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MECHANICS

On the Structure of a Stable (in the Lyapunov Sense) Attractor

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An urgent problem in the Lyapunov stability theory [1–3] is finding criteria that make it possible to distinguish simple-attracting sets (simple attractors) from strange-attracting sets (strange attractors) [4–6]. There is a principal distinction in the behavior of dynamic systems with simple and strange attractors, because the former systems are regular, while the latter ones are random.

In this paper, it is shown that if the invariant set $A \subset R^n$ of a dynamic system φ: $R^n \longrightarrow R^n$ is compact, attracting, stable in the Lyapunov sense, and there exists a trajectory, whose closure is dense in A, then the set A is a stable torus and, hence, the dynamic system will be regular.

According to the assumption of Ruelle and Takens, all trajectories of strange attractors are unstable in the Lyapunov sense, and this was taken by these authors as a starting point for explaining the turbulence phenomenon. They proceed from the following turbulence definition: "… the motion of a liquid medium is turbulent if this motion is described by the integral curve of a vector field, which tends to an nonempty set A being not an equilibrium state or a closed orbit" [6]. Theorems proved by us elucidate the attractor structure in various systems in the case of the absence of the turbulence in the Ruelle–Takens sense.

Following [7, 8], we refer to the invariant set $M \subset R^n$ of the dynamic system φ: $R^n \longrightarrow R^n$ as stable in the Lyapunov sense as $t \longrightarrow +\infty$ provided that the condition

$$\forall \varepsilon > 0 \ \exists \delta > 0, \ \ d(m, x) < \delta \Rightarrow d(\varphi(t, m), \varphi(t, x)) < \varepsilon$$

is satisfied at each point m of this set for all $t \in R^+$, where $x \in M$ and d is the metric of the space R^n.

In other words, the stability of a set in the Lyapunov sense implies that all points of this set are stable in the Lyapunov sense. We note, that the equilibrium state, periodic trajectory, and almost-periodic trajectory with a compact closure are sets stable in the Lyapunov sense.

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We will denote positive and negative limiting sets of a point $x \in R^n$ as $\omega(x)$ and $\alpha(x)$, respectively. An open sphere of a radius δ with the center at the point $y \in R^n$ is denoted as $B_\delta(y)$.

In [8], the following statement on the uniform stability of the compact set $M \subset R^n$ is established.

Statement 1. *Let M be a compact set stable in the Lyapunov sense as $t \longrightarrow +\infty$, and $m_1 \in M$. Then, for each number $\varepsilon > 0$, there exists a number $\delta > 0$, such that*

$$d(m_1, m_2) < \delta \Rightarrow d(\varphi(t, m_1), \varphi(t, m_2)) < \varepsilon \ \forall t \in R^+, \quad m_2 \in M. \tag{1}$$

Next, we formulate lemmas required in what follows.

Lemma 1. *Let*: (*1*) *A be a compact set stable in the Lyapunov sense as $t \longrightarrow +\infty$*; (*2*) *$A$ be an attractor as $t \longrightarrow +\infty$, i.e., for arbitrary neighborhood U of the set A, there exist a neighborhood V of the set A, such that*

$$\varphi(t, V) \subset U \ \forall t \in R^+ \quad and \quad \omega(x) \subset A \ \forall x \in V; \tag{2}$$

(*3*) *there exists a trajectory $C(a)$, $a \in A$, which is dense everywhere in the set A. Then, for an arbitrary $\lambda > 0$, there exists a number $\delta(\lambda) > 0$, such that*

$$\forall b_1, b_2 \in A \tag{3}$$
$$d(b_1, b_2) < \delta(\lambda) \Rightarrow d(\varphi(t, b_1), \varphi(t, b_2)) < \lambda \ \forall t \in R.$$

Lemma 1 is a direct corollary of Statement 1, as well as of the definition of the attractor, and the presence in the set A of the trajectory dense everywhere.

Corollary 1. *For arbitrary two points $b_1, b_2 \in A$, $b_1 \neq b_2$, the relation*

$$\exists \mu > 0, \ d(\varphi(t, b_1), \varphi(t, b_2)) > \mu \ \forall t \in R \tag{4}$$

takes place.

Lemma 2. *Let the prerequisites of Lemma* 1 *be satisfied. Then, the set A is the minimum set of almost periodic* (*in the Bohr sense*) *trajectories.*

Proof. We prove initially, that

$$\exists a \in A \quad \overline{C^+(a)} = A. \tag{5}$$

By virtue of condition (3) of Lemma 1,

$$\exists a \in A \quad \overline{C(a)} = A.$$

We assume that if $b \in \alpha(a)$, then $\omega(b) \supset C(a)$. Indeed, let $x = \varphi(a, t_1)$, and we choose a number $\varepsilon > 0$. Since the point $b \in \alpha(a)$ is stable in the Lyapunov sense, then, for an arbitrary number $c \in R$, there exist $t_2 < c$ and $t_1 > t_2$, such that $d(\varphi(t_2 + t, a), \varphi(t, b)) > \varepsilon$ $\forall t \in R^+$ and $d(x, \varphi(t_1 - t_2, b)) > \varepsilon$. This proves the inclusion. Since $\alpha(a)$ is invariant and closed, then $\alpha(a) \supset \omega(a)$ and, thus, $\omega(a) \supset C(a)$. Since $\overline{C^+(a)} \supset \omega(a)$, then $\overline{C^+(a)} = A$. Thus, statement (5) is established.

