi_i, \psi_j \rangle = \delta_{ij} \quad (j = 1, ..., m)$. Therefore, we conclude that a linear unbiased estimator $\hat{\mathbf{s}}$ in (8) has a representation:

$$\hat{\mathbf{s}}(\cdot) = [\mathbf{P}(\mathbf{L}, \mathbf{L}^*)y](\cdot), \tag{9}$$

where the subspace $\mathbf{L}^* = \mathbf{L}^*(\hat{\mathbf{s}})$ is defined by the relation

$$\mathbf{L}^* = \mathbf{H} \ominus \mathbf{L}_*,$$

and the subspace $\mathbf{L}_* = \mathbf{L}_*(\hat{\mathbf{s}})$ is defined by the relation

$$\mathbf{L}_* = \overline{\text{sp}}\{\psi_j, j = 1, ..., m\}.$$

We need to assume that

$$\mathbf{L}_* \subset D_T.$$

It is clear that

$$\mathbf{L} + \mathbf{L}^* = \mathbf{H}, \quad \mathbf{L} \cap \mathbf{L}^* = \mathbf{0}, \tag{10}$$

because $\langle \varphi_j, \psi_j \rangle = \delta_{ij} \quad (j = 1, ..., m)$. We denote $\mathbf{P} = \mathbf{P}(\mathbf{L}, \mathbf{L}^*)$. It is easy to see that the conjugate operator $\mathbf{P}^*$ is a projector onto the subspace $\mathbf{L}_*$ parallel to the subspace $\mathbf{L}^\perp = \mathbf{H} \ominus \mathbf{L}$:

$$\mathbf{P}^* \, \mathbf{h} = \begin{cases} \mathbf{h} & \text{if} \quad \mathbf{h} \in \mathbf{L}_*, \\ \mathbf{0} & \text{if} \quad \mathbf{h} \in \mathbf{L}^\perp, \end{cases} \tag{11}$$

where the conjugate operator B^* is defined by the relation

$$\langle B \cdot, \cdot \rangle = \langle \cdot, B^* \cdot \rangle.$$

5. Accuracy of estimation

Assume that the estimator $\hat{\mathbf{s}}$ is a linear unbiased estimator, $\hat{\mathbf{s}} = \mathbf{P}y$, where $\mathbf{P} = \mathbf{P}_{\hat{\mathbf{s}}} = \mathbf{P}(\mathbf{L}, \mathbf{L}^*)$ for a subspace $\mathbf{L}^*$ such that (10) holds. The accuracy of the estimation can be measured by the value of

$$\sigma_T^2(\hat{\mathbf{s}}, \mathbf{s}) = \mathbf{E}\, ||\hat{\mathbf{s}} - \mathbf{s}||^2.$$

We denote by $\mathbf{R}_x$ the covariance operator of a process x :

$$\mathbf{E}\, \{\langle x[h_1], x[h_2] \rangle\} = \langle R_x \, h_1, h_2 \rangle, \quad h_1, h_2 \in \mathbf{H}.$$

By (1), we have $\hat{\mathbf{s}} - \mathbf{s} = \mathbf{P}x$ and (see (6))

$$[\mathbf{P}x](\cdot) = \sum_{j=1}^{m} \overline{x[\psi_j]} \cdot \varphi_j(\cdot)$$

for an orthonormal basis $\{\varphi_j, j = 1, ..., m\}$ of the subspace $\mathbf{L}$ and for a basis $\{\psi_j, j = 1, ..., m\}$ of the space $\mathbf{L}^*$ such that $\langle \varphi_i, \psi_j \rangle = \delta_{ij}$ $(i, j = 1, ..., m)$. Therefore,

$$\sigma_T^2(\hat{s}, s) = \mathbf{E}\, ||\mathbf{P}x||^2 = \mathbf{E}\, ||\sum_{j=1}^{m} \overline{x[\psi_j]} \cdot \varphi_j(\cdot)||^2 = \sum_{j=1}^{m} \mathbf{E}\, |x[\psi_j]|^2. \tag{12}$$

Thus, we have

$$\sigma_T^2(\hat{s}, s) = \sum_{j=1}^{m} \langle R_x \psi_j, \psi_j \rangle = \sum_{j=1}^{m} \langle R_x P^* \varphi_j, P^* \varphi_j \rangle = \sum_{j=1}^{m} \langle \mathbf{P} R_x P^* \varphi_j, \varphi_j \rangle.$$

As soon as $\mathbf{P}^* h = \mathbf{0}$ for $h \in \mathbf{L}^\perp$, we obtain

$$\sigma_T^2(\hat{s}, s) = \text{tr} \mathbf{P} \mathbf{R}_x \mathbf{P}^*, \tag{13}$$

where $\text{tr}\, A$ is the trace of an operator A.

