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Characterizing the complex hyperbolic space by Kähler homogeneous structures

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1. *Introduction*

The Kähler case of Riemannian homogeneous structures [**3**, **15**, **18**] has been studied in [**1**, **2**, **6**, **7**, **13**, **16**], among other papers. Abbena and Garbiero [**1**] gave a classification of Kähler homogeneous structures, which has four primitive classes $\mathscr{K}_1, \dots, \mathscr{K}_4$ (see [**6**, theorem 5·1] for another proof and Section 2 below for the result). The purpose of the present paper is to prove the following result:

Theorem 1·1. *A simply connected irreducible homogeneous Kähler manifold admits a nonvanishing Kähler homogeneous structure in Abbena–Garbiero's class* $\mathscr{K}_2 \oplus \mathscr{K}_4$ *if and only if it is the complex hyperbolic space equipped with the Bergman metric of negative constant holomorphic sectional curvature.*

We thus have a situation similar to the Riemannian case, where a connected, simply connected and complete Riemannian manifold admits a non-vanishing homogeneous structure of first class if and only if it is isometric to the hyperbolic space ([**18**, theorem 5·2]), and similarly we obtain a vector field ξ (see (4·4) in Section 4) which is the complex analogue of the vector field in the Riemannian case satisfying $\nabla_X \xi = g(X,\xi)\,\xi - g(\xi,\xi)\,X$, ([**8**, **18**]), for Riemannian homogeneous structures of first class and models of negative constant (ordinary) sectional curvature. Moreover, this suggests the possibility of a quaternionic analogue to Theorem 1·1 for models of negative constant quaternionic sectional curvature and also of a Cayleyan analogue.

On the other hand, similarly to the Riemannian case [**18**, p. 55], one has a solvable group acting simply transitively on the relevant domain (see Remark 4·2(i)), thus explaining why positive (holomorphic sectional) curvature is not detected.

2. *Preliminaries and notations*

As is well known, Ambrose and Singer proved in [**3**] that *a connected, simply connected and complete Riemannian manifold* (M, g) *is homogeneous if and only if it admits a Riemannian homogeneous structure*, i.e. a $(1,2)$ tensor field S satisfying

$$\tilde{\nabla} g = 0, \quad \tilde{\nabla} R = 0, \quad \tilde{\nabla} S = 0, \tag{2·1}$$

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where $\tilde{\nabla} = \nabla - S$, ∇ denotes the Levi–Civita connection and R the curvature tensor of ∇. We write $S_{XYZ} = g(S_X Y, Z)$. Then, from $\nabla g = 0$ it follows that the condition $\tilde{\nabla} g = 0$ is equivalent to $S_{XZY} = -S_{XYZ}$. We set

$$R_{XY} Z = \nabla_{[X,Y]} Z - \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z, \quad R_{XYZW} = g(R_{XY} Z, W),$$
$$R_{XY}(Z, W) = R_{XYZW}, \quad X, Y, Z, W \in \mathscr{X}(M).$$

In the sequel $X, Y, Z, W, U, \xi, \zeta$, will stand for vector fields on a C^∞ manifold M. We denote by F the Kähler form, defined by $F(X, Y) = g(X, JY)$, by r the Ricci tensor and by s the scalar curvature.

Sekigawa [**17**] proved that *a connected, simply connected and complete almost Hermitian manifold* (M, g, J) *is homogeneous if and only if it admits an almost Hermitian homogeneous structure*, i.e. a $(1,2)$ tensor field S satisfying the conditions (2·1) and $\tilde{\nabla} J = 0$.

We further suppose that (M, g, J) is Kähler, so $\nabla J = 0$. As $\tilde{\nabla} J = 0$, the condition $\nabla J = 0$ is equivalent to $S_{XJYJZ} = S_{XYZ}$. Hence, if S is an almost Hermitian homogeneous structure satisfying that invariance condition, then the manifold is Kähler ([**1**, theorem 5·8]). This property is equivalent to saying that S belongs to the vector space

$$\mathscr{S}(V)_+ = \{S \in \otimes^3 V^* : S_{XYZ} = -S_{XZY} = S_{XJYJZ}\}.$$

Definition 2·1. A *Kähler homogeneous structure* on a Kähler manifold (M, g, J) is an almost Hermitian homogeneous structure S on M such that $S_x \in \mathscr{S}(T_x M)_+$ for all $x \in M$.

