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ON AN APPROACH TO THE APPROXIMATION OF A SOLUTION OF PROBLEMS FOR THE HEAT-CONDUCTION EQUATION WITHOUT BOUNDARY-VALUE OR INITIAL CONDITIONS

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We propose an approach to the approximation of solutions of problems for the heat-conduction equation without initial or boundary-value conditions. The solution is given as a sum of odd and even functions, and this allows one to reconstruct conditions missed in the initial setting of the problem. The method is illustrated with test examples. Bibliography: 1 title.

1. Setting of the problem

The process of heat conduction along a closed bar is a well-known example of a problem for the heat-conduction equation without boundary-value conditions. Similar problems also arise in the numerical integration of multidimensional heat equations in cylindrical or spherical coordinates if economical difference schemes (the locally one-dimensional Samarskii scheme, Peaceman–Raeckford scheme, etc.) are used. At the same time, in conducting thermal physics studies, there appears a need to solve problems for heat conduction equations without initial conditions. Most often such problems are related to determining the temperature field in a solid due to certain external factors being periodic in time, e.g., a periodic heat source that works for an extended period of time. It does not matter what the initial temperature is, the heat-conduction process "forgets" it as time passes, which means that the periodic temperature field is formed only by the boundary conditions and the energy sources. In this article, we give a common approach to both problems to reconstruct the conditions missed in the initial setting of the problem.

2. Problems without initial conditions

In a domain Ω with a boundary Γ, we consider a linear parabolic-type equation:

$$\frac{\partial u}{\partial t} = Lu, \qquad \bar{x} \in \Omega; \qquad Lu = \operatorname{div}(k \operatorname{grad} u) - qu + f, \qquad t > -\infty, \tag{1}$$

where $\bar{x} = \{ x_i, i = 1, \dots, n \}$ are spatial coordinates, t is time; $k = k(\bar{x}, t)$, $q = q(\bar{x}, t)$, and $f = f(\bar{x}, t)$. On the boundary Γ, the function $u(\bar{x}, t)$ satisfies linear boundary-value conditions of the first kind:

$$u|_\Gamma = \varphi(s, t), \qquad t > -\infty, \ s \in \Gamma, \tag{2}$$

(the second- and third-type boundary-value problems can be considered in a similar way).

We state the problem as follows: for $t > 0$, find a function $u(\bar{x}, t)$ that satisfies Eq. (1) and boundary-value conditions (2). We assume that the data of the problem (the coefficients in Eq. (1) and the function $\varphi(s,t)$) are defined in certain classes of functions for which the existence and uniqueness of a solution in the classical sense holds. It is worthwhile to make a remark coming from physics: one can assume that certain initial conditions for Eq. (1) are given at $t = -\infty$ and, after an infinite period of time, $-\infty < t < 0$, they cease to be important for the current heat state at time $t > 0$.

We represent the solution of the problem as a sum of functions $u^+(\bar{x}, t)$ and $u^-(\bar{x}, t)$, which are even and odd with respect to the argument t, $u(\bar{x}, t) = u^+(\bar{x}, t) + u^-(\bar{x}, t)$; we also represent the operator L as a sum of operators, $L = L^+ + L^-$, where L^+ and L^- are operators such that for any even and odd functions $g^+(x,t)$ and $g^-(x,t)$, respectively, we have $L^+ g^+$ and $L^- g^-$ are even functions of t and $L^+ g^-$

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and L^-g^+ are odd. This can be done if the coefficients k, q, and f of the operator L are written as $z(\bar{x},t) = z^+(\bar{x},t) + z^-(\bar{x},t)$, where $z(\bar{x},t)$ is one of the functions $k = k(\bar{x},t)$, $q = q(\bar{x},t)$, and $f = f(\bar{x},t)$. Consider an auxiliary system

$$\frac{\partial u^+}{\partial t} = L^+u^- + L^-u^+, \qquad \frac{\partial u^-}{\partial t} = L^+u^+ + L^-u^-, \qquad \bar{x} \in \Omega, \tag{3}$$

with boundary-value conditions

$$u^+(\bar{x},t)\big|_\Gamma = \varphi^+(s,t), \qquad u^-(\bar{x},t)\big|_\Gamma = \varphi^-(s,t), \tag{4}$$

where $\varphi^+(s,t) + \varphi^-(s,t) = \varphi(s,t)$.

