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Complex locally uniform rotundity of Musielak-Orlicz spaces

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Abstract The concepts of complex locally uniform rotundity and complex locally uniformly rotund point are introduced. The sufficient and necessary conditions of them are given in complex Musielak-Orlicz spaces.

Keywords: complex locally uniform rotundity, complex locally uniformly rotund point, Musielak-Orlicz spaces.

In recent years many mathematicians[1—6] have devoted to the geometric theory of complex Banach spaces, because its applications in harmonic analysis, operator theory, Banach algebra, C^* algebra, differential equation, quantum mechanics, liquid mechanics and so on are irreplaceable by the geometric theory of real Banach spaces. For a comprehensive description one may refer to ref. [1]. In 1994 Thorp and Whielely introduced the concept of complex extreme point. In 1975 Globevnic[2] introduced the concepts of complex strict rotundity and complex uniform rotundity.

Let $[X, \|\cdot\|]$ be a complex Banach space. A point $x \in S(X)$ is called a complex extreme point of $B(x)$ if $y \in X$, $\|x + \lambda y\| \leqslant 1(|\lambda \leqslant 1|)$ imply $y = 0$. If every point in $S(X)$ is a complex extreme point of $B(X)$, then X is said to be complex strict rotund. Furthermore, if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x, y \in X, \ \|y\| > \varepsilon, \ \|x + \lambda y\| \leqslant 1(|\lambda| \leqslant 1) \Rightarrow \|x\| \leqslant 1 - \delta,$$

then X is said to be a complex uniformly rotund space. Clearly, we can get the equivalent descriptions of complex extreme point and complex uniform rotundity as follows: A point $x \in S(X)$ is called a complex extreme point provided that $\max\limits_{|\lambda|=1} \|x + \lambda y\| > 1$ for any $0 \neq y \in X$; X is complex uniformly rotund provided that for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\max\limits_{|\lambda|=1} \|x + \lambda y\| \geqslant 1 + \delta$ for all $x, y \in X$ satisfying $\|x\| = 1$ and $\|y\| > \varepsilon$.

By such descriptions it is natural to introduce the concepts of complex locally uniformly rotund point and complex locally uniform rotundity.

Definition 1. A point $x \in S(X)$ is called a complex locally uniformly rotund point of $B(X)$ provided that for any $\varepsilon > 0$, there exists $\delta = \delta(x, \varepsilon) > 0$ such that $\max\limits_{|\lambda|=1} \|x + \lambda y\| \geqslant 1 + \delta$ for any $y \in X$ satisfying $\|y\| > \varepsilon$.

Definition 2. X is called a complex locally uniformly rotund space provided that every point in $S(X)$ is a complex locally uniformly rotund point of $B(X)$.

Wu and Sun[3—5] have discussed complex extreme point, complex strict rotundity and complex uniform rotundity of vector-valued Musielak-Orlicz function space $L_M(X)$ and found the sufficient and necessary conditions of them. In this paper, we will give the sufficient and necessary conditions of complex locally uniformly rotund point and complex locally uniform rotundity in $L_M(X)$.

Let (T,Σ,μ) be a finite nonatomic measurable space. Suppose that a function $M(t,u)$: $T\times[0,\infty)\to[0,\infty]$ satisfies

(*) for μ-a.e. $t\in T, M(t,0)=0$; $\lim\limits_{u\to\infty}M(t,u)=\infty$ and $0<M(t,u')<\infty$ for some $u'>0$;

(**) for μ-a.e. $t\in T$, $M(t,u)$ is convex on $[0,\infty)$ with respect to u;

(***) for each $u\in[0,\infty)$, $M(t,u)$ is a Σ-measurable function of t on T.

Moreover for a given complex Banach spaces $(X,\|\cdot\|)$, we denote by X_T the set of all strongly Σ-measurable functions from T to X, and for each $x\in X_T$, define the modular of x by $\rho_M(x)=\int_T M(t,\|x(t)\|)\mathrm{d}t$. The linear set $L_M=\{x\in X_T:\rho_M(\lambda x)<\infty$ for some $\lambda>0\}$ endowed with norm

$$\|x\|_M=\inf\left\{c>0:\ \rho_M\left(\frac{x}{c}\right)\leqslant 1\right\}$$

is a complex Banach space. We call it vector-valued Musielak-Orlicz space and denote it by $L_M(X)$. For $x\in L_M(X)$, write $\xi_M(x)=\inf\{c>0:\ \rho_M(x/c)<\infty\}$. It is proved that $\xi_M(x)=\lim\limits_{n\to\infty}\|x|_{T_n}\|_M$, where $T_n=\{t\in T:\ \|x(t)\|>n\}$.