We now show that the set A is the minimum one. To do this, we choose $b \in A$ and $\varepsilon > 0$. Then, by virtue of arguments proved above, there exists the point $a_1 \in A$, such that $\overline{C^+(a_1)} = A$ and $d(\varphi(t, a_1), \varphi(t, b)) < \varepsilon$ $\forall t \in R^+$. Since ε is arbitrary, then $\overline{C^+(b)} = A$.

Because the point b is arbitrary, each half-trajectory of the set A is dense everywhere in A, and hence, the set A consists of almost periodic trajectories. Thus, Lemma 2 is proved.

Let A be an attractor as $t \longrightarrow +\infty$. Denote the attraction set as $\Pi(A)$, such that

$$\Pi(A) = \{x \in R^n: \omega(x) \in A\}.$$

In addition, for an arbitrary point $a \in A$, we define

$$K(a) = \{x \in \Pi(A): d((\varphi(t, x), \varphi(t, a)) \longrightarrow 0, (t \longrightarrow +\infty))\}.$$

It is easy to show that for different points a_1 and a_2, the sets $K(a_1)$ and $K(a_2)$ are nonintersecting and

$$\Pi(A) = \bigcup\{K(a): a \in A\}. \tag{6}$$

We now establish equality (6). Indeed, let $x \in \Pi(A)$ and $\tau_n \longrightarrow +\infty$. Since $\omega(x) \subset A$, then we can choose a subsequence $s_n \longrightarrow +\infty$, such that $\varphi(s_n, x) \longrightarrow y \in A$ as $s_n \longrightarrow +\infty$. We define the sequence $y_{s_n} = \varphi(s_n, y)$. Since $y_{s_n} \in A$, then we can choose the convergent subsequence y_{s_k}, such that $y_{s_k} \longrightarrow a$ as $k \longrightarrow +\infty$. The point a is stable in the Lyapunov sense. Therefore, $d(\varphi(s_k, a)\ \varphi(s_k, x)) \longrightarrow 0$ as $k \longrightarrow +\infty$. Consequently, from the stability of the point a, it follows that $x \in K(a)$. Thereby, the validity of relation (6) is proved.

Lemma 3. *The attraction set $\Pi(A)$ is open and stable in the Lyapunov sense as $t \longrightarrow +\infty$.*

Proof. The property of openness for the set $\Pi(A)$ is established in [8]. We will prove that the set $\Pi(A)$ is stable in the Lyapunov sense. By virtue of Statement 1, for an arbitrary $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$, such that

$$a, b \in A,$$
$$d(a, b) < \delta \Rightarrow d(\varphi(t, a), \varphi(t, b)) < \varepsilon \ \forall t \in R^+.$$

Let $\varepsilon > 0$ and $y \in \Pi(A)$. We choose the number $\lambda > 0$, such that $B_\lambda(y) \in \Pi(A)$ and the inequality

$$r(\varphi(t, x), A) < \frac{1}{2}\delta\left(\frac{\varepsilon}{2}\right),\ x \in B_\lambda(y),\ t > \tau$$

is satisfied for a certain $\tau > 0$, where r is the distance from the point to the set.

The existence of the number τ follows from the definition of the attractor. Due to the continuity of the dynamic system, we can choose $\lambda_1 < \lambda$, such that the following expressions hold:

$$d(\varphi(t, x), \varphi(t, y)) < \frac{1}{2}\delta\left(\frac{\varepsilon}{2}\right),$$
$$x \in B_{\lambda_1}(y), \quad t \in [0, \tau].$$

Hence,

$$d(\varphi(t, x), \varphi(t, y)) < \varepsilon \ \forall (t, x) \in R^+ \times B_{\lambda_1}(y).$$

Thus, Lemma 3 is proved.

Let A be a set for which conditions (*1*)–(*3*) of Lemma 1 are satisfied. We show that if $a \in A$ then, for each neighborhood N_1 of the point a, there exists the neighborhood $N_2 \subset N_1$, such that

$$x \in (N_2 \cap K(b)) \Rightarrow b \in N_1. \tag{7}$$

Actually, let N_1 be the neighborhood of a point $a \in A$. We may take that N_1 is an open sphere $B_\lambda(a)$ with the radius λ and $B_\lambda(a) \subset \Pi(A)$. From the property of stability in the Lyapunov sense of the point a, it follows that

$$\exists \delta_1 > 0$$
$$x \in B_{\delta_1}(a) \Rightarrow d(\varphi(t, x), \varphi(t, a)) < \frac{1}{2}\delta(\lambda) \ \forall t \in R^+. \tag{8}$$

The choice of the number $\delta(\lambda)$ is performed according to (3). Let $x \in B_{\delta_1}(a) \cap K(b)$. For large values of s, we have $d(\varphi(s, b), \varphi(s, x)) < \frac{1}{2}\delta(\lambda)$. Consequently,

$$d(\varphi(-s, \varphi(s, b)), \varphi(-s, \varphi(s, a))) = d(b, a) < \lambda.$$

Implication (7) follows from the inequality obtained and condition (3) of Lemma 1.