6. The best linear unbiased estimator

We also consider another Hilbert structure on $\mathbf{H}$. Let $\mathbf{H}_\star$ be a Hilbert space with inner product

$$\langle \varphi, \psi \rangle_\star \ = \ \langle \mathbf{R}_x \varphi, \psi \rangle \ = \ \mathbf{E} x[\varphi]\overline{x[\psi]}$$

and norm $||\cdot||_\star^2 \ = \ \langle \cdot, \cdot \rangle_\star$.

We note that, without loss of generality, we may assume that

$$\mathbf{R}_x \varphi \neq \mathbf{0} \quad \text{for any } \varphi \in L.$$

Let $\mathbf{L}^\perp$ be the orthogonal complement (in the space $\mathbf{H}$) of the subspace $\mathbf{L}$, let $\mathbf{L}_\star$ be the orthogonal complement (in the space $\mathbf{H}_\star$) of the subspace $\mathbf{L}^\perp$, and let $\mathbf{L}^\star$ be the orthogonal complement (in the space $\mathbf{H}$) of the subspace $\mathbf{L}_\star$. We put $\mathbf{P}_\star = \mathbf{P}(\mathbf{L}, \mathbf{L}^\star)$. It is easy to see that

$$\{\mathbf{P}_\star\}^* \varphi = \begin{cases} \varphi & \text{if} \quad \varphi \in \mathbf{L}_\star, \\ \mathbf{0} & \text{if} \quad \varphi \in \mathbf{L}^\perp \end{cases} \tag{14}$$

and

$$\mathbf{P}_\star \varphi = \begin{cases} \varphi & \text{if} \quad \varphi \in \mathbf{L}, \\ \mathbf{0} & \text{if} \quad \varphi \in \mathbf{L}^\star. \end{cases} \tag{15}$$

Consider an estimator $\mathrm{s}^\star \ = \ \mathbf{P}_\star \, y$.

Proposition 1. *The estimator* $\mathrm{s}^\star$ *is the best linear unbiased estimator for* s*, i.e.,*

$$\mathbf{E}\, \mathrm{s}^\star \ = \ \mathrm{s} \quad \textit{and} \quad \sigma_T^2(\hat{\mathrm{s}},\ \mathrm{s}) \geq \sigma_T^2(\mathrm{s}^\star,\ \mathrm{s})$$

for any linear unbiased estimator $\hat{\mathrm{s}}$.

Proof. The estimator $\mathrm{s}^\star$ is an unbiased estimator. We put $\hat{\mathrm{s}} \ = \ \mathbf{P}y$, where $\mathbf{P} = \mathbf{P}(\mathbf{L},\ \mathbf{L}^\star)$. Let $\{\varphi_j,\ j = 1, 2, ..., m\}$ be an orthonormal basis of the space $\mathbf{L}$ and $\psi_j = \mathbf{P}\varphi_j$ $(j = 1, ..., m)$. Thus,

$$\varphi_j \ = \ \psi_j \ + \varphi^j, \qquad \text{where } \varphi^j \in \mathbf{L}^\perp. \tag{16}$$

In this case (see (12), (14)),

$$\sigma_T^2(\hat{\mathrm{s}},\ \mathrm{s}) \ = \ \sum_{j=1}^{m} \mathbf{E}\,|x[\psi_j]|^2 \ = \ \sum_{j=1}^{m} ||\psi_j||_\star^2. \tag{17}$$

