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THE CANARY TREE REVISITED

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Abstract. We generalize the result of Mekler and Shelah [3] that the existence of a canary tree is independent of ZFC + GCH to uncountable regular cardinals. We also correct an error from the original proof.

§1. Introduction. In [3] A. Mekler and S. Shelah defined the notion of a canary tree. A tree T is called a canary tree if it is of cardinality 2^{ω} and of height ω_1 with no uncountable branches such that whenever a closed unbounded set is forced into a bistationary subset of ω_1 without adding reals also an ω_1-branch is added into the tree T. The main result in [3] is that the existence of a canary tree is independent of ZFC + GCH.

When we studied the proof of the consistency of the existence of a canary tree we observed that there is a minor flaw in the proof. Namely, the partial order used does not add the desired order-preserving functions. By modifying the partial order the problem can be fixed. We will do that and we will also generalize the theorem to regular uncountable cardinals κ, i.e., we show that it is consistent with GCH that there is a tree T of cardinality 2^{κ} and of height κ^+ such that there are no κ^+-branches in the tree and whenever a κ-stationary subset of κ^+ is destroyed without adding new subsets of κ, a κ^+-branch is added into the tree T. In addition we show that it is consistent with GCH that there are no κ-canary trees. Both consistency results can be obtained by forcing notions which preserve all cardinals.

Our proof is longer than the one in [3] partly because we are not able to utilize the general theory of proper forcing, especially the iteration lemma, but we have to prove everything "from scratch".

It should be noted that it is possible that there is a κ-stationary subset S of κ^+ such that whenever a κ-cub is forced into S new subsets of κ are added into the universe [2]. Anyhow, if GCH holds then it is possible to force a κ-cub into any κ-stationary subset of κ^+ without adding small subsets.

In addition to the destruction of κ-bistationary sets, κ-canary trees have connections to other problems, too. Assume GCH. Let κ be a regular cardinal and let η_κ be a κ-dense linear order of cardinality κ. Let λ denote κ^+. For a subset S of λ

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define the linear order $\Phi(S)$ to be $\sum_{\alpha<\lambda} \tau_\alpha$ where

$$\tau_\alpha = \begin{cases} 1+\eta_\kappa & \text{if } \alpha \in S \cap S^\lambda_\kappa, \\ \eta_\kappa & \text{otherwise.} \end{cases}$$

Recall that a tree U is a universal non-equivalence tree for a model $\mathfrak{A}$ if for every model $\mathfrak{B}$ in the same vocabulary as $\mathfrak{A}$ and of the same cardinality as $\mathfrak{A}$ the following holds: If $\mathfrak{A}$ and $\mathfrak{B}$ are non-isomorphic, then player $\forall$ has a winning strategy in the Ehrenfeucht-Fraïssé game between $\mathfrak{A}$ and $\mathfrak{B}$ in which player $\forall$ has to go up the tree U move by move. (We say that the length of the game is U.) Now the existence of a κ-canary tree is equivalent to the existence of a universal non-equivalence tree for $\Phi(S)$, $S \subseteq \lambda$ (see [3, p. 211], and [9, p. 126] for a similar result for Abelian groups).

Our notation is fairly standard, but a few words are in order here. Suppose $\kappa \leq \lambda$ are cardinals. The Cohen forcing which adds λ many subsets of κ is denoted by $\mathrm{Fn}(\lambda, 2, \kappa)$. By S^λ_κ we denote the set of ordinals of cofinality κ that are strictly less than λ.

Suppose further that κ is a regular cardinal. A subset C of λ is κ-closed if every $\delta \in S^\lambda_\kappa$ for which $C \cap \delta$ is unbounded in δ is in C. The set C is a κ-cub subset of λ if it is κ-closed and unbounded in λ. A subset S of λ is κ-stationary if it intersects every κ-cub subset of λ. The notions of κ-bistationary and κ-costationary are defined in the obvious way. A simple fact worth noting is that a subset S of λ is κ-stationary if and only if $S \cap S^\lambda_\kappa$ is stationary.

Let p be a pair. The first component of p is denoted by $1\mathrm{st}(p)$ and the second component by $2\mathrm{nd}(p)$.

The standard name of an object x in the ground model is denoted by $\check{x}$. We usually omit the check when it is clear from the context that the standard name is meant.

Let P be a partial order and G a P-generic set. The interpretation of a P-name τ by G is denoted by $\tau[G]$. A subset B of P is **pre-dense** in P if for every condition $p \in P$ there is a condition $q \in B$ which is compatible with p. A **nice name for a subset of** τ is a name of the form

$$\bigcup \{\{\pi\} \times A_\pi \mid \pi \in \mathrm{dom}(\tau)\}$$

where each A_π is an antichain in P. An important property of nice names is that if μ is name for a subset of σ then there is a nice name τ for a subset of σ such that $\Vdash \tau = \mu$.

Let S be a subset of κ^+. By $T(S)$ we denote the tree obtained by ordering the sequences

$$\left\{ b \in \bigcup_{\alpha<\lambda} {}^{(\alpha+1)}S \;\middle|\; b \text{ is strictly increasing and } \kappa\text{-closed} \right\}$$

by end-extension.

Suppose that T is a tree. The elements of T at level α is denoted by $\mathrm{Lev}_T(\alpha)$. Let $t \in T$. By $\mathrm{ht}(t)$ we denote the height of the element t, and by $\mathrm{pred}(t)$ we denote the predecessors of t.