The classification of Kähler homogeneous structures was obtained by Abbena and Garbiero in [**1**, theorem 2·1]. We recall their result here: let V be a $2n$-dimensional real vector space (which is the model for the tangent space at any point of a manifold equipped with a Kähler homogeneous structure) endowed with a complex structure J and a Hermitian inner product $\langle , \rangle$, that is, $J^2 = -I$, $\langle JX, JY \rangle = \langle X, Y \rangle$, $X, Y \in V$, where I denotes the identity isomorphism of V.

If $\dim V \geqslant 6$, $\mathscr{S}(V)_+$ *decomposes into the direct sum of the following subspaces invariant and irreducible under the action of the group* $U(n)$:

$$\mathscr{K}_1 = \{S \in \mathscr{S}(V)_+ : S_{XYZ} = \tfrac{1}{2}(S_{YZX} + S_{ZXY} + S_{JYJZX} + S_{JZXJY}),\ c_{12}(S) = 0\};$$
$$\mathscr{K}_2 = \{S \in \mathscr{S}(V)_+ : S_{XYZ} = \langle X, Y \rangle \theta_1(Z) - \langle X, Z \rangle \theta_1(Y) + \langle X, JY \rangle \theta_1(JZ)$$
$$- \langle X, JZ \rangle \theta_1(JY) - 2\langle JY, Z \rangle \theta_1(JX),\ \theta_1 \in V^*\};$$
$$\mathscr{K}_3 = \{S \in \mathscr{S}(V)_+ : S_{XYZ} = -\tfrac{1}{2}(S_{YZX} + S_{ZXY} + S_{JYJZX} + S_{JZXJY}),\ c_{12}(S) = 0\};$$
$$\mathscr{K}_4 = \{S \in \mathscr{S}(V)_+ : S_{XYZ} = \langle X, Y \rangle \theta_2(Z) - \langle X, Z \rangle \theta_2(Y) + \langle X, JY \rangle \theta_2(JZ)$$
$$- \langle X, JZ \rangle \theta_2(JY) + 2\langle JY, Z \rangle \theta_2(JX),\ \theta_2 \in V^*\};$$

$X, Y, Z \in V$, where c_{12} is defined by $c_{12}(S)(X) = \Sigma_{i=1}^{2n} S_{e_i e_i, X}$, for any $X \in V$, $\{e_1, \ldots, e_{2n}\}$ denotes an arbitrary orthonormal basis of V and

$$\theta_1(X) = \frac{1}{2(n-1)} c_{12}(S)(X), \quad \theta_2(X) = \frac{1}{2(n+1)} c_{12}(S)(X), \quad X \in V.$$

If $\dim V = 4$, then $\mathscr{S}(V)_+ = \mathscr{K}_2 \oplus \mathscr{K}_3 \oplus \mathscr{K}_4$. If $\dim V = 2$, then $\mathscr{S}(V)_+ = \mathscr{K}_4$.

3. *The class* $\mathscr{K}_2 \oplus \mathscr{K}_4$

We can write the class $\mathscr{K}_2 \oplus \mathscr{K}_4$ as

$$\begin{aligned}\mathscr{K}_2 \oplus \mathscr{K}_4 = \{S \in \mathscr{S}(V)_+ : S_{XYZ} &= \langle X, Y\rangle (\theta_1+\theta_2)(Z) \\ &-\langle X, Z\rangle (\theta_1+\theta_2)(Y) + \langle X, JY\rangle (\theta_1+\theta_2)(JZ) \\ &-\langle X, JZ\rangle (\theta_1+\theta_2)(JY) + 2\langle Y, JZ\rangle (\theta_1-\theta_2)(JX)\}. \end{aligned} \qquad (3{\cdot}1)$$

Remark 3·1. We recall that $\mathscr{K}_2 \oplus \mathscr{K}_4$ is the sum of the isotypic components $\mathscr{K}_2, \mathscr{K}_4$ of the smallest dimension in the decomposition of $\mathscr{S}(V)_+$ above. In fact, one has $\mathscr{K}_2 \cong \mathscr{K}_4 \cong V \cong V^*$ and the respective dimensions of $\mathscr{K}_1, \ldots, \mathscr{K}_4$ are (see [**1**, p. 382]) $n(n+1)(n-2), 2n, n(n-1)(n+2), 2n$. It is thus reasonable that $\mathscr{K}_2 \oplus \mathscr{K}_4$ corresponds to spaces of (negative) constant holomorphic sectional curvature, which are scarce in all homogeneous Kähler spaces.