Hence,

$$\sigma_T^2(\hat{\mathrm{s}},\ \mathrm{s}) \ = \ \sum_{j=1}^{m} ||\varphi_j - \varphi^j||_\star^2 \ \geq \ \sum_{j=1}^{m} ||\mathbf{P}_\star^*\varphi_j||_\star^2,$$

because (see (14)) $\mathbf{P}_\star^*$ is the orthoprojector of the space $\mathbf{H}_\star$ onto the subspace $\mathbf{L}_\star$, which is the orthogonal complement (in the space $\mathbf{H}_\star$) of the subspace $\mathbf{L}^\perp$. It is obvious that

$$\sigma_T^2(\mathrm{s}^\star,\ \mathrm{s}) \ = \ \sum_{j=1}^{m} ||\mathbf{P}_\star^*\varphi_j||_\star^2. \tag{18}$$

Thus, we have

$$\sigma_T^2(\hat{\mathrm{s}},\ \mathrm{s}) \geq \sigma_T^2(\mathrm{s}^\star,\ \mathrm{s}).$$

7. The least-squares method in the estimation problem

The least-squares estimator $\mathbf{s}^*$ is defined by the relation

$$\mathbf{s}^* = \arg \inf_{\mathbf{s} \in \mathbf{L}} ||y[\cdot] - s[\cdot]||,$$

$$\text{where} \quad \mathbf{s} = vs \quad \text{and} \quad ||g[\cdot]|| = \sup_{\varphi \in D_T} \frac{|g[\varphi]|}{||\varphi||}.$$

It is easy to see that

$$\mathbf{s}^* = \mathbf{P}(\mathbf{L})\, y, \qquad \text{that is,} \qquad \mathbf{P}_{\mathbf{s}^*} = \mathbf{P}(\mathbf{L}),$$

where $\mathbf{P}(\mathbf{L})$ is the operator of orthogonal projection in the space $\mathbf{H}$ onto a subspace $\mathbf{L}$. In the general case, the least-squares estimator $\mathbf{s}^*$ is not the best linear unbiased estimator.

Proposition 2. *Assume that the subspaces* $\mathbf{L}$ *and* $\mathbf{L}^{\perp}$ *are orthogonal in the metric of the Hilbert space* $\mathbf{H}_{\star}$. *Then the least-squares estimator* $\mathbf{s}^*$ *is the best linear unbiased estimator.*

Proof. In this case (see (14)) we have $\mathbf{P}_{\mathbf{s}^*} = \mathbf{P}(\mathbf{L}) = \mathbf{P}_{\star} = \mathbf{P}_{\mathbf{s}^{\star}}$; therefore $\mathbf{s}^* = \mathbf{s}^{\star}$.

8. Comparison of the accuracy of a linear unbiased estimator with the accuracy of the best linear unbiased estimator

We compare the accuracy of an estimator $\hat{\mathbf{s}} = \mathbf{P}y$, where $\mathbf{P} = \mathbf{P}(\mathbf{L}, \mathbf{L}^*)$, with the accuracy of the best linear unbiased estimator $\mathbf{s}^{\star}$. It was stated in Sec. 6 (see (17) and (18)) that

$$\sigma_T^2(\hat{\mathbf{s}}, \mathbf{s}) = \sum_{j=1}^{m} ||\mathbf{P}^*\varphi_j||_{\star}^2, \qquad \sigma_T^2(\mathbf{s}^{\star}, \mathbf{s}) = \sum_{j=1}^{m} ||\mathbf{P}_{\star}^*\varphi_j||_{\star}^2,$$

where $\{\varphi_j,\ j = 1, ..., m\}$ is an orthonormal basis of $\mathbf{L}$. Since

$$\varphi_j = \psi_j + \varphi^j, \quad \text{where} \quad \varphi^j \in \mathbf{L}^{\perp} \quad \text{and} \quad \psi_j = \mathbf{P}_{\star}^*\varphi_j,\ (j = 1, ..., m),$$

it follows that

$$\mathbf{P}^*\varphi_j = \mathbf{P}^*[\psi_j + \varphi^j] = \mathbf{P}^*[\psi_j] = \mathbf{P}^*[\mathbf{P}_{\star}^*\varphi_j] \quad (j = 1, ..., m).$$

Thus, we have

$$\mathbf{P}^*\varphi_j = \mathbf{P}^*[\mathbf{P}_{\star}^*\varphi_j] \quad (j = 1, ..., m).$$