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§2. A problem in the original proof. For the reader's convenience we now describe the partial order used in [3].

Let Q_0 be the set of functions such that a function f is in Q_0 if and only if

$$\operatorname{dom}(f) \subseteq S^{\omega_1}_{\omega} \text{ is countable and}$$
$$\forall \delta \in \operatorname{dom}(f)(f(\delta) \in {}^{\delta}\delta).$$

Order Q_0 by reverse inclusion. Clearly Q_0 is ω_1-closed, and of cardinality ω_1 as we assume GCH.

From a Q_0-generic set G_0 we define a tree $\mathscr{T}(G_0)$ as follows:

$$\mathscr{T}(G_0) = \left\{ t \in {}^{<\omega_1}\omega_1 \mid \forall \delta \leq \operatorname{dom}(t)\left(t \restriction \delta \notin \operatorname{ran}(\bigcup G_0)\right)\right\}.$$

One can easily verify that $\mathscr{T}(G_0)$ is of cardinality 2^{ω} and in $V[G_0]$ there are no ω_1-branches in $\mathscr{T}(G_0)$. Hence the tree $\mathscr{T}(G_0)$ will be a canary tree if for every bistationary subset S of ω_1 we can add an order preserving function from $T(S)$ to $\mathscr{T}(G_0)$ without adding an ω_1-branch to $\mathscr{T}(G_0)$. The partial order $P(S, G_0)$, which we define below, is designed to add the needed function for $T(S)$ where S is a bistationary subset of ω_1. First an auxiliary notion of an S-node is defined. A node $t \in \mathscr{T}(G_0)$ is an **S-node** if for every $\delta \in S^{\omega_1}_{\omega}\backslash S$ less than or equal to $\operatorname{dom}(t)$ it holds that $t \restriction \delta \notin {}^{\delta}\delta$. It should be noted that if t is an S-node then $t\restriction\delta$ is an S-node for every $\delta \in \operatorname{dom}(t)$ and t has successors of arbitrary height which are S-nodes. Now we define the partial order $P(S, G_0)$. The elements of $P(S, G_0)$ are pairs $\langle g, X\rangle$ such that X is a countable subset of $\bigcup_{\alpha<\omega_1} {}^{(\alpha+1)}\omega_1$, g is an order preserving partial mapping from $T(S)$ to the S-nodes of $\mathscr{T}(G_0)$ the domain of which is a countable subtree of $T(S)$ and the following conditions hold:

$$\forall c \in \operatorname{dom}(g) \forall t \in X\left(t \nsubseteq g(c)\right),$$
$$(1) \qquad \forall \langle c_i \mid i \in \omega\rangle \in {}^{\omega}\operatorname{dom}(g)\left(\langle c_i \mid i \in \omega\rangle \text{ increasing} \to \bigcup_{i\in\omega} g(c_i) \in \mathscr{T}(G_0)\right).$$

For a condition $\langle g, X\rangle \in P(S, G_0)$, let

$$o(\langle g, X\rangle) = \sup\{\operatorname{dom}(t) \mid t \in X \vee t \in \operatorname{ran}(g)\}$$

and $\operatorname{dom}(\langle g, X\rangle) = \operatorname{dom}(g)$. A condition $\langle h, Y\rangle \in P(S, G_0)$ extends $\langle g, X\rangle$ if and only if

$$g \subseteq h,$$
$$X \subseteq Y,$$
$$\forall c \in \operatorname{dom}(h)\backslash\operatorname{dom}(g)\left(\operatorname{dom}(h(c)) > o(\langle g, X\rangle)\right).$$

Claim 2.1. *Suppose that G_0 is a Q_0-generic set and S is a bistationary subset of ω_1 in some forcing extension which contains G_0. Then for every $t \in T(S)$, the set*

$$D_t = \left\{\langle g, X\rangle \in P(S, G_0) \mid \exists t' \geq t \forall t^* \geq t' \forall \langle g^*, X^*\rangle \in P(S, G_0)\right.$$
$$\left.\left(t^* \in \operatorname{dom}(g^*) \to \langle g, X\rangle \perp \langle g^*, X^*\rangle\right)\right\}$$

is dense in $P(S, G_0)$.

PROOF. Suppose that $t \in T(S)$ and $p \in P(S, G_0)$. Let $q = \langle g, X\rangle \in P(S, G_0)$ be an extension of p such that for some $t'' \geq t$, $t'' \in \mathrm{dom}(g)$. (If there are no such t'' and q, we are done.) Let $\alpha = o(q)$. Choose an ascending sequence $\langle \beta_i \mid i < \omega\rangle$ of ordinals such that $\beta_i \in S$, $\beta_0 > \alpha$,

$$\{s \in T(S) \mid s \geq (t'' \frown \langle \beta_0\rangle)\} \cap \mathrm{dom}(g) = \emptyset$$

and $\beta = \bigcup_{i\in\omega} \beta_i \in S$. For $n \in \omega$, let $b_n = t'' \frown \langle \beta_i \mid i \leq n\rangle$. Choose two sequences $\langle u_i \mid i \in \omega\rangle$ and $\langle u'_i \mid i \in \omega\rangle$ of elements of $\mathscr{T}(G_0)$ such that