LEMMA 3·2. *A simply connected complete irreducible Kähler manifold* $(M = G/H, g, J)$ *of real dimension* $2n \geqslant 4$ *which admits a nonvanishing Kähler homogeneous structure* $S \in \mathscr{K}_2 \oplus \mathscr{K}_4$, *is Einstein.*

Proof. Let ξ be the vector field dual to the 1-form $\theta = \theta_1+\theta_2$ and ζ the vector field dual to the 1-form $\theta_1-\theta_2$, both with respect to the metric. From (3·1) we have

$$S_X Y = g(X, Y)\,\xi - g(Y, \xi)\,X - g(X, JY)\,J\xi + g(JY, \xi)\,JX - 2g(JX, \zeta)\,JY. \qquad (3{\cdot}2)$$

Taking the $\tilde{\nabla}_Z$ derivative of this formula, applying Ambrose–Singer's equations (2·1) and Sekigawa's equation $\tilde{\nabla} J = 0$ and contracting with $g(W, \cdot)$, we obtain

$$\begin{aligned} g(X, Y)\,g(\tilde{\nabla}_Z \xi, W) - g(\tilde{\nabla}_Z \xi, Y)\,g(X, W) - g(X, JY)\,g(\tilde{\nabla}_Z J\xi, W) \\ + g(\tilde{\nabla}_Z \xi, JY)\,g(JX, W) - 2g(JX, \tilde{\nabla}_Z \zeta)\,g(JY, W) = 0. \end{aligned}$$

Since $\dim M \geqslant 4$, we deduce

$$\tilde{\nabla}\xi = 0, \quad \tilde{\nabla}\zeta = 0. \qquad (3{\cdot}3)$$

Thus, from $\tilde{\nabla} g = 0$ we have $g(\xi, \xi) = \text{const}, g(\zeta, \zeta) = \text{const}$. If $\xi = 0$, $\zeta \neq 0$, we also have

$$R_{XY}\,\zeta = 4(g(X, J\zeta)\,g(Y, \zeta) - g(X, \zeta)\,g(Y, J\zeta))\,J\zeta. \qquad (3{\cdot}4)$$

The second Ambrose–Singer condition in (2·1) can be written as

$$(\nabla_X R)_{YZWU} = -R_{S_X YZWU} - R_{YS_X ZWU} - R_{YZS_X WU} - R_{YZWS_X U}. \qquad (3{\cdot}5)$$

Suppose $\xi \neq 0$. Applying Bianchi's second identity to (3·5) and then substituting (3·2), we obtain

$$\begin{aligned}\mathfrak{S}_{XYZ}\{2g(X, \xi)\,R_{YZWU} + g(X, W)\,R_{YZ\xi U} + g(X, U)\,R_{YZW\xi} - 2g(X, JY)\,R_{J\xi ZWU} \\ - g(X, JW)\,R_{YZJ\xi U} - g(X, JU)\,R_{YZWJ\xi}\} = 0. \end{aligned} \qquad (3{\cdot}6)$$

Contracting (3·6) with respect to X and W and applying Bianchi's first identity, we deduce

$$\begin{aligned}(2n+2)\,R_{ZY\xi U} = &-2g(Y, \xi)\,r(Z, U) + 2g(Z, \xi)\,r(Y, U) \\ &+ 2g(Y, JZ)\,r(J\xi, U) - g(Y, U)\,r(Z, \xi) + g(Y, JU)\,r(Z, J\xi) \\ &+ g(Z, U)\,r(Y, \xi) - g(Z, JU)\,r(Y, J\xi). \end{aligned} \qquad (3{\cdot}7)$$