Therefore,

$$||\mathbf{P}^*\varphi_j||_{\star} \leq ||\mathbf{P}^*||_{\star} \cdot ||\mathbf{P}_{\star}^*\varphi_j||_{\star},$$

where for an operator A we denote by $||A||_{\star}$ the uniform operator norm in the space $\mathbf{H}\star$. Finally, we obtain

$$\sigma_T^2(\mathbf{s}^{\star}, \mathbf{s}) \leq \sigma_T^2(\hat{\mathbf{s}}, \mathbf{s}) \leq ||\mathbf{P}^*||_{\star} \cdot \sigma_T^2(\mathbf{s}^{\star}, \mathbf{s}). \tag{19}$$

Now we state and prove a simple geometric result. We assume that $\mathbf{L}$ and $\mathbf{M}$ are subspaces of a Hilbert space $\mathbf{H}$ with inner product $(\cdot, \cdot)$ and norm $||\cdot||$. Assume that

$$\overline{\mathbf{L} + \mathbf{M}} = \mathbf{H} \quad \text{and} \quad \mathbf{L} \cap \mathbf{M} = 0.$$

Denote

$$\cos(\mathbf{L}, \mathbf{M}; \mathbf{H}) = \sup_{h_1 \in \mathbf{L}, h_2 \in \mathbf{M}} \frac{|(\mathbf{h}_1, \mathbf{h}_2)|}{||h_1|| \cdot ||h_2||}.$$

Let an operator $\mathbf{P}$ be defined by the relation

$$\mathbf{Ph} = \begin{cases} \mathbf{h} & \text{if} \quad \mathbf{h} \in \mathbf{L}, \\ \mathbf{0} & \text{if} \quad \mathbf{h} \in \mathbf{M}. \end{cases}$$

Lemma. *The uniform operator norm* $||\mathbf{P}||$ *of the operator* $\mathbf{P}$ *in the space* $\mathbf{H}$ *satisfies the relation*

$$||\mathbf{P}||^2 \;\le\; \frac{1}{1-\cos^2(\mathbf{L},\mathbf{M};\mathbf{H})}. \tag{20}$$

Proof. Assume that a linear set $\mathbf{L}+\mathbf{M}$ is dense in $\mathbf{H}$. Therefore,

$$||\mathbf{P}|| \;=\; \sup_{\mathbf{h_L}\in\mathbf{L},\mathbf{h_M}\in\mathbf{M}} \frac{||\mathbf{h_L}||}{||\mathbf{h_L}-\mathbf{h_M}||}.$$

It is clear that

$$\inf_{\mathbf{h_M}\in\mathbf{M}} ||\mathbf{h_L}-\mathbf{h_M}|| \;=\; ||\mathbf{h_L}-\mathbf{P}(\mathbf{L})\mathbf{h_L}|| \quad \text{and}$$

$$\inf_{\mathbf{h}\in\mathbf{L}} ||\mathbf{h}-\mathbf{P}(\mathbf{L})\mathbf{h}||^2 \;=\; (1-\cos^2(\mathbf{L},\mathbf{M};\mathbf{H}))||\mathbf{h}||^2.$$

Thus, we obtain (20).

Proposition 3. *For any linear unbiased estimator* $\hat{\mathrm{s}} \;=\; \mathbf{P}(\mathbf{L},\mathbf{L}^*)\,y$, *the following inequalities hold:*

$$\sigma_T^2(\mathrm{s}^*,\ \mathrm{s}) \;\le\; \sigma_T^2(\hat{\mathrm{s}},\ \mathrm{s}) \;\le\; \frac{1}{1-\cos^2(\mathbf{L}_*,\mathbf{L}^{\perp};\mathbf{H}_*)}\cdot\sigma_T^2(\mathrm{s}^*,\ \mathrm{s}). \tag{21}$$

Proof. The proposition follows from (19) and (21).