$$\begin{gathered}\forall u \in \mathscr{T}(G_0)\big(u \geq u_0 \rightarrow \forall s \in X(s \not\leq u)\big),\\ \forall u \in \mathscr{T}(G_0)\big(u \geq u'_0 \rightarrow \forall s \in X(s \not\leq u)\big),\\ u_0 > g(t''),\ u'_0 > g(t''),\ u_0 \perp u'_0,\\ \mathrm{dom}(u_0) = \mathrm{dom}(u'_0) > \beta,\end{gathered}$$

and for each $i \in \omega$,

$$\begin{gathered}u_i \text{ and } u'_i \text{ are } S\text{-nodes},\\ \mathrm{dom}(u_i) = \mathrm{dom}(u'_i),\\ u_i < u_{i+1},\ u'_i < u'_{i+1},\\ \mathrm{dom}(u_{i+1}) > \bigcup \mathrm{ran}(u_i),\ \mathrm{dom}(u'_{i+1}) > \bigcup \mathrm{ran}(u'_i),\\ \bigcup_{i\in\omega} \mathrm{dom}(u_i)\ \Big(= \bigcup_{i\in\omega} \mathrm{dom}(u'_i)\Big) \notin S.\end{gathered}$$

This is possible since X is countable and every S-node has S-node continuations arbitrarily high in $\mathscr{T}(G_0)$. As at most one branch is cut in $\mathscr{T}(G_0)$ at every limit level, $\bigcup_{i\in\omega} u_i \in \mathscr{T}(G_0)$ or $\bigcup_{i\in\omega} u'_i \in \mathscr{T}(G_0)$ (or both). We may assume that $u = \bigcup_{i\in\omega} u_i \in \mathscr{T}(G_0)$. Let $q_1 = \langle g_1, X\rangle$ where $g_1 = g \cup \{\langle b_i, u_i\rangle \mid i \in \omega\}$. Clearly $q_1 \in P(S, G_0)$ and $q_1 \leq q$. Since $\beta \in S$, the sequence $t' = \bigcup_{i\in\omega} b_i \frown \langle\beta\rangle \in T(S)$. Suppose $t^* \geq t'$. Let g_2 be any extension of g_1 such that $t^* \in \mathrm{dom}(g_2)$. Then $g_2(t^*) \geq u$. Let $\delta = \bigcup_{i\in\omega} \mathrm{dom}(u_i)$. Since $u \in {}^{\delta}\delta$ and $\delta \notin S$, $g_2(t^*)$ is not an S-node. ⊣

Suppose that G is a $P(S, G_0)$-generic set. By the claim above, it is clear that the the domain of the function $f = \bigcup\{g \mid \exists X(\langle g, X\rangle \in G)\}$ is not dense in $T(S)$; in fact, the set

$$\big\{t \in T(S) \mid \forall t' \in T(S)\big(t' > t \rightarrow t' \notin \mathrm{dom}(f)\big)\big\}$$

is dense in $T(S)$. Hence f can not be the desired order preserving function from $T(S)$ to $\mathscr{T}(G_0)$. Thus it seems difficult to prove that there is one in $V[G_0][G]$.

One can try to fix the problem by requiring in (1) that $\bigcup_{i\in\omega} g(c_i)$ must be an S-node instead of just requiring that it is in $\mathscr{T}(G_0)$. Of course this does not resolve the problem but just moves it.

§3. **A fix and a generalization.** In this section we fix the problem we found and at the same time we generalize the theorem.

We start by defining the concept of a κ-canary tree.

DEFINITION 3.1. *Let κ be a regular cardinal. A tree T is a κ-**canary tree** if the following conditions hold*:

(i) *The tree T is of cardinality 2^κ and of height κ^+.*
(ii) *There are no κ^+-branches in T.*
(iii) *Whenever a κ-closed and unbounded set is forced into a κ-bistationary subset of κ^+ without forcing new subsets of κ, a κ^+-branch is forced into T.*

Note that Condition (iii) is equivalent with the following: If $S \subseteq S^{\kappa^+}_\kappa$ is a stationary subset of κ^+ and a cub subset of κ^+ is forced into $\kappa^+ \backslash S$ without adding subsets of cardinality at most κ, then a κ^+-branch is added into T.

REMARK 3.2. *A tree T is a canary tree if and only it it is an ω-canary tree.* ⊣

Next we define the notion of κ-proper forcing and prove few crucial properties which we shall need in the sequel. The definition of κ-proper forcing is a direct generalization of that of proper forcing [7, p. 102]. The following definition of a generic condition is, of course, from [7, p. 101].

DEFINITION 3.3. *Let N be and elementary submodel of $(H(\chi), \in)$ and let $P \in N$ be a partial order. A condition $q \in P$ is an $\langle N, P\rangle$-**generic condition** if for every dense subset D of P that is in N the set $N \cap D$ is pre-dense below q.*

DEFINITION 3.4. *Assume κ is a cardinal with $\kappa^{<\kappa} = \kappa$. A partial order P is κ-**proper** if the following hold*:

(i) *It is κ-closed.*
(ii) *For every χ large enough and for every $N \prec (H(\chi), \in)$ of cardinality κ if*

$$P \in N,$$
$$\kappa + 1 \subseteq N, \ {}^{<\kappa}N \subseteq N,$$
$$N \cap \kappa^+ \text{ is an ordinal},$$

and $p \in P \cap N$ then there is an $\langle N, P\rangle$-generic condition q with $q \leq p$.