Contracting (3·7) with regard to Y and U, we obtain $r(Z,\xi) = (s/2n)\,g(Z,\xi)$. Putting $a = 1/(2n+2)$, $b = s/2n$, we can write (3·7) as

$$\frac{1}{a}R_{\xi U} = 2\theta \wedge r(U) + 2b\theta(JU)\,F + bU^\flat \wedge \theta + b(JU)^\flat \wedge (\theta \circ J). \tag{3·8}$$

On the other hand, from Bianchi's first identity one has $R_{WU}(J\xi,\cdot\,) = R_{\xi JW}(U,\cdot\,) - R_{\xi JU}(W,\cdot\,)$. Thus we can write (3·6) as

$$\begin{aligned} 2\theta \wedge R_{WU} + W^\flat \wedge R_{\xi U} - U^\flat \wedge R_{\xi W} + 2F \wedge (R_{\xi JU}(W,\cdot\,) - R_{\xi JW}(U,\cdot\,)) \\ + (JW)^\flat \wedge R_{\xi JU} - (JU)^\flat \wedge R_{\xi JW} = 0. \end{aligned} \tag{3·9}$$

Substituting (3·8) in (3·9) and denoting by $\Xi(U)$ the right hand side of (3·8), we deduce

$$\begin{aligned} \frac{2}{\alpha}\theta \wedge R_{WU} + W^\flat \wedge \Xi(U) - U^\flat \wedge \Xi(W) + 2F \wedge (i_W(\Xi(JU)) - i_U(\Xi(JW))) \\ + (JW)^\flat \wedge \Xi(JU) - (JU)^\flat \wedge \Xi(JW) = 0. \end{aligned} \tag{3·10}$$

Taking $W = \xi$ in this formula we obtain

$$(2g(\xi,\xi)\,F + \theta \wedge (\theta \circ J)) \wedge (r(JU) - b(JU)^\flat) = 0.$$

Contracting this formula with ξ, in account of $r(\xi,\cdot\,) = bg(\xi,\cdot\,)$ we deduce

$$g(\xi,\xi)\,(\theta \circ J) \wedge (r(JU) - b(JU)^\flat) = 0.$$

Contracting with $J\xi$ we have $g(\xi,\xi)^2\,(r(JU) - b(JU)^\flat) = 0$. Since $\xi \neq 0$ we conclude that $r = bg$, that is (M, g, J) is Einstein.

Now suppose $\xi = 0, \zeta \neq 0$. It is immediate to see that $[\zeta, J\zeta] = -2g(\zeta,\zeta)\,J\zeta$, so that M has a 2-dimensional involutive distribution, which is also parallel as follows from (3·2) and $\tilde{\nabla}\zeta = 0$ in (3·3). Thus ([**5**, proposition 10·21]) the holonomy representation $\mathrm{Hol}\,(g)$ leaves invariant a subspace of dimension 2. So, by the De Rham theorem ([**5**, theorem 10·41]), we conclude that M is holomorphically isometric to a Kähler product, but then M would not be irreducible.

Theorem 3·3. *A simply connected irreducible homogeneous Kähler manifold of real dimension $2n \geqslant 4$ admitting a nonvanishing Kähler homogeneous structure $S \in \mathscr{K}_2 \oplus \mathscr{K}_4$ is holomorphically isometric to a bounded symmetric domain of negative constant holomorphic sectional curvature.*

Proof. Suppose $\xi \neq 0$. Since by the Lemma the manifold is Einstein (3·10) gives us

$$\theta \wedge \left(\frac{1}{ab}R_{WU} + U^\flat \wedge W^\flat + (JU)^\flat \wedge (JW)^\flat - 2F(W,U)\,F\right) = 0. \tag{3·11}$$

Contracting this formula with ξ we deduce, as $k = g(\xi,\xi)$,

$$\begin{aligned} \frac{k}{ab}R_{WU} - k(W^\flat \wedge U^\flat + (JW)^\flat \wedge (JU)^\flat + 2F(W,U)\,F) = \theta \wedge \Bigg(\frac{1}{ab}R_{WU}(\xi,\cdot\,) \\ + \theta(U)\,W^\flat - \theta(W)\,U^\flat + \theta(JU)\,(JW)^\flat - \theta(JW)\,(JU)^\flat - 2F(W,U)\,(\theta \circ J)\Bigg). \end{aligned} \tag{3·12}$$