Write $e(t)=\sup\{u\geqslant 0:\ M(t,u)=0\}$, $E(t)=\sup\{u\geqslant 0:\ M(t,u)<\infty\}$.

It is true that $e(t)$ and $E(t)$ are Σ-measurable with respect to t (see Proposition 5.1 in ref. [6]).

$M(t,u)$ is said to satisfy the Δ-condition on $T_0(M\in\Delta(T_0))$ if there exist $k\geqslant 1$ and a nonnegative measurable function $\delta(t)$ such that $\int_{T_0}M(t,\delta(t))\mathrm{d}t<\infty$ and $M(t,2u)\leqslant kM(t,u)$ ($t\in T_0$ a.e.; $u\geqslant\delta(t)$). From $M\in\Delta(T_0)$, it is easy to verify that for any $h>0$, there exists a nonnegative measurable integrable function $\delta_0(t)$ on T_0, and $\bar{k}>0$, such that

$$M(t,hu)\leqslant\bar{k}M(t,u)+\delta_0(t),\qquad t\in T_0\ \text{a.e.}$$

If $M\in\Delta(T)$, we write $M\in\Delta$.

Lemma 1. Let X be a complex Banach space. If

$$x,y\in X\ \text{and}\ \|x+y\|+\|x-y\|+\|x+iy\|+\|x-iy\| =\sum_{k=\pm1,\pm i}\|x+ky\|\leqslant 4(1+\delta)\|x\|,$$

then $\max\limits_{|\lambda|\leqslant 1}\left\|x+\frac{\lambda}{2}y\right\|\leqslant(1+13\sqrt{\delta})\|x\|$.

Proof. Pick $f\in X^*$ such that $\|f\|_{X^*}=1$, $f(x)=\|x\|$. Since

$$4\|x\|=4f(x)=4\mathrm{Re}f(x)=\mathrm{Re}f(4x)=\sum_k\mathrm{Re}f(x+ky)\leqslant\sum_k|f(x+ky)|\leqslant\sum_k\|x+ky\|\leqslant 4(1+\delta)\|x\|,$$

we have

$$\|x+ky\|-|f(x+ky)|\leqslant 4\delta\|x\|;\ |f(x+ky)|-\mathrm{Re}f(x+ky)\leqslant 4\delta\|x\|,\quad k=\pm1,\pm i.$$

Observing that $\| y \| \leqslant \frac{1}{4}\sum_k \| kx + y \| = \frac{1}{4}\sum_k \| x + ky \| \leqslant (1+\delta)\| x \|$, we can deduce

$$\mathrm{Re}f(x+ky) \leqslant \mathrm{Re}f(x) + |f(y)| \leqslant \| x \| + \| y \| \leqslant (2+\delta)\| x \| \leqslant 3\| x \| .$$

Combining this with $\mathrm{Im}f(x) = 0$, we can obtain

$$\begin{aligned} \mathrm{Im}kf(x) &= \mathrm{Im}f(x+ky) \leqslant (|f(x+ky)|^2 - \mathrm{Re}^2 f(x+ky))^{\frac{1}{2}} \\ &\leqslant ((\mathrm{Re}f(x+ky) + 4\delta\| x \|)^2 - \mathrm{Re}^2 f(x+ky))^{\frac{1}{2}} \\ &= (8\delta\| x \| \mathrm{Re}f(x+ky) + 16\delta^2 \| x \|^2)^{\frac{1}{2}} \\ &\leqslant (24\delta \| x \|^2 + 16\delta^2\| x \|^2)^{\frac{1}{2}} \leqslant \sqrt{40\delta}\,\| x \| . \end{aligned}$$

Taking $k = \pm 1, \pm i$, respectively, we obtain

$$|\mathrm{Im}f(y)| \leqslant \sqrt{40\delta}\,\| x \|, \quad |\mathrm{Re}f(y)| \leqslant \sqrt{40\delta}\,\| x \| .$$

It follows that $|f(y)| \leqslant \sqrt{80\delta}\,\| x \|$.