REMARK 3.5. (i) *A partial order P is proper if and only if it is ω-proper.*
(ii) *Suppose that $\kappa^{<\kappa} = \kappa$ and a partial order P is κ^+-closed, or it is κ-closed and has the κ^+-c.c. Then P is κ-proper.* ⊣

This direct generalization has a drawback, namely it follows that κ-properness is not necessarily preserved under iteration with κ^+-support (by λ-support we mean what some authors would call $< \lambda$-support). An example of this is the partial order due to L. Stanley and Shelah [4] demonstrating the failure of a generalization of Martin's axiom.We shortly describe the situation. Let us consider sequences of length ω_2 of functions from ω_1 to ω_1 and order these sequences by defining $\bar{g} <^* \bar{f}$ if and only if for all $\alpha < \beta < \omega_2$, $\bar{g}(\alpha)$ and $\bar{g}(\beta)$ differ at zero or they differ before $\bar{f}(\alpha)$ and $\bar{f}(\beta)$ do. Assuming CH it can be shown that there is no infinite $<^*$ descending sequence. On the other hand, for every sequence $\bar{f}$ there is an ω_1-closed partial order $P(\bar{f})$ having the ω_2-c.c. such that a $P(\bar{f})$-generic set G introduces a sequence $\bar{g} <^* \bar{f}$. With an iteration of length ω we can force an infinite $<^*$ descending sequence. Since every step is ω_1-closed, the iterated partial order is also ω_1-closed. Hence CH holds in the forcing extension. It follows that ω_2 must be collapsed. By the remark above, the partial orders used in the iteration are ω_1-proper, but the iterated partial order is not as the lemma below shows. For some positive versions consult [8].

LEMMA 3.6. *Suppose κ is an uncountable cardinal with $\kappa^{<\kappa} = \kappa$, P is a κ-proper partial order and G is a P-generic set. Then for any set $A \in V[G]$ of ordinals of cardinality κ, there is a set $C \in V$ of cardinality κ that covers A, i.e., $A \subseteq C$.*

PROOF. Let G be a P-generic set. If suffices to prove that for every function f from κ to ordinals that is in $V[G]$, there is a set $C \in V$ of cardinality κ that covers $\operatorname{ran}(f)$. Towards a contradiction assume that

(2.1) f is a function from κ to ordinals in $V[G]$ such that for every subset C of $\sup(\operatorname{ran}(f))$ of cardinality κ in V it holds that $\operatorname{ran}(f) \not\subseteq C$.

Let $\dot{f}$ be a P-name for a function from κ to ordinals with $f = \dot{f}[G]$. Let $p \in G$ be a condition that forces (2.1).

Let $N \prec (H(\chi), \in)$ where χ is large enough such that

$$|N| = \kappa,\ \kappa + 1 \subseteq N,\ {}^{<\kappa}N \subseteq N,$$
$$\dot{f}, p, P \in N,$$
$$N \cap \kappa^+ = \delta \text{ is an ordinal.}$$

Let $\alpha < \kappa$. Then the set

$$D_\alpha = \{r \in P \mid \exists \beta (r \Vdash \dot{f}(\alpha) = \beta)\}$$

is dense in P and definable from $\dot{f}$, α and P. Thus it is in N. Let

$$C_\alpha = \{\beta \in \mathrm{Ord} \mid \exists r \in P \cap N (r \Vdash \dot{f}(\alpha) = \beta)\}.$$

Clearly $C_\alpha \subseteq N$. Let q be an $\langle N, P\rangle$-generic extension of p.

Since $N \cap D_\alpha$ is pre-dense below q, q forces "$\dot{f}(\alpha) \in C_\alpha$". It follows that

$$q \Vdash \operatorname{ran}(\dot{f}) \subseteq C$$

where $C = \bigcup_{\alpha<\kappa} C_\alpha$. Clearly C is in V. Since C_α is a subset of N, it is of cardinality at most κ. Hence C is of cardinality at most κ. This contradicts the assumption that p forces "$\operatorname{ran}(\dot{f})$ is not covered by any set of cardinality κ that is in V". ⊣

COROLLARY 3.7. *Suppose κ is an uncountable cardinal with $\kappa^{<\kappa} = \kappa$ and P is κ-proper. Then κ^+ is a regular cardinal in every P-generic extension.*

LEMMA 3.8. *Suppose κ is an uncountable cardinal with $\kappa^{<\kappa} = \kappa$, and P is κ-proper. Then forcing with P preserves stationary subsets of κ^+. Especially it preserves κ-stationary subsets of κ^+.*

PROOF. Suppose $S \subseteq \kappa^+$ is stationary. Then there is $\mu < \kappa^+$ such that $S \cap S^{\kappa^+}_\mu$ is a stationary subset of κ^+. So we may assume that $S \subseteq S^{\kappa^+}_\mu$. Towards a contradiction assume that $\dot{C}$ is a P-name and p a condition such that

$$p \Vdash \dot{C} \subseteq \kappa^+ \text{ is a cub and } \dot{C} \cap S = \emptyset.$$

There are two cases to consider.