On the other hand, from Bianchi's first identity we deduce $R_{WU}(\xi,\cdot\,) = R_{\xi U}(W,\cdot\,) - R_{\xi W}(U,\cdot\,)$. Substituting these two summands by their expression from (3·8) and then substituting the expression for $R_{WU}(\xi, \cdot)$ in (3·12) we obtain

$$\frac{1}{ab} R_{WU} = W^\flat \wedge U^\flat + (JW)^\flat \wedge (JU)^\flat + 2F(W, U) F,$$

from which

$$R_{YZWU} = \frac{s}{4n(n+1)} \{g(Y, W)\, g(Z, U) - g(Y, U)\, g(Z, W) + g(Y, JW)\, g(Z, JU) \\ - g(Y, JU)\, g(Z, JW) + 2g(Y, JZ)\, g(W, JU)\}.$$

That is, (M, g, J) is a model of constant holomorphic sectional curvature $c = s/n(n+1)$. To determine the sign of s, we prove that (for $\xi = 0$ or not) if (M, g, J) is a space of constant holomorphic sectional curvature c, then $\zeta = 0$ and $c = -g(\xi, \xi)$. In fact, comparing the expressions for $R_{XJX}\xi$ respectively obtained from substitution in the usual expression

$$R_{XY} Z = c\{g(X, Z)\, Y - g(Y, Z)\, X + g(JX, Z)\, JY - g(JY, Z)\, JX + 2g(X, JY)\, JZ\}$$

and from $\nabla_X \xi = S_X \xi$, since $n \geqslant 2$ we obtain the system of equations

$$g(X, \xi)\, g(JX, \zeta) - g(JX, \xi)\, g(X, \zeta) = 0, \tag{3·13}$$

$$cg(X, \xi) = -g(\xi, \xi)\, g(X, \xi - \zeta), \tag{3·14}$$

$$0 = (2g(\xi, \zeta) - g(\xi, \xi) - c)\, g(X, X) + g(X, \xi)\, g(X, \zeta) \\ + g(JX, \xi)\, g(JX, \zeta) - 2g(X, \zeta)^2 - 2g(JX, \zeta)^2. \tag{3·15}$$

From (3·14) we have $g(\xi, \xi)\, \zeta = (g(\xi, \xi) + c)\, \xi$ which, if it is true, implies (3·13). In (3·15) we can take X orthogonal to ζ and $J\zeta$, so $g(\xi, \xi) + c = 2g(\xi, \zeta)$, thus $g(\xi, \xi)\, (g(\xi, \xi) + c) = 2g(\xi, g(\xi, \xi)\, \zeta) = 2g(\xi, \xi)\, (g(\xi, \xi) + c)$, hence

$$g(\xi, \xi)\, (g(\xi, \xi) + c) = 0. \tag{3·16}$$

Suppose $g(\xi, \xi) = 0$. As $\xi = 0$, (3·15) simplifies to

$$cg(X, X) + 2g(X, \zeta)^2 + 2g(JX, \zeta)^2 = 0.$$

Since $\dim M \geqslant 4$, we can take X orthogonal to ζ and to $J\zeta$, so $cg(X, X) = 0$, that is, the manifold is flat. From (3·4) we deduce

$$g(X, \zeta)\, g(JY, \zeta) - g(Y, \zeta)\, g(JX, \zeta) = 0.$$

Taking $X = \zeta$ we have $g(\zeta, \zeta)\, \zeta = 0$, which implies $\zeta = 0$, but then S vanishes.

On the other hand, if $g(\xi, \xi) \neq 0$, then (3·16) gives $g(\xi, \xi) + c = 0$; hence $\zeta = 0$.

The case $\xi = 0$, $\zeta \neq 0$ has been already discarded in the previous lemma.