Furthermore we have

$$\begin{aligned} \| x + ky \| &\leqslant |f(x+ky)| + 4\delta\| x \| \leqslant |f(x)| + |f(y)| + 4\delta \| x \| \\ &\leqslant \| x \| + 9\sqrt{\delta}\,\| x \| + 4\delta\| x \| \leqslant (1 + 13\sqrt{\delta})\| x \|, \quad k = \pm 1, \pm i. \end{aligned}$$

For $|\lambda| \leqslant 1$, we may only consider the case of $\mathrm{Re}\lambda \geqslant 0$, $\mathrm{Im}\lambda \geqslant 0$. Then

$$\begin{aligned} \left\| x + \lambda \frac{y}{2} \right\| &\leqslant \frac{1}{2}(\| x + (\mathrm{Re}\lambda) y \| + \| x + (\mathrm{Im}\lambda) i y \|) \\ &\leqslant \frac{1}{2}(\| x + y \| + \| x + iy \|) \leqslant (1 + 13\sqrt{\delta})\| x \| . \end{aligned}$$

Lemma 2. If $M \in \Delta$, then $\| x_n \|_M \to 1 \Leftrightarrow \rho_M(x_n) \to 1$.

Proof. If there exists $\varepsilon > 0$ such that $\| x_n \|_M \leqslant 1 - \varepsilon$, then we have $\rho_M(x_n) \leqslant \| x_n \|_M \leqslant 1 - \varepsilon$.

If $\rho_M(x_n) \leqslant 1 - \varepsilon$ and $\| x_n \|_M \to 1$, combining this with $\sup\limits_n \rho_M(2x_n) < \infty$, we obtain a contradiction

$$\begin{aligned} 1 &= \rho_M\left(\frac{x_n}{\| x_n \|_M}\right) = \rho_M\left(\left(\frac{1}{\| x_n \|_M} - 1\right)2x_n + \left(2 - \frac{1}{\| x_n \|_M}\right)x_n\right) \\ &\leqslant \left(\frac{1}{\| x_n \|_M} - 1\right)\rho_M(2x_n) + \left(2 - \frac{1}{\| x_n \|_M}\right)\rho_M(x_n) \\ &\leqslant \left(\frac{1}{\| x_n \|_M} - 1\right)\sup_n \rho_M(2x_n) + \left(2 - \frac{1}{\| x_n \|_M}\right)(1-\varepsilon) \to 1 - \varepsilon . \end{aligned}$$

If $\| x_n \| \geqslant 1 + \varepsilon$, then we have $\rho_M(x_n) \geqslant \| x_n \|_M \geqslant 1 + \varepsilon$.

If $\rho_M(x_n) \geqslant 1 + \varepsilon$ and $\| x_n \|_M \to 1$, combining this with $\sup\limits_n \rho_M\left(2\frac{x_n}{\| x_n \|_M}\right) < \infty$, we obtain a contradiction

$$\begin{aligned} 1 + \varepsilon &\leqslant \rho_M(x_n) = \rho_M\left((2 - \| x_n \|_M)\frac{x_n}{\| x_n \|_M} + (\| x_n \|_M - 1)\frac{2X_n}{\| x_n \|_M}\right) \\ &\leqslant (2 - \| x_n \|_M)\rho_M\left(\frac{x_n}{\| x_n \|_M}\right) + (\| x_n \|_M - 1)\rho_M\left(\frac{2x_n}{\| x_n \|_M}\right) \to 1 . \end{aligned}$$

(We have known that the conditions $M \in \Delta$ and $e(t) = 0$ (a.e.) imply $\rho_M(x_n) \to 1 \Leftrightarrow \| x_n \|_M \to 1$ in Proposition 5.13 of ref. [6]. This lemma shows that the condition $e(t) = 0$ (a.e.) is not necessary.)

Theorem 1. A point $x \in S(L_M(X))$ is a complex locally uniformly rotund point if and only if

(i) $\| x(t) \| = E(t)$ $(a.e.)$ *or* $\rho_M(x) = 1$;

(ii) $\| x(t) \| \geqslant e(t)$ $(a.e.)$;

(iii) for any $\varepsilon > 0$, $D > 0$, there exist $\delta > 0$ such that

$$\| y \mid_{A(x,y,\delta)} \|_M < \frac{\varepsilon}{3}, \quad \| y_{B(x,y,\delta)} \|_M < \frac{\varepsilon}{3} \text{ for any } y \in L_M(X), \text{ where}$$

$$A(x,y,\delta) = \left\{ t \in T : \sum_k \| x(t) + ky(t) \| \leqslant 4(1+\delta) \| x(t) \| \right\},$$

$$B(x,y,\delta) = \{ t \in T : \| x(t) \| = 0,\ M(t,1) \leqslant D,\ \| y(t) \| \leqslant \delta \}.$$

(iv) If $S \in (0,1)$, $T_0 \subset T$, $\int_{T_0} M\left(t, \frac{\| x(t) \|}{1-s}\right) \mathrm{d}t < \infty$, then $M \in \Delta(T_0)$.