(A) (This is due to Shelah [5]) Suppose $\mu < \kappa$. Let $<^*$ be a well-ordering of P. Since $S^{\kappa^+}_{<\kappa} \in I[\kappa^+]$ (see [6]), also $S^{\kappa^+}_\mu \in I[\kappa^+]$. Let the sequence $\bar{a} = \langle a_\xi \mid \xi < \kappa^+\rangle$

and the cub subset E of κ^+ witness this. Choose $N \prec (H(\chi), \in)$, where χ is large enough, of cardinality κ such that

$$\kappa + 1 \subseteq N,$$
$$N \cap \kappa = \delta \in S \cap E,$$
$$\bar{a}, p, P, S, \dot{C}, <^* \in N.$$

As $\delta \in S \cap E$, there is $e \subseteq \delta$ of order type $\mathrm{cf}(\delta) = \mu$ such that $\bigcup e = \delta$ and for every $\xi < \delta$ there is $\zeta < \delta$ with $e \cap \xi = a_\zeta$. Thus $e \cap \xi \in N$ for every $\xi < \delta$. Let $\{\delta_\xi \mid \xi < \mu\}$ be a strictly increasing enumeration of e. Define by recursion a descending sequence $\langle p_\xi \mid \xi < \mu\rangle$ such that $p_0 = p$ and

$p_{\xi+1}$ is the $<^*$-first p' stronger than p_ξ such that for some $\gamma_\xi > \delta_\xi$, $p' \Vdash \gamma_\xi \in \dot{C}$,

if ξ is a limit ordinal, then p_ξ is the $<^*$-first p' stronger than p_ζ for each $\zeta < \xi$.

This is possible since P is κ-closed and $\mu < \kappa$. For every $\zeta < \mu$, the initial segment $\langle p_\xi \mid \xi \leq \zeta\rangle$ is definable from p, P, $<^*$, $\dot{C}$ and $\{\delta_\xi \mid \xi \leq \zeta\}$. Thus each $p_\xi \in N$. Which in turn yields $\gamma_\xi \in N$. Since P is κ-closed, there is a lower bound q for $\langle p_\xi \mid \xi < \mu\rangle$. Now q forces "$\dot{C}$ is unbounded in δ", and therefore it forces "$\delta \in \dot{C}$". This contradicts the assumption that $p \Vdash \dot{C} \cap S = \emptyset$, as $\delta \in S$.

(B) Suppose that $\mu = \kappa$. Let χ be large enough and $N \prec (H(\chi), \in)$ as required in Definition 3.4 such that $S, \dot{C}, p \in N$ and $N \cap \kappa^+ = \delta \in S$. Since N is an elementary submodel of $(H(\chi), \in)$, we have

$$N \models p \Vdash_P \text{“}\dot{C} \subseteq \kappa^+ \text{ is a cub”}.$$

For $\alpha < \delta$, let

$$D_\alpha = \{q \in P \mid q \leq p \wedge \exists \beta < \kappa^+(\alpha < \beta \wedge q \Vdash \beta \in \dot{C})\} \cup \{q \in P \mid q \perp p\}.$$

Clearly D_α is a dense subset of P and in N. Let q be a (N, P)-generic extension of p. We claim that q forces "$\dot{C}$ is unbounded in δ". Towards a contradiction assume that $q' \leq q$ forces "$\dot{C}$ is bounded in δ". We may assume that q' decides the supremum of $\dot{C} \cap \delta$. Let this be γ. As q is (N, P)-generic, there is $r \in D_\gamma \cap N$ compatible with q'. Let q'' be a common extension of r and q'. As $r \in N$, there is $\beta \in N \cap \kappa^+$ greater than γ such that r forces "$\beta \in \dot{C}$". Since q'' extends r, it forces this too. Now q'' forces

$$\dot{C} \cap \delta \subseteq \gamma \wedge \dot{C} \cap \delta \nsubseteq \gamma$$

which is absurd.

Since q extends p and p forces "$\dot{C}$ is a cub", q forces "$\delta \in \dot{C}$". But this contradicts the assumption $p \Vdash \dot{C} \cap S = \emptyset$, as $\delta \in S$.

This completes the proof as we reached a contradiction in both cases. ⊣

REMARK 3.9 (Shelah [5]). *If there is a supercompact cardinal, then there is a forcing extension in which there are regular cardinals $\lambda > \kappa$, a stationary set $S \subseteq S^\lambda_\kappa$ and a κ^+-closed partial order P such that S is not a stationary subset of λ in any P-generic extension.* ⊣

Now we turn our attention to the main theorem.

THEOREM 3.10. *Assume* GCH. *Suppose κ is a regular cardinal. Then there is a partial order P such that in every P-generic extension there is a κ-canary tree, all cardinals are preserved and* GCH *still holds.*

PROOF. The proof is rather long, so we have divided it into claims. First we define the partial order mentioned in the theorem, and then in the claims we prove various properties it has.

Let Q_0 be a collection of functions such that f is in Q_0 if and only if

$$\mathrm{dom}(f) \subseteq S^{\kappa^+}_\kappa,\ |\mathrm{dom}(f)| \leq \kappa,$$
$$\forall \delta, \eta \in \mathrm{dom}(f)\big(f(\delta) \in {}^\delta\delta \wedge \big(\delta < \eta \rightarrow f(\delta) \not\subseteq f(\eta)\big)\big).$$

Order Q_0 by reverse inclusion. Then Q_0 is κ^+-closed, and of cardinality κ^+ as we assumed GCH.