Let the functions $u^+(\bar{x},t)$ and $u^-(\bar{x},t)$ satisfy Eq. (3) and boundary-value conditions (4). Then the function $u(\bar{x},t)$ satisfies Eq. (1) and boundary-value conditions (2), and it is a solution of the problem for $t > 0$. Consider the time grid $\omega_\tau = \{\, t_k = k\tau, k = 0,1,2,\dots\}$, and introduce the following notation: $y_k(\bar{x})$ are values of the function y on the grid at the moment $t = t_k$; $y_{\bar{t}}$ is the time difference derivative of the function $y(\bar{x})$. We use the Rote scheme for system (3):

$$y^+_{\bar{t}} = L^+y^- + L^-y^+, \qquad y^-_{\bar{t}} = L^+y^+ + L^-y^-, \qquad \bar{x} \in \Omega, \tag{5}$$

where $y^+(\bar{x}) = y^+_k(\bar{x})$ and $y^-(\bar{x}) = y^-_k(\bar{x})$ are discrete analogues of the functions $u^+(\bar{x},t)$ and $u^-(\bar{x},t)$ at the moment $t = t_k$. Further, we use the following properties of even and odd functions:

$$\left.\frac{\partial u^+}{\partial t}\right|_{t=0} = 0; \qquad u^-(\bar{x},0) = 0.$$

Then $y^-_0 = 0$ and $y^+_1 = y^+_0$ up to $O(\tau^2)$. Let us use these approximation properties in (5) for $k = 0$. We get the following system of two equations:

$$L^+y^-_1 + L^-y^+_1 = 0, \qquad y^-_1 = \tau\left(L^+y^+_1 + L^-y^-_1\right). \tag{6}$$

Solving system (6) with the boundary-value conditions $y^+_1\big|_\Gamma = \varphi^+(s,\tau)$ and $y^-_1\big|_\Gamma = \varphi^-(s,\tau)$, we find the grid functions y^+_1 and y^-_1. In this way we reconstruct the initial conditions for a solution of system (1) for $t > \tau$, since $u(\bar{x},0) \approxeq y^+_0 + y^-_0 = y^+_1$.

System (6) is of elliptic type and can be solved by using the grid method. The Rote method used above suggests that one apply the implicit difference scheme for the integration of problem (1), (2). From this point of view, the calculation consumption for reconstructing the initial conditions by solving system (6) using the grid method is not burdensome, since the number of operations is equivalent to one step in solving Eq. (1) using the implicit difference scheme.

To illustrate the method discussed, let us give several test examples.

Example 1. Find a solution of the equation $u_t = 0.5u_{xx}$ for $0 \le x \le 1$ and $t > 0$ under the boundary-value conditions $u(0,t) = t$ and $u(1,t) = 1 + t$.

According to the above method, we rewrite system (3) and boundary-value conditions (4) as

$$\begin{gathered}\frac{\partial u^+}{\partial t} = \frac{\partial^2 u^-}{\partial x^2}; \qquad \frac{\partial u^-}{\partial t} = \frac{\partial^2 u^+}{\partial x^2}; \\ u^+(0,t) = 0, \qquad u^+(1,t) = 1; \qquad u^-(0,t) = t, \qquad u^-(1,t) = t.\end{gathered}$$

To reconstruct the initial conditions in this example, we can avoid the approximation of the operator on the spatial grid. Thus, in place of (5), similarly to the Rote method, we write, instead of (5), the system of difference equations in the following form:

$$\frac{y^+_1 - y^+_0}{\tau} = \frac{1}{2}\frac{\partial^2 u^-}{\partial x^2}; \qquad \frac{y^-_1 - y^-_0}{\tau} = \frac{1}{2}\frac{\partial^2 u^+}{\partial x^2}; \qquad 0 < x < 1,$$

with additional conditions corresponding to the symmetry and anti-symmetry of the functions $u^+(x,t)$ and $u^-(x,t)$, respectively,