Proof. Necessity

If (i) is not true, then $\rho_M(x) < 1$ and $\mu\{t \in T : \| x(t) \| < E(t)\} > 0$. Take $b > 0$ such that $T_0 = \{t \in T : \| x(t) \| + b < E(t)\}$ is a positive measurable set and

$$\int_{G \setminus T_0} M(t, \| x(t) \|) \mathrm{d}t + \int_{T_0} M(t, \| x(t) \| + b) \mathrm{d}t \leqslant 1.$$

Take $\theta \in S(X)$, and let $y(t) = b\theta \mid_{T_0}$. Then $y \neq 0$. But for any λ, $|\lambda| \leqslant 1$, $\rho_M(x + \lambda y) \leqslant 1$, which contradicts the fact that x is a complex extreme point.

If (ii) is not true, then $\mu\{t \in T : \| x(t) \| < e(t)\} > 0$. Take $b > 0$ such that $T_b = \{t \in T : \| x(t) \| + b \leqslant e(t)\}$ is a positively measurable set. Take $\theta \in S(X)$ and let $y(t) = b\theta \mid_{T_b}$. Then $y \neq 0$. But for any λ satisfying $|\lambda| \leqslant 1$, we have

$$\rho_M(x + \lambda y) \leqslant \rho_M(x \mid_{T \setminus T_b}) + \int_{T_b} M(t, \| x(t) \| + b) \mathrm{d}t = \rho_M(x \mid_{T \setminus T_b}) \leqslant 1.$$

Hence $\| x + \lambda y \|_M \leqslant 1$ $(|\lambda| \leqslant 1)$, which contradicts the fact that x is a complex extreme point.

If (iii) is not true, then there exist $\varepsilon > 0$, $D > 0$, for any n, there exists $y_n \in L_M(X)$ such that $\left\| y_n \mid_{A\left(x, y_n, \frac{1}{n}\right)} \right\| \geqslant \frac{\varepsilon}{3}$ or $\left\| y_n \mid_{B\left(x, y_n, \frac{1}{n}\right)} \right\| \geqslant \frac{\varepsilon}{3}$, where

$$A_n := A\left(x, y_n, \frac{1}{n}\right) = \left\{ t \in T : \sum_k \| x(t) + ky_n(t) \| \leqslant 4\left(1 + \frac{1}{n}\right) \| x(t) \| \right\},$$

$$B_n := B\left(x, y_n, \frac{1}{n}\right) = \left\{ t \in T : \| x(t) \| = 0,\ M(t,1) \leqslant D,\ \| y_n(t) \| \leqslant \frac{1}{n} \right\}.$$

If $\| y_n \mid_{A_n} \| \geqslant \frac{\varepsilon}{3}$, by Lemma 1, for $t \in A_n$, we have

$$\left\| x(t) + \frac{\lambda}{2} y_n(t) \right\| \leqslant \left(1 + 13\sqrt{\frac{1}{n}}\right) \| x(t) \|.$$

Furthermore, we have

$$\rho_M\left(\frac{x + \frac{\lambda}{2} y_n \mid A_n}{1 + 13\sqrt{\frac{1}{n}}} \right) = \int_{T \setminus A_n} M\left(t, \frac{\| x(t) \|}{1 + 13\sqrt{\frac{1}{n}}}\right) \mathrm{d}t + \int_{A_n} M\left(t, \frac{\left\| x(t) + \frac{\lambda}{2} y_n \right\|}{1 + 13\sqrt{\frac{1}{n}}}\right) \mathrm{d}t$$

$$\leqslant \int_T M(t, \| x(t) \|)dt \leqslant 1.$$

This shows $\left\| x + \frac{\lambda}{2} y_n |_{A_n} \right\|_M \leqslant 1 + 13\sqrt{\frac{1}{n}} \to 1$. But $\left\| \frac{y_n}{2} |_{A_n} \right\|_M \geqslant \frac{\varepsilon}{6}$, which contradicts the fact that x is a complex locally unformly rotund point.