In a forcing extension V_1 which contains a Q_0-generic set G_0 we define a tree of functions

$$\mathscr{T}(G_0) = \big\{f \in {}^{<\kappa^+}\kappa^+ \mid \forall \delta \in S^{\kappa^+}_\kappa \big(f\restriction\delta \notin \mathrm{ran}(\textstyle\bigcup G_0)\big)\big\}.$$

Suppose that S is a κ-bistationary subset of κ^+ in V_1. We associate a partial order to S and G_0, but first we have to redefine the notion of an S-node. A node $t \in \mathscr{T}(G_0)$ is an S-**node** if for every $\delta \in S^{\kappa^+}_\kappa \setminus S$ less than or equal to $\mathrm{dom}(t)$ it holds that $t\restriction\delta \notin {}^\delta\delta$. For a partial function g from $T(S)$ to $\mathscr{T}(G_0)$, let $o(g) = \sup\{\mathrm{dom}(t) \mid t \in \mathrm{ran}(g)\}$. Now we can define the partial order $P(S, G_0)$. Let $P(S, G_0)$ be the collection of pairs $\langle g, X\rangle$ that satisfy the following conditions:

(3.1) The element g is an order preserving partial function of cardinality at most κ from $T(S)$ to $\mathscr{T}(G_0)$ the domain of which is closed under initial segments.

(3.2) The element X is a partial function from κ^+ to $\bigcup_{\alpha<\kappa^+} {}^{(\alpha+1)}\kappa^+$ of cardinality at most κ such that

$$o(g) \cap S^{\kappa^+}_\kappa \subseteq \mathrm{dom}(X),$$
$$\forall \alpha \in \mathrm{dom}(X) \cap S^{\kappa^+}_\kappa \big(X(\alpha) \subseteq (\textstyle\bigcup G_0)(\alpha)\big).$$

(3.3) For all $t \in \mathrm{dom}(g)$, $\mathrm{dom}(g(t)) = \sup(\mathrm{ran}(t))$.

(3.4) For all $t \in \mathrm{dom}(g)$, $g(t)$ is an S-node.

(3.5) For all $t \in \mathrm{dom}(g)$ and $\alpha \in \mathrm{dom}(X)$, $X(\alpha) \not\subseteq g(t)$.

(3.6) For all strictly increasing sequences $\langle t_\zeta \mid \zeta < \kappa\rangle$ of elements of $\mathrm{dom}(g)$, it holds that $\bigcup_{\zeta<\kappa} g(t_\zeta) \in \mathscr{T}(G_0)$.

A condition $\langle g, X\rangle$ is stronger than a condition $\langle h, Y\rangle$ if and only if $h \subseteq g$ and $Y \subseteq X$.

The partial order $P(S, G_0)$ is κ-closed since the union of a descending sequence of elements of $P(S, G_0)$ clearly satisfies Conditions (3.1) – (3.5), and if the length of the sequence is less than κ, Condition (3.6) does not set any new requirements.

Let $\varepsilon = \kappa^{++}$. Finally, we define an iterated forcing notion

$$\bar{Q} = \langle P_\alpha, \dot{Q}_\beta \mid \alpha \leq \varepsilon, \beta < \varepsilon\rangle$$

with κ^+-support (i.e., supports of conditions are of cardinality $< \kappa^+$) as follows: Let $\langle \dot{S}_\beta \mid 0 < \beta < \varepsilon\rangle$ be an enumeration of forcing names such that $\dot{S}_\beta$ is a P_β-name for a κ-bistationary subset of κ^+. Let $\dot{Q}_\beta$ be a P_β-name such that

$$\Vdash_\beta \dot{Q}_\beta = P\big(\dot{S}_\beta, \dot{G}(0)\big)$$

where $\dot{G}(0)$ is a name for the Q_0 part of the generic set.

We will show that $\bar{Q}$ is κ-proper. A crucial property in the proof is that the partial order does not force new subsets of cartinality at most κ. This makes it possible to generalize the proof that a single step is κ-proper to the whole iteration.

CLAIM 3.11. *Suppose that G_0 is a Q_0-generic set and S is a κ-stationary subset of κ^+ in a forcing extension V_1 which contains G_0. If G is a $P(S, G_0)$-generic set over V_1, then $\bigcup\{g \mid \exists X(\langle g, X\rangle \in G)\}$ is an order preserving function from $T(S)$ to $\mathscr{T}(G_0)$.*

PROOF. It suffices to prove that for every $t \in T(S)$, the set

$$D_t = \{\langle g, X\rangle \in P(S, G_0) \mid t \in \mathrm{dom}(g)\}$$

is dense in $P(S, G_0)$. So let $p = \langle g, X\rangle \in P(S, G_0)$ such that $t \notin \mathrm{dom}(g)$. Let $\alpha = \sup(\mathrm{ran}(t))$ and $B = \{u \in \mathrm{dom}(g) \mid u \leq_{T(S)} t\}$.