Remark 3·4. In the conditions of Lemma 3·2, for $\xi = 0$, $\zeta \neq 0$, substituting (3·2) in (3·5), as $\nabla J = 0$ we have

$$(\nabla_X R)_{YZWU} = 2g(JX, \zeta)\, (R_{JYZWU} + R_{YJZWU} + R_{YZJWU} + R_{YZWJU}) = 0.$$

Thus (M, g, J) is Hermitian symmetric. As we have seen, it is holomorphically isometric to a Kähler product, with a 2-dimensional factor, which is a space of negative constant curvature $-4g(\zeta, \zeta)$, as it follows from (3·4).

Now, there are ([**10**, pp. 518–520]) four non-flat non-compact Hermitian symmetric surfaces: $SU(1,1)/S(U(1)\times U(1))$, $SO^*(4)/U(2)$, $SO_0(2,1)/SO(2)\times 1$ and $Sp(1)/U(1)$. All of them are isomorphic and can be identified to the real hyperbolic space $SO_0(2,1)/SO(2)\times 1$. Taking the half-plane Poincaré model $H=\{z=x+iy\in\mathbb{C}:y>0\}$ for it, we have the metric and the complex structure (particular cases of (4·2) and (4·3) below)

$$g=-\frac{dx^2+dy^2}{cy^2},\quad J=\frac{\partial}{\partial y}\otimes dx-\frac{\partial}{\partial x}\otimes dy.$$

It is immediate that the vector field $\zeta=-(cy/2)\,\partial/\partial y$ (a particular case of (4·4) below) satisfies the equation $\nabla_X\zeta=2g(X,J\zeta)\,J\zeta$, ∇ being the Levi–Civita connection of g. We conclude that in the simply connected reducible case one can also consider spaces holomorphically isometric to products of the 2-dimensional Siegel domain model of the complex hyperbolic space, that is, the above half-plane Poincaré model, equipped with the Kähler homogeneous structure S given by $S_X Y=2g(X,J\zeta)\,JY$, and any other Hermitian symmetric space, endowed with its standard vanishing Kähler homogeneous structure as a Hermitian symmetric space.

4. *Proof of Theorem* 1·1

As for the converse, it suffices to give a non-vanishing Kähler homogeneous structure on any model of negative constant holomorphic sectional curvature. Thus, to finish the proof we explicitly give such a structure on the Siegel domain

$$D_+=\left\{(z=x+iy,u^1,\dots,u^n)\in\mathbb{C}^{n+1}:y-\sum_{k=1}^{n}u^k\,\bar{u}^k>0\right\},$$

equipped with the Kähler structure obtained by a convenient Cayley transform ([**14**, p. 5]) from that of the unit open ball in $\mathbb{C}^{n+1}$ ([**9**, p. 227] or [**12**, p. 169], changing the sign in the second summand of the numerator in the last expression of the metric). A calculation shows that

$$\begin{aligned} g_+=-\frac{1}{c(y-\sum u^k\bar{u}^k)^2}\{&dz\,d\bar{z}+4(y-\textstyle\sum u^k\bar{u}^k)\sum du^k\,d\bar{u}^k\\ &+2i(dz\textstyle\sum u^k d\bar{u}^k-d\bar{z}\sum\bar{u}^k du^k)+4(\sum\bar{u}^k du^k)(\sum u^k d\bar{u}^k)\}.\end{aligned}$$

(D_+,g_+,J_+) is a Kähler manifold, with

$$J_+=i\left(\frac{\partial}{\partial z}\otimes dz+\sum_{k=1}^{n}\frac{\partial}{\partial u^k}\otimes du^k-\frac{\partial}{\partial\bar{z}}\otimes d\bar{z}-\sum_{k=1}^{n}\frac{\partial}{\partial\bar{u}^k}\otimes d\bar{u}^k\right).$$