If $\| y_n |_{B_n} \|_M \geqslant \frac{\varepsilon}{3}$, letting $z_n = y_n |_{B_n}$ we have $\| z_n \|_M \geqslant \frac{\varepsilon}{3}$. But for any λ satisfying $|\lambda| \leqslant 1$

$$\begin{aligned}\rho_M(x + \lambda z_n) &= \int_{T \setminus B_n} M(t, \| x(t) \|)dt + \int_{B_n} M(t, \| x(t) + \lambda y_n(t) \|)dt \\ &\leqslant \int_{T \setminus B_n} M(t, \| x(t) \|)dt + \int_{B_n} M(t, \| y_n(t) \|)dt \\ &\leqslant \int_{T \setminus B_n} M(t, \| x(t) \|)dt + \frac{1}{n} \sup_{t \in B_n} M(t,1) \mu B_n \leqslant \rho_M(x) + \frac{D\mu T}{n} \leqslant 1 + \frac{D\mu T}{n},\end{aligned}$$

furthermore, we have $\| x + \lambda z_n \|_M \leqslant 1 + \frac{D}{n} \mu T \to 1 \ (n \to \infty)$, which contradicts the fact that x is a complex locally uniformly rotund point.

If (iv) is not true, then there are $s \in (0,1)$ and $T_0 \subset T$ such that $\int_{T_0} M\left(t, \frac{\| x(t) \|}{1-s}\right) dt < \infty$ and $M \notin \Delta$ on T_0. By Theorem 5.5 in ref. [6], we can construct $y = y |_{T_0} \in L_M(X)$ satisfying $\rho_M(y) < 1$ and $\xi_M(y) = 1$. Let

$$y_n(t) = \begin{cases} sy(t), & t \in T_n = \{t \in T_0 : \| y(t) \| > n\}, \\ 0, & \text{otherwise}. \end{cases}$$

Obviously $\mu T_n \to 0$, since

$$\begin{aligned}\rho_M(x + \lambda y_n) &\leqslant \rho_M(x) + \int_{T_n} M(t, \| x(t) + \lambda y_n(t) \|)dt \\ &\leqslant \rho_M(x) + \int_{T_n} M\left(t, (1-s) \frac{\| x(t) \|}{1-s} + sy(t)\right) dt \\ &\leqslant 1 + (1-s) \int_{T_n} M\left(t, \frac{\| x(t) \|}{1-s}\right) dt + s \int_{T_n} M(t, \| y(t) \|)dt \\ &\to 1.\end{aligned}$$

But $\| y_n \|_M \geqslant s \| y |_{T_n} \|_M \geqslant s\xi_M(y) = s$, which contradicts the fact that x is a complex locally uniformly rotund point.

Sufficiency. We shall consider the following two cases:

I. $$\| x(t) \| = E(t) \text{ a.e.}$$

For any $\varepsilon > 0$, by condition (iii), there exists $\delta > 0$ such that $\left\| y |_{A(x,y,\delta)} \right\|_M < \frac{\varepsilon}{3}$ for any $y \in L_M(X)$. Then for $\| y \|_M \geqslant \varepsilon$, we have $\left\| y |_{T \setminus A(x,y,\delta)} \right\|_M \geqslant \frac{2}{3}\varepsilon$. For $t \in T \setminus A(x,y,\delta)$, we have $\frac{1}{4(1+\delta)} \sum_k \| x(t) + ky(t) \| > \| x(t) \| = E(t)$.a.e.

Combining this with $\mu(T \setminus A(x,y,\delta)) > 0$, we have

$$\frac{1}{4} \sum_k \left(\frac{x + ky}{1 + \delta} \right) \geqslant \int_{T \setminus A(x,y,\delta)} M\left(t, \frac{1}{4(1+\delta)} \sum_k \| x(t) + ky(t) \| \right) dt = \infty.$$

Hence, $\max_k \rho_M\left(\frac{x+ky}{1+\delta}\right)=\infty$. This shows $\max_k \|x+ky\|_M \geqslant 1+\delta$.