Let

$$\begin{aligned} b &= \sup(B), \\ b' &= \bigcup_{u \in B} g(u), \\ \delta &= \sup\{\sup(\mathrm{ran}(u)) \mid u \in B\}. \end{aligned}$$

If $b \in B$ or $\mathrm{cf}(\delta) < \kappa$, then obviously $b' \in \mathscr{T}(G_0)$. If $b \notin B$ and $\mathrm{cf}(\delta) = \kappa$, then Condition (3.6) ensures that $b' \in \mathscr{T}(G_0)$. So $b' \in \mathscr{T}(G_0)$. As $t \in T(S)$ and $b \leq t$, we have $\delta \in S$. By Condition (3.3), $\mathrm{dom}(b') = \delta$. Since $g(u)$ is an S-node for every $u \in B$ and $\delta \in S$, b' is also an S-node. (Here Condition (3.3) forbids the trick we used in Claim 2.1.) If $B = \emptyset$, let $b' = \emptyset$. Let $c' \in \mathscr{T}(G_0)$ be an S-node continuation of b' with $\mathrm{ht}(c') = \alpha$ such that for all $\gamma \in \mathrm{dom}(X)$, $X(\gamma) \not\subseteq \mathrm{pred}(c')$. Since $|\mathrm{dom}(X)| \leq \kappa$, there is such a node c'. Let mapping g' be defined for every $u \in \mathrm{pred}(t)\backslash B$ by

$$g'(u) = \text{the unique element in } \mathrm{pred}(c') \cap \mathrm{Lev}_{\mathscr{T}(G_0)}(\sup(\mathrm{ran}(u))).$$

It follows from the fact that c' is an S-node that $g'(u)$ is an S-node for every $u \in \mathrm{pred}(t)\backslash B$. Let X' be defined for every $\gamma \in ((\alpha + 1)\backslash \mathrm{dom}(X)) \cap S^{\kappa^+}_{\kappa}$ by

$$X'(\gamma) = (\bigcup G_0)(\gamma)\restriction\xi$$

where $\xi = \max(o(g), \sup\{\mathrm{ht}(s) \mid s \in (\bigcup G_0)(\gamma) \cap \mathrm{pred}(c')\}) + 1$, i.e., we take an initial segment of $(\bigcup G_0)(\gamma)$ that is longer than anything in $\mathrm{ran}(g)$ and long enough to diverge from $\mathrm{pred}(c')$. Let $q = \langle g \cup g', X \cup X'\rangle$. Clearly $q \in D_t$. ⊣

DEFINITION 3.12. *For each $\alpha \leq \varepsilon$, let*

$$P'_\alpha = \{p \in P_\alpha \mid \forall\zeta \in \mathrm{supt}(p)\exists x \in V(p(\zeta) = \check{x})\}.$$

CLAIM 3.13. *For every $\alpha \leq \varepsilon$ the following hold:*

(i) *P_α is κ-proper.*
(ii) *Forcing with P_α does not add subsets of cardinality at most κ.*
(iii) *P'_α is a dense sub-order of P_α.*

PROOF. The proof is by induction on α. If $\alpha = 1$, then $P_\alpha \cong Q_0$, and the properties (i) and (ii) follow as Q_0 is κ^+-closed. Property (iii) is obvious.

Suppose that $\alpha < \varepsilon$ and that for every $\zeta < \alpha$ the claim holds for P_ζ. First we prove that P_α is κ-proper. Let $N \prec (H(\chi), \in)$ such that

$$|N| = \kappa,\ \kappa + 1 \subseteq N,\ {}^{<\kappa}N \subseteq N,$$
$$\kappa^+ \cap N = \delta \text{ is an ordinal,}$$
$$P_\alpha \in N.$$

Let $p \in P_\alpha \cap N$. We need to find an $\langle N, P_\alpha\rangle$-generic extension q of p. For that purpose let $\langle D_\zeta \mid \zeta < \kappa\rangle$ enumerate the dense subsets of P_α that are in N. Choose a descending sequence $\langle p_\zeta \mid \zeta < \kappa\rangle$ such that

$$p_0 = p,$$
$$p_{\zeta+1} \in D_\zeta \cap N.$$

This is possible since P_α is κ-closed and ${}^{<\kappa}N \subseteq N$. By the induction hypothesis, for all $\zeta < \alpha$, forcing with P_ζ does not add subsets of cardinality at most κ. Hence for every $\xi < \kappa$ and $\zeta \in \operatorname{supt}(p_\xi)$, the set

$$D_{\xi,\zeta} = \{r \in P_\alpha \mid r \text{ decides the value of } p_\xi(\zeta)\}$$

is dense in P_α and definable from p_ξ and ζ. As P_α is a κ^+-support iterated forcing notion, $\operatorname{supt}(p_\xi) \subseteq N$ for each $\xi < \kappa$. Thus $D_{\xi,\zeta}$ is in N for every $\zeta \in \operatorname{supt}(p_\xi)$. But if a condition $r \in P_\alpha$ decides the value of $p_\xi(\zeta)$, then $r\restriction\zeta$ will decide the value, too. For this reason the following holds in $H(\chi)$:

$$\forall\xi < \kappa\forall\zeta \in \operatorname{supt}(p_\xi)\exists\xi' < \kappa\exists g^\zeta_\xi \in V\exists X^\zeta_\xi \in V\big(p_{\xi'}\restriction\zeta \Vdash_\zeta p_\xi(\zeta) = \langle g^\zeta_\xi, X^\zeta_\xi\rangle\big).$$