$$y_1^+(x) = y_0^+(x); \qquad y_0^-(x) = 0.$$

Then $y_{\bar{x}x}^- = 0$ and $y_1^-(x) = C_1 x + C_2$. The constants C_1 and C_2 are determined from the boundary conditions for the function $u^-(x,t)$ at $t = \tau$: $y_1^-(0) = \tau$, $y_1^-(1) = \tau$; hence $y_1^-(x) = \tau$ and the equation for the function y_1^+ can be written as $y_{\bar{x}x,1}^+ = 2$. Integrating this equation with the boundary-value conditions $y_1^+(0) = 0$ and $y_1^+(1) = 1$, we find the solution $y_1^+(x) = x^2$. Thus, we reconstruct the initial condition for the function $u(x,t)$ in the form $y(x,0) = y_0^+ + y_0^- = x^2$.

To estimate the accuracy of the obtained solution, define at $t = -100$ an initial condition by

$$u(-100, x) = 0, \qquad 0 \le x \le 1,$$

and solve the problem under consideration by using the finite-difference method on the time interval $-100 \le t \le 0$ by applying the implicit three-node scheme. Since the Fourier number Fo that corresponds to the time interval equal to 100, is much greater than one (Fo = 200), we believe that at $t = -100$, the solution of the problem does not depend on the initial condition given at $t = -100$. In numerical calculations, we used a grid with 20 spatial nodes and the step $\tau = 1$ for the time variable. The numerical experiments showed that the relative error between the initial conditions determined by the above method and the function $y(x,0) = x^2$ is $1.4517 \cdot 10^{-8}$ for all nodes of the spatial grid.

Example 2. Find a solution of the equation $u_t = u_{xx}$ on the interval $(0,1)$ for $t > 0$ with the following boundary conditions: $u(0,t) = 0$, $u(1,t) = A \sin \alpha t$.

According to the above methods, the function that recovers the initial conditions for this example is

$$y(x,0) = (x^3 - x)\frac{A \sin \alpha\tau}{\tau}. \tag{7}$$

To estimate the accuracy of this solution, similarly to the first example, the finite-difference method was used on the time interval $-200 < t < 0$ with $A = 1000$ and the initial condition $u(x,-200) = 0$, $0 \le t \le 1$. The obtained solution was compared with (7). The following grid parameters were chosen: $\tau = 0.0065$ and $h = 0.05$. Table 1 shows values of relative errors $z(x) = |(y(x) - u(x))/y(x)|$ at interior nodes of the grid for different values of the parameter α, which corresponds to the frequency of the periodic temperature change in the boundary-value condition at $x = 1$.

TABLE 1. Relative Reconstruction Error for the Initial Conditions

x	$a = 0,1$	$a = 0,2$	$a = 0,4$	$a = 0,8$
0,1	$1.9552 \cdot 10^{-4}$	$7.8216 \cdot 10^{-4}$	$3.1294 \cdot 10^{-3}$	$1.2531 \cdot 10^{-2}$
0,2	$1.9247 \cdot 10^{-4}$	$7.6987 \cdot 10^{-4}$	$3.0802 \cdot 10^{-3}$	$1.2332 \cdot 10^{-2}$
0,3	$1.8739 \cdot 10^{-4}$	$7.4959 \cdot 10^{-4}$	$2.9989 \cdot 10^{-3}$	$1.2004 \cdot 10^{-2}$
0,4	$1.8039 \cdot 10^{-4}$	$7.2159 \cdot 10^{-4}$	$2.8867 \cdot 10^{-3}$	$1.1552 \cdot 10^{-2}$
0,5	$1.7157 \cdot 10^{-4}$	$6.8629 \cdot 10^{-4}$	$2.7452 \cdot 10^{-3}$	$1.0982 \cdot 10^{-2}$
0,6	$1.6105 \cdot 10^{-4}$	$6.4419 \cdot 10^{-4}$	$2.5765 \cdot 10^{-3}$	$1.0302 \cdot 10^{-2}$
0,7	$1.4899 \cdot 10^{-4}$	$5.9593 \cdot 10^{-4}$	$2.3832 \cdot 10^{-3}$	$0.9524 \cdot 10^{-2}$
0,8	$1.3557 \cdot 10^{-4}$	$5.4225 \cdot 10^{-4}$	$2.1682 \cdot 10^{-3}$	$0.8660 \cdot 10^{-2}$
0,9	$1.2102 \cdot 10^{-4}$	$4.84 \cdot 10^{-4}$	$1.9350 \cdot 10^{-3}$	$0.7724 \cdot 10^{-2}$

This numerical experiment shows that, similarly to Example 1, a sufficiently high reconstruction accuracy is reached for the initial condition, although, if the oscillation frequency of the boundary temperature increases, the accuracy of the method decreases.