II. $$\rho_M(x)=1.$$

If x is not a complex locally uniformly rotund point, then there exist $\varepsilon>0$, $y_n \in L_M^0(X)$ and $\|y_n\|_M>\varepsilon$ such that

$$\|x+ky_n\|_M \leqslant 1+\frac{1}{n}, \quad n=1,2,\cdots;\ k=\pm 1,\ \pm i.$$

By condition (iii), there exists $\delta \in (0,1/2)$ such that for any $y \in L_M^0(X)$, $\|y|_{A(x,y,\delta)}\|_M < \varepsilon/3$, where

$$A(x,y,\delta)=\left\{t \in T: \sum_k \|x(t)+ky(t)\| \leqslant 4(1+2\delta)\|x(t)\|\right\}.$$

Let $A_n=\{x, y_n, \delta\}$, $B_n=T \setminus A_n$. Then $\|y_n|_{B_n}\|_M>2\varepsilon/3 \quad (n=1,2,\cdots)$.

We shall consider the following two cases:

(II-1) $\inf_n \int_{B_n} M(t,(1+\delta)\|x(t)\|)\mathrm{d}t=\alpha>0$,

$$\rho_M\left(\frac{x}{1+1/n}\right)=\int_{A_n} M\left(t, \frac{\|x(t)\|}{1+1/n}\right)\mathrm{d}t+\int_{B_n} M\left(t, \frac{\|x(t)\|}{1+1/n}\right)\mathrm{d}t$$

$$\leqslant \int_{A_n} M\left(t, \frac{1}{4(1+1/n)}\sum_k \|x(t)+ky_n(t)\|\right)\mathrm{d}t+\frac{1}{1+2\delta}$$

$$\int_{B_n} M\left(t, \frac{1}{4(1+1/n)}\sum_k \|x(t)+ky_n(t)\|\right)\mathrm{d}t$$

$$\leqslant \int_{T} M\left(t, \frac{1}{4(1+1/n)}\sum_k \|x(t)+ky_n(t)\|\right)\mathrm{d}t-\frac{2\delta}{1+2\delta}$$

$$\int_{B_n} M\left(t, \frac{1}{4(1+1/n)}\sum_k \|x(t)+ky_n(t)\|\right)\mathrm{d}t$$

$$\leqslant \frac{1}{4}\sum_k \int_T M\left(t, \frac{x(t)+ky_n(t)}{1+1/n}\right)\mathrm{d}t-\frac{2\delta}{1+2\delta}\int_{B_n} M\left(t, \frac{\|1+2\delta\|}{1+1/n}\|x(t)\|\right)\mathrm{d}t$$

$$\leqslant \frac{1}{4}\sum_k \rho_M\left(\frac{x+ky_n}{1+1/n}\right)-\frac{2\delta}{1+2\delta}\int_{B_n} M(t,(1+\delta)\|x(t)\|)\mathrm{d}t$$

$$\leqslant 1-\frac{2\delta}{1+2\delta}a.$$

For large n satisfying $\frac{1+2\delta}{1+1/n}>1+\delta$, the above inequality is true. Letting $n \rightarrow \infty$, we have $1=\rho_M(x) \leqslant 1-\frac{2\delta}{1+2\delta}a$, a contradiction.

(II-2) $$\inf_n \int_{B_n} M(t,(1+\delta)\|x(t)\|)\mathrm{d}t=0.$$

By passing to a subsequence, we may assume $\sum_{n=1}^{\infty}\int_{B_n} M(t,(1+\delta)\|x(t)\|)\mathrm{d}t<\infty$.

Let $B=\bigcup_{n=1}^{\infty} B_n$. Then we have $\int_B M(t,(1+\delta)\|x(t)\|)\mathrm{d}t<\infty$. By condition (iv), we have

$M \in \Delta(B)$. Then for $6/\varepsilon$, there exist $\bar{D} > 0$ and a measurable nonnegative function $\delta_0(t)$, $\int_B \delta_0(t)\mathrm{d}t < \infty$ such that

$$M(t, 6u/\varepsilon) \leqslant \bar{D}M(t,u) + \delta_0(t), \ t \in B.$$

Take $\eta > 0$ such that $\int_e \delta_0(t)\mathrm{d}t \leqslant \frac{1}{2}$ when $e \subset B$, $\mu e < \eta$. For this η, take $D > 0$ such that $\mu\{t \in T: M(t,1) > D\} < \eta/3$. By condition (iii), we can take $\delta' > 0$ such that for any $y \in L_M(X)$, $\| y 1_{C(x,y,\delta')} \|_M < \varepsilon/3$, where

$$C(x,y,\delta') = \{t \in T: \| x(t) \| = 0, \ M(t,1) \leqslant D, \ \| y(t) \| \leqslant \delta'\}.$$

Let $H_n = B_n \setminus C(x, y_n, \delta')$. Then $\| y_n 1_{H_n} \|_M > \varepsilon/3$ $(n = 1,2,\cdots)$.