The Riemannian manifold (D_+,g_+) is homogeneous, hence complete. Since (D_+,g_+,J_+) is connected, simply connected and complete, it is a model of negative constant holomorphic sectional curvature. Furthermore, we seek for a homogeneous structure S of the type

$$S_X Y=g(X,Y)\,\xi-g(\xi,Y)\,X-g(X,JY)\,J\xi+g(\xi,JY)\,JX.$$

Since $\tilde{\nabla}\xi=0$, we must find a vector field ξ on D_+ satisfying

$$\nabla_X\xi=g(X,\xi)\,\xi-g(\xi,\xi)\,X-g(X,J\xi)\,J\xi. \tag{4·1}$$

Putting $u^k = v^k + iw^k$, the metric g_+ and the complex structure J_+ are written as

$$\begin{aligned} g_+ = -\frac{1}{c[y-\sum((v^k)^2+(w^k)^2)]^2}\{dx^2+dy^2+4[y-\sum_{j\neq k}((v^j)^2+(w^j)^2)] \\ \times\sum((dv^k)^2+(dw^k)^2)-4[dx\sum(w^k dv^k - v^k dw^k) - dy\sum(v^k dv^k + w^k dw^k)] \\ +8\sum_{k,l}[(v^k v^l + w^k w^l)(dv^k dv^l + dw^k dw^l) + (v^l w^k - v^k w^l)(dv^k dw^l - dw^k dv^l)]\}, \end{aligned} \quad (4{\cdot}2)$$

$$J_+ = \frac{\partial}{\partial y}\otimes dx - \frac{\partial}{\partial x}\otimes dy + \sum\left(\frac{\partial}{\partial w^k}\otimes dv^k - \frac{\partial}{\partial v^k}\otimes dw^k\right). \quad (4{\cdot}3)$$

A long but straightforward computation shows that the vector field

$$\xi = -\frac{c}{2}(y-\sum u^k \bar{u}^k)\frac{\partial}{\partial y} \quad (4{\cdot}4)$$

satisfies (4·1) for all X. Specifically, if X is given by

$$X = \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y} + \sum\left(\gamma_k\frac{\partial}{\partial v^k} + \delta_k\frac{\partial}{\partial w^k}\right),$$

we obtain

$$\begin{aligned} \nabla_X\xi = \frac{c}{4}\Bigg\{2\left[(\alpha+\sum(\delta_k v^k - \gamma_k w^k))\frac{\partial}{\partial x} + \sum(\gamma_k v^k + \delta_k w^k)\frac{\partial}{\partial y}\right] \\ + \sum\left(\gamma_k\frac{\partial}{\partial v^k} + \delta_k\frac{\partial}{\partial w^k}\right)\Bigg\}. \end{aligned}$$

Hence, (D_+, g_+, J_+) admits the Kähler homogeneous structure S given by

$$\begin{aligned} S_X Y = -\frac{c}{2}(y-\sum u^k\bar{u}^k)\Bigg\{g(X,Y)\frac{\partial}{\partial y} - g\left(\frac{\partial}{\partial y}, Y\right)X \\ + g(X, JY)\frac{\partial}{\partial x} + g\left(\frac{\partial}{\partial y}, JY\right)JX\Bigg\}. \end{aligned} \quad (4{\cdot}5)$$

According to Heintze's Theorem ([**11**, theorem 4]), a connected homogeneous Kähler manifold of negative curvature is holomorphically isometric to the complex hyperbolic space. Hence

Corollary 4·1. *A connected homogeneous Kähler manifold of real dimension $2n \geqslant 4$ and negative curvature admits a nonvanishing Kähler homogeneous structure $S \in \mathscr{K}_2 \oplus \mathscr{K}_4$.*

Remark 4·2. (i) The Siegel domain D_+ thus admits at least two Kähler homogeneous structures: $S = 0$, as D_+ is a realization of the noncompact Hermitian symmetric space $U(1,n)/U(1)\times U(n)$; and the non-vanishing structure S in (4·5), corresponding to the fact that the solvable group $\mathbb{C}H^n$ (see [**11**, p. 33]) acts simply transitively on D_+ (see [**11**, p. 32], [**5**, p. 181], [**4**, p. 92]) by holomorphic isometries. Hence, we can identify D_+ with $\mathbb{C}H^n$ and thus it is immediate to obtain the expression of the canonical connection on the reductive homogeneous space (D_+, g_+, J_+) identified to the solvable group $\mathbb{C}H^n$.

(ii) For $\dim M = 2$ only the class $\mathscr{K}_4$ remains. This case has been studied by Abbena and Garbiero [**1**, p. 391].

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