9. The spectral representation. The reproducing kernel $G_t^f(u,\,v)$ of the space $\mathbf{H}_T(f)$

Now we consider the spectral representation of the process x (see [1] for details):

$$x[\varphi] \;=\; \int_{-\infty}^{\infty} \hat{\varphi}(u)\ Z(du),$$

where $Z(\cdot)$ is an orthogonal measure such that

$$\mathbf{E}\,Z(A)\cdot\overline{Z(B)} \;=\; \int_{A\cap B} f(u)\ du, \quad \mathbf{E}\,Z(A)=0.$$

Denote by $\mathbf{X}_a^b$ a subspace of the space $L^2(dP)$ such that

$$\mathbf{X}_a^b \;=\; \overline{\mathrm{sp}}\{x[\varphi]:\ \operatorname{supp}\varphi\subset[a,b]\}$$

(where by $\overline{\mathrm{sp}}\{M\}$ we denote the closure of the linear manifold of M) and put $\mathbf{X} \;=\; \mathbf{X}_{-\infty}^{\infty}$. It is well known that the relation $U\,x[\varphi] \;=\; \hat{\varphi}$ defines an isometry $U\ :\mathbf{X}\to L_f^2$.

Let $\mathbf{H}_T(f)$ be a linear set of all entire functions of degree less than T that belongs to the space L_f^2. M. Krein's alternative states that if $\overline{\mathbf{H}_T(f)} \;\ne\; L_f^2$, then $\mathbf{H}_T(f)$ is a closed subspace of the space L_f^2 and the functional $l_z:\ l_z(\varphi) \;=\; \varphi(z)$ is bounded on $\mathbf{H}_T(f)$. It should be mentioned that

$$U\,\mathbf{X}_{-T}^{T} \;=\; \mathbf{H}_T(f).$$

Denote by $G_T^f(u,v)$ the reproducing kernel of the space $\mathbf{H}_T(f)$:

$$\big(\varphi(\cdot),\ G_T^f(z,\cdot)\big) \;=\; \varphi(z).$$

It is clear that

$$G_T^f(u,v) = \sum_{j=1}^{\infty} \overline{\varphi_j(u)} \cdot \varphi_j(v), \qquad ||G_T^f(v,\cdot)||_f^2 = G_T^f(v,\, v),$$

where $\{\varphi_j,\, j = 1, 2, ...\}$ is an orthonormal basis of the space $\mathbf{H}_T(f)$ and

$$|h(v)|^2 \leq G_T^f(v,\, v) \cdot ||h(\cdot)||_f^2 \qquad \text{for any function} \quad h \in \mathbf{H}_T(f). \tag{22}$$

For example, if the function $f(t) \equiv 1$ we have

$$G_T^{\mathbf{1}}(u,v) = \frac{\sin T(u-v)}{\pi(u-v)}, \qquad ||G_T^{\mathbf{1}}(u,\,\cdot)||_{\mathbf{1}}^2 = G_T^{\mathbf{1}}(u,\, u) = \frac{T}{\pi}, \tag{23}$$

where $\mathbf{1}(t) \equiv 1$. We denote by $\Phi_T(v,u)$ the Fejér kernel:

$$\Phi_T(v,u) = \frac{\sin^2 T(u-v)}{\pi T\,(u-v)^2}.$$

Proposition 4. *Suppose that $f, \frac{1}{f} \in L^2_{\frac{1}{1+u^2}}$. Then*

$$\frac{T}{\pi} \cdot \Big(\int_{-\infty}^{\infty} \Phi_T(v,u) \cdot f(u)\, du \Big)^{-1} \leq G_T^f(v,v) \leq \frac{t}{\pi} \cdot \int_{-\infty}^{\infty} \Phi_T(v,u) \cdot \frac{1}{f(u)}\, du. \tag{24}$$

Proof. The function $G_T^{\mathbf{1}}(v,\,\cdot)$ is the reproducing kernel of the space $\mathbf{H}_T(\mathbf{1})$, therefore

$$\Big(\varphi(\cdot),\, g(\cdot)\Big)_f = \varphi(v) \quad (\varphi \in \mathbf{H}_T(f) \cap \mathbf{H}_T(\mathbf{1})\,)$$

$$\text{for the function} \qquad g(u) = \frac{G_T^{\mathbf{1}}(v,\, u)}{f(u)}.$$

Since the linear set $\mathbf{H}_T(f) \cap \mathbf{H}_T(\mathbf{1})$ is dense in $\mathbf{H}_T(f)$, we have

$$G_T^f(v,\,\cdot) = \mathbf{P}_T(f)\, g, \quad \text{where} \quad \mathbf{P}_T(f)$$

is the orthoprojector in L_f^2 onto $\mathbf{H}_T(f)$. Thus, we conclude that

$$G_T^f(v,\, v) = ||G_T^f(v,\,\cdot)||_f^2 \leq ||\mathbf{P}_T(f) g||_f^2 = \frac{T}{\pi} \int_{-\infty}^{\infty} \Phi_T(v,u) \cdot \frac{1}{f(u)}\, du. \tag{25}$$