(II-2-1) $\qquad \mu H_n < \eta \ (n = 1,2,\cdots)$.

Since $\left\| \frac{3}{\varepsilon} y_n 1_{H_n} \right\|_M \geqslant 1$ and $H_n \subset B_n \subset B$, we have

$$1 \leqslant \rho_M\left(\frac{3}{\varepsilon} y_n 1_{H_n}\right) = \int_{H_n} M\left(t, \frac{3}{\varepsilon} \| y_n(t) \| \right)\mathrm{d}t \leqslant \int_{H_n} M\left(t, \frac{6}{\varepsilon} \frac{\| y_n(t) \|}{(1 + 1/n)}\right)\mathrm{d}t$$
$$\leqslant \int_{H_n} \left(\bar{D}M\left(t, \frac{\| y_n(t) \|}{1 + 1/n}\right) + \delta_0(t)\right)\mathrm{d}t \leqslant \bar{D}\int_{H_n} M\left(t, \frac{\| y_n(t) \|}{1 + 1/n}\right)\mathrm{d}t + \frac{1}{2}.$$

Then $\int_{H_n} M\left(t, \frac{\| y_n(t) \|}{1 + 1/n}\right)\mathrm{d}t \geqslant \frac{1}{2\bar{D}}$. With the same method as in the proof of (II-1), we have

$$\rho_M\left(\frac{x}{1 + 1/n}\right) \leqslant \frac{1}{4}\sum_k \rho_M\left(\frac{x + ky_n}{1 + 1/n}\right)$$
$$- \frac{2\delta}{1 + 2\delta}\int_{H_n} M\left(t, \frac{1}{4(1 + 1/n)}\right)\sum_k \| x(t) + ky_n(t) \|)\mathrm{d}t$$
$$\leqslant 1 - \frac{2\delta}{1 + 2\delta}\int_{H_n} M\left(t, \frac{\| y_n(t) \|}{1 + 1/n}\right)\mathrm{d}t$$
$$\leqslant 1 - \frac{2\delta}{1 + 2\delta} \cdot \frac{1}{2\bar{D}}.$$

Letting $n \to \infty$, we have $1 \leqslant 1 - \frac{\delta}{(1 + 2\delta)\bar{D}}$, a contradiction.

(II-2-2) $\qquad \mu H_n \geqslant \eta \ (n = 1,2,\cdots)$.

Since $e(t) > 0 \Rightarrow M(t,(1 + \delta)e(t)) > 0$; $e(t) = 0 \Rightarrow M(t, \delta'/2) > 0$, there exists $b > 0$ such that $\mu\{t \in T: e(t) > 0$ and $M(t,(1 + \delta)e(t)) < b$ or $e(t) = 0$ and $M(t,\delta'/2) < b\}$ $< \eta/3$. Combining this with $\mu\{t \in T: M(t,1) > D\} < \eta/3$, we obtain $\mu\Omega_n \geqslant \eta/3$, where

$$\Omega_n = \{t \in H_n: M(t,1) \leqslant D, \ e(t) > 0 \Rightarrow M(t,(1 + \delta)e(t)) \geqslant b,$$
$$e(t) = 0 \Rightarrow M(t,\delta'/2) \geqslant b\}.$$

When $t \in \Omega_n$ and $e(t) > 0$, we have

$$M\left(t, \frac{1}{4(1 + 1/n)}\sum_k \| x(t) + ky_n(t) \| \right) \geqslant M(t,(1 + \delta) \| x(t) \|)$$
$$\geqslant M(t,(1 + \delta)e(t)) \geqslant b;$$

when $t \in \Omega_n$ and $e(t) = 0$, we have

$$M\left(t, \frac{1}{4(1 + 1/n)}\sum_k \| x(t) + ky_n(t) \| \right) \geqslant M\left(t, \frac{\| y_n(t) \|}{1 + 1/n}\right) \geqslant M(t,\delta'/2) \geqslant b.$$

So we always have $\int_{\Omega_n} M\left(t, \frac{1}{4(1+1/n)} \sum_k \| x(t) + ky_n(t) \| \right) \mathrm{d}t \geqslant \frac{b\eta}{3}$. Then

$$\rho_M\left(\frac{x}{1+1/n}\right) \leqslant 1 - \frac{2\delta}{1+2\delta} \int_{\Omega_n} M\left(t, \frac{1}{4(1+1/n)} \sum_k \| x(t) + ky_n(t) \| \right) \mathrm{d}t$$

$$\leqslant 1 - \frac{2\delta}{1+2\delta} \cdot \frac{b\eta}{3}.$$

Letting $n \to \infty$, we have $1 \leqslant 1 - \frac{2b\delta\eta}{3(1+2\delta)}$, a contradiction.