The approach suggested for the reconstruction of initial conditions can easily be extended to the heat-conduction equation with an operator Lu of a more general form, including the equation for conduction-convection energy transport.

3. The problem without boundary-value conditions

For the equation

$$c\frac{\partial u}{\partial t} = Lu, \qquad Lu = \frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial t}\right) - qu + f, \qquad t > 0, \tag{8}$$

where $c = c(x,t)$, $k = k(x,t)$, $q = q(x,t)$, and $f = f(x,t)$ on the segment $x \in [0,1]$, find a function $u(x,t)$ satisfying the initial condition

$$u(x,0) = \varphi_0(x), \qquad x \in [0,1], \tag{9}$$

and the following periodicity condition:

$$u(x,t) = u(x+1,t), \quad \text{for all } x \in [0,1],\ t > 0. \tag{10}$$

Suppose problem (8)–(10) has a unique classical solution. We look for the function $u(x,t)$ in the form of a sum of even and odd functions with respect to the argument x: $u(\bar{x},t) = u^+(\bar{x},t) + u^+(\bar{x},t)$.

Consider the system of equations

$$\begin{aligned} c^+\frac{\partial u^+}{\partial t} + c^-\frac{\partial u^-}{\partial t} &= L^+u^+ + L^-u^-, \\ c^-\frac{\partial u^+}{\partial t} + c^+\frac{\partial u^-}{\partial t} &= L^+u^- + L^-u^+, \end{aligned} \tag{11}$$

where the operators L^+ and L^- are defined in such a way that for all $g^+(x,t)$ and $g^-(x,t)$ (the even and odd functions, respectively), L^+g^+ and L^-g^- are even and L^+g^- and L^-g^+ are odd functions of the argument x:

$$L^+g = \frac{\partial}{\partial x}\left(k^+\frac{\partial g}{\partial x}\right) - q^+g + f^+, \qquad L^-g = \frac{\partial}{\partial x}\left(k^-\frac{\partial u}{\partial x}\right) - q^-g + f^-.$$

Here c, k, q, and f, with indices (+) and (−), are even (+) and odd (−) components of the functions $c(x,t)$, $k(x,t)$, $q(x,t)$, and $f(x,t)$.

To Eq. (11), we add the initial conditions

$$u^+(x,0) = \varphi_0^+(x), \qquad u^-(x,0) = \varphi_0^-(x), \qquad 0 \le x \le 1, \tag{12}$$

where $\varphi_0^+(x) + \varphi_0^-(x) = \varphi_0(x)$. Let us determine the boundary-value conditions for the functions $u^+(x,t)$ and $u^-(x,t)$ using the properties of even and odd functions as well as condition (10) that the solution is periodic:

$$\left.\frac{\partial u^+}{\partial x}\right|_{x=0} = \left.\frac{\partial u^+}{\partial x}\right|_{x=1} = 0; \qquad u^-(0,t) = u^-(1,t) = 0. \tag{13}$$

Hence initial conditions (12) and boundary-value conditions (13) are completely defined for system (11), so that a solution of this problem can be obtained in both analytical and numerical ways, for example, using the finite-element method in numerical calculations. If the coefficients of an equation are varying, then the above approach for solving problem without boundary-value conditions is not an alternative to the cyclic sweep method [1], since it requires the integration of a system of two heat-conduction equations. However, it can be useful to determine the boundary-value conditions at the stage of setting up the problem to find then a solution of the problem with the help of another numerical or analytical method.

References

1. A. A. Samarskii, *A Theory of Difference Schemes* [in Russian], Nauka, Moscow (1977).