The following theorem also takes place:

Theorem 1. *Let the invariant set $A \subset R^n$ of the dynamic system φ: $R^n \longrightarrow R^n$ be compact, attracting as $t \longrightarrow +\infty$, stable in the Lyapunov sense as $t \longrightarrow +\infty$, and there exist in A a trajectory everywhere dense.*

Then, the set A is a torus. In particular, if A is a hyperbolic set, then it will be either an equilibrium state or a closed trajectory.

Proof. By lemmas 1 and 2, the set A is a minimum set of almost-periodic trajectories. We impart to A the structure of a compact topologic group, which is always possible [8]. The set is compact, and, consequently, by the Pontryagin theorem [9], the commutative connected finite-dimensional topologic group is locally homeomorphic to the set being the Cartesian product $\Gamma_1 \times \Gamma_2$. Here, Γ_1 is the compact zero-dimensional topological group and Γ_2 is the n-dimensional set homeomorphic to the sphere $|x| < 1$. The set Γ_1 is discrete or perfect. A perfect zero-dimensional set from R^n is known from [8] to be the Cantor set. Therefore, A is a local disk or a product of the Cantor set by an n-dimensional element. It follows from the Pontryagin theorem that if A is connected locally, then A is the Cartesian product of n circumferences, i.e., A is an n-dimensional torus T^n.

We now establish that A possesses the property of a local connectedness. To do this, we assume the contrary. Then, each point $a \in A$ has a neighborhood N_1, such that $A \cap N_1$ is a product of a n-dimensional element and a Cantor set. Let N_2 be the connected neighborhood of a point $a \in A$. Since for any neighborhood N_1, there exist a neighborhood $N_2 \subset N_1$, such that

$$x \in N_2,\ x \in K(b) \Rightarrow b \in N_1,$$

then we can assume that

$$N_2 \subset \cup\{K(b) \colon b \in A \cap N_1\}.$$

Due to the property of the intersection $A \cap N_1$, A can be decomposed into a sum of two sets A_i ($i = 1, 2$), such that

$$A_i \cap N_2 = \varnothing,$$
$$d(a_1, a_2) > c > 0 \ \forall a_i \in A_i \ \ (i = 1, 2).$$

We now assume that

$$U_i = \cup\{K(b) \colon b \in A_i\} \cap N_2.$$

Next, we show that sets U_i are open. Indeed, let $y \in K(b) \cap N_2$ and $b \in A_i$. According to Lemma 3, the set $\Pi(A)$ is stable in the Lyapunov sense. Hence, we have

$$\exists \varepsilon > 0,$$
$$d(x, y) < \varepsilon \Rightarrow d(\varphi(t, x), \varphi(t, y)) < \frac{1}{2}\delta(\lambda),$$

where $\delta(\lambda)$ is a number corresponding to the number $\lambda > 0$ chosen in the same manner as in the item (3) of Lemma 1. If $x \in B_\varepsilon(y) \cap N_2 \cap K(a)$, then for a sufficiently large t, we have $d(\varphi(t, a), \varphi(t, b)) < \delta(\lambda)$. Hence, $d(a, b) < \lambda$. From the last inequality and statement (7), it follows that $x \in U_i$, $U_1 \cap U_2 = \varnothing$, $U_1 \cup U_2 = N_2$, and the set N_2 is a combination of two open nonempty sets, which contradicts to the connectedness of the set N_2. The contradiction obtained proves the local connectedness of the set A. Therefore, the set A is a torus. The theorem is proved.

Theorem 1 is generalized to the case of a connected metric space X and a finite-dimensional attractor $A \subset X$. Namely, the following theorem takes place.

Theorem 2. *Let X be a locally connected metric space. Let the invariant set $A \subset X$ of a dynamic system φ: $X \longrightarrow X$ be finite-dimensional, attracting as $t \longrightarrow +\infty$, stable in the Lyapunov sense as $t \longrightarrow +\infty$, and a trajectory dense everywhere there exist in A. Then, the set A is a topological torus. In particular, if A is hyperbolic set, then it will be either an equilibrium state, or a closed trajectory.*

The proof of Theorem 2 is performed in the same manner as that of Theorem 1.

Comments to Theorem 2. It was established by V. V. Nemytskiĭ and V. V. Stepanov [8], that for any compact metric group G, there exists a dynamic system, such that G is a minimal set of almost-periodic trajectories in this dynamic system. Therefore, any such a group can be a stable attractor, and hence, the condition of Theorem 2 stating that the set A is contained in a certain closed locally connected metric space cannot be relaxed.

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