Now we apply (22) to the function $h(u) = \frac{1}{\sqrt{T\pi}} \frac{\sin T(v-u)}{(v-u)}$, which belongs to $\mathbf{H}_T(f)$, and, therefore,

$$\frac{T}{\pi} = |h(v)|^2 \leq G_T^f(v,\, v) \cdot ||h(\cdot)||_f^2 = G_T^f(v,\, v) \cdot \int_{-\infty}^{\infty} \Phi_T(v,u) \cdot f(u)\, du. \tag{26}$$

Relation (24) follows from (25) and (26).

We put

$$\mu_f(v) = \sup_{T \geq 0} \int_{-\infty}^{\infty} \Phi_T(v,u) f(u)\, du \cdot \int_{\infty}^{\infty} \Phi_T(v,u) \frac{1}{f(u)}\, du. \tag{27}$$

It is clear that $\mu_f(v) \geq 1$. Proposition 4 implies the following proposition.

Proposition 5. *Assume that $f, \frac{1}{f} \in L^2_{\frac{1}{1+u^2}}$. Then*

$$\frac{T}{\pi \mu_f(v)} \cdot \int_{-\infty}^{\infty} \Phi_T(v,u) \frac{1}{f(u)}\, du \leq G_T^f(v,\, v) \leq \frac{T}{\pi} \cdot \int_{-\infty}^{\infty} \Phi_T(v,u) \frac{1}{f(u)}\, du \tag{28}$$

and

$$\frac{T}{\pi} \cdot \Big(\int_{-\infty}^{\infty} \Phi_T(v,u) f(u)\, du \Big)^{-1} \leq G_T^f(v,\, v) \leq \frac{T \mu_f(v)}{\pi} \cdot \Big(\int_{-\infty}^{\infty} \Phi_T(v,u) f(u)\, du \Big)^{-1}. \tag{29}$$

Proposition 6. *Assume that $f, \frac{1}{f} \in L^2_{\frac{1}{1+u^2}}$. Then*

$$\frac{T}{\pi \mu_f(v)} \leq G_T^f(v,\, v) \cdot G_t^{\frac{1}{f}}(v,\, v) \leq \frac{T \mu_f(v)}{\pi}. \tag{30}$$

10. The estimation problem for a signal whose Fourier transform is a finite measure

Now we consider the case where the unknown signal $\mathrm{s}(t)$ may be represented in the form

$$\mathrm{s}(t) = \sum_{j=1}^{m} \theta_j \cdot \exp\{itu_j\}, \tag{31}$$

where $\Lambda = \{u_j\,(j = 1, 2, ..., m)\}$. We assume that Λ is a known finite subset of the real line. Denote by $\mathbf{L}$ the linear set of all functions $s(t)$ that can be represented in the form (31) and put

$$\mathbf{L}_T = \mathbf{1}_{[-T,T]}(t) \cdot \mathbf{L}.$$

We assume that $\mathbf{H}$ is an L^2 space on the interval $[-T, T]$ and

$$s[\varphi] = (\varphi, \mathrm{s}).$$

Assume that we observe the random variables

$$y[\varphi] = s[\varphi] + x[\varphi], \qquad \varphi \in D_T.$$

Theorem. *Assume that the spectral density f of the process x satisfies the condition*

$$\lim_{T\to\infty} \int_{-\infty}^{\infty} \Phi_T(v,u) f(u)\,du \cdot \int_{\infty}^{\infty} \Phi_T(v,u) \frac{1}{f(u)}\,du = 1 \quad \textit{for} \quad v \in \Lambda.$$

Then

$$\frac{\sigma_T^2(\mathrm{s}^*,\ \mathrm{s})}{\sigma_T^2(\mathrm{s}^\star,\ \mathrm{s})} \to 1 \quad \textit{as} \quad T \to \infty.$$

Proof. It is easy to note that the theorem follows from Propositions 3–5 (for details, see [4]).

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