Theorem 2. $L_M(X)$ is complex locally uniformly rotund if and only if

(i) $M \in \Delta k$; (ii) $e(t) = 0(a.e.)$; (iii) for $x \in S(L_M(X))$ and any $\varepsilon > 0$, there exists $\delta > 0$, such that $\left\| y|_{A(x,y,\delta)} \right\|_M \leqslant \frac{\varepsilon}{3}$ for any $y \in L_M(X)$, where

$$A(x,y,\delta) = \{t \in T: \frac{1}{4} \sum_k \| x(t) + ky(t) \| \leqslant (1+\delta) \| x(t) \| \}.$$

Proof. Necessity.

If (i) is not true, it is clear that there exists $x \in S(L_M(X))$, $\| x(t) \| < E(t)$ (a.e.) and $\rho_M(x) < 1$. By condition i) in Theorem 1, x is not a complex locally uniformly rotund point.

If (ii) is not true, then $T_0 = \{t \in T: e(t) > 0\}$ is a positively measurable set. Take $b > 0$, $T' \subset T \setminus T_0$ such that $\int_{T'} M(t,b) \mathrm{d}t = 1$. Take $\mu_0 \varepsilon = S(X)$, $x = \frac{e(t)}{2} u_0 \big|_{T_0} + bu_0|_{T'}$. Then $\rho_M(x) = 1$. Hence $\| x \|_M = 1$. But $\mu\{t \in T: \| x(t) \| < e(t)\} \geqslant \mu T_0 > 0$, which shows that x is not a complex locally uniformly rotund point by condition (ii) of Theorem 1.

If (iii) is not true, by the conditon (iii) in Theorem 1, x is not a complex locally uniformly rotund point.

Sufficiency. Given $x \in L_M(X)$, for any $y \in L_M(X)$ satisfying $\| y \|_M \geqslant \varepsilon$, we write $A = \left\{t \in T: \sum_k \| x(t) + ky(t) \| \leqslant 4(1+\delta) \| x(t) \| \right\}$, where $\delta > 0$ is defined by condition (iii). Then $\| y|_{T \setminus A} \|_M > \frac{2\varepsilon}{3}$. By (i), (ii) of Theorem 2 and Proposition 5.13 in ref. [6], $\| y|_{T \setminus A} \|_M > \frac{2\varepsilon}{3}$ implies $\rho_M(y|_{T \setminus A}) \geqslant \varepsilon'$ (ε' only depends on ε). Then

$$1 = \rho_M(x) \leqslant \frac{1}{4} \sum_k \rho_M(x+ky) - \frac{\delta}{1+\delta} \int_{T \setminus A} M(t, \| y(t) \|) \mathrm{d}t \leqslant 1 - \frac{\delta}{2} \varepsilon'.$$

We have $\max_k \rho_M(x+ky) \geqslant 1 + \frac{\delta\varepsilon'}{2}$. By Lemma 2 there is $r > 0$ (r only depends on δ, ε') such that $\max_k \| x + ky \|_M \geqslant 1 + r$.

Remark. Comparing this Theorem with Theorems 5.19 and 5.20 in ref. [6], we know that complex uniform rotundity is essentially stronger than complex locally uniform rotundity and the latter is essentially stronger than complex strict rotundity. In addition to conditons (i) and (ii) in Theorem 2, the complex strict rotundity and complex uniform rotundity of X are added to the criteria of complex strict rotundity and complex uniform rotundity of $L_M(X)$ respectively. So people guess that the third condition of complex locally uniform rotundity of $L_M(X)$ is the complex locally uniform rotundity of X. But it is not true. The sense of condition (iii) in Theorem 2 is that, roughly speaking, for every $x \in S(L_M(X))$, the set $\{x(t): t \in T \setminus T_0\}$ is complex uniformly rotund in X, where T_0 is a set with arbitrarily small measure.

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