Since N is an elementary submodel of $H(\chi)$ and $\operatorname{supt}(p_\xi) \subseteq N$ for every $\xi < \kappa$, we have that $g^\zeta_\xi, X^\zeta_\xi \in N$. Let $g_\zeta = \bigcup_{\xi<\kappa} g^\zeta_\xi$ and $X_\zeta = \bigcup_{\xi<\kappa} X^\zeta_\xi$. (If $\zeta < \alpha$ and $\zeta \notin \operatorname{supt}(p_\xi)$, then let $g^\zeta_\xi = X^\zeta_\xi = \emptyset$.) Then $g_\zeta, X_\zeta \in V$. Since $\delta > \kappa$, we can choose $t \in {}^\delta\delta$ such that $t\restriction\kappa \notin N$. Now define q as follows: Let the support of q be $\bigcup_{\xi<\kappa}\operatorname{supt}(p_\xi)$ and for each $\zeta \in \operatorname{supt}(q)$, set

$$q(\zeta) = \begin{cases} \bigcup_{\xi<\kappa} p_\xi(0) \cup \{\langle\delta, t\rangle\} & \text{if } \zeta = 0 \\ \langle g_\zeta, X_\zeta\rangle & \text{otherwise.} \end{cases}$$

(Actually standard names should be used.) We prove by induction on $\zeta \leq \alpha$ that $q\restriction\zeta$ is a condition. As for every $\xi < \kappa$, the condition $p_\xi(0) \in N$ and it is of cardinality at most κ, $p_\xi(0) \subseteq N$. Thus, by a density argument, we have $\bigcup_{\xi<\kappa}\operatorname{dom}(p_\xi(0)) = \delta$. Since ${}^{<\kappa}N \subseteq N$, $\operatorname{cf}(\delta) = \kappa$. It follows from these observations that $q(0) \in Q_0$, whence $q\restriction 1$ is a condition.

Suppose that ζ is a limit ordinal and for all $\zeta' < \zeta$, $q\restriction\zeta'$ is a condition. It follows from the definition of the κ^+-support iterated forcing notion that $q\restriction\zeta$ is a condition.

Suppose that $\zeta = \beta + 1$ and $\zeta \in \operatorname{supt}(q)$. Then $q\restriction\beta$ forces "$q(\beta)$ satisfies Conditions (3.1) – (3.5)" since it forces "$p_\xi(\beta) = \langle g^\beta_\xi, X^\beta_\xi\rangle$" for every $\xi < \kappa$. So if it did not force all Conditions (3.1) – (3.5), then there would be $\xi' < \kappa$ such that $q\restriction\beta$ forces "$\langle g^\beta_{\xi'}, X^\beta_{\xi'}\rangle$ is not a condition" from which it would follow that $p_{\xi'}$ is not a condition. To see that Condition (3.6) is satisfied, let $\langle t_\xi \mid \xi < \kappa\rangle$ be an increasing sequence of elements of $\operatorname{dom}(g_\beta)$. Let $u = \bigcup_{\xi<\kappa} g_\beta(t_\xi)$ and $\gamma = \operatorname{dom}(u)$. Then $\operatorname{cf}(\gamma) = \kappa$. There are three cases to consider:

(A) Suppose that for some $\xi^* < \kappa$ every t_ξ is in $\operatorname{dom}(g^\beta_{\xi^*})$. It follows from the fact that $q \restriction \beta \Vdash p_{\xi^*}(\beta) = \langle g^\beta_{\xi^*}, X^\beta_{\xi^*} \rangle$ that $q \restriction \beta \Vdash u \in \mathcal{T}(\dot{G}(0))$.
(B) Suppose $\gamma < \delta$. Then there are $\xi^*, \xi' < \kappa$ with

$$o(g^\beta_{\xi^*}) > \gamma,\ \operatorname{dom}(g^\beta_{\xi^*}(t_{\xi'})) > \operatorname{dom}(X^\beta_{\xi^*}(\gamma)).$$

It follows from $\operatorname{cf}(\gamma) = \kappa$ that $\gamma \in \operatorname{dom}(X^\beta_{\xi^*})$. Thus, by Condition (3.5), we have $X^\beta_{\xi^*}(\gamma) \not\subseteq g^\beta_{\xi^*}(s)$ for all $s \in \operatorname{dom}(g^\beta_{\xi^*})$. Therefore $X_\beta(\gamma) \not\subseteq u$, and $q \restriction \beta$ forces "$u \in \mathcal{T}(\dot{G}(0))$" as required.
(C) Suppose $\gamma = \delta$. Since $\operatorname{dom}(u) = \delta > \kappa$ there are $\xi^*, \xi' < \kappa$ such that $\operatorname{dom}(g^\beta_{\xi^*}(t_{\xi'})) > \kappa$. But $g^\beta_{\xi^*}, t_{\xi'} \in N$ yields $g^\beta_{\xi^*}(t_{\xi'}) \in N$. It follows from the definition of g_β that $g_\beta(t_{\xi'}) = g^\beta_{\xi^*}(t_{\xi'})$. By the choice of t, we have $t \restriction \kappa \notin N$. Thus $g_\beta(t_{\xi'}) \not\subseteq t$, and hence $u \not\subseteq t$. As $q \restriction \beta$ forces "t is the only branch that is cut at level δ in $\mathcal{T}(\dot{G}(0))$", it also forces "$u \in \mathcal{T}(\dot{G}(0))$" as required.

So $q \restriction \beta$ forces "$q(\beta) \in \dot{Q}_\beta$" and hence $q \restriction \zeta$ is a condition. By the construction of q it is the required $\langle N, P_\alpha \rangle$-generic condition.

The proof that P_α does not add subsets of cardinality κ is almost a verbatim copy of the proof that P_α is κ-proper. Suppose that the set R is in V and τ is a P_α-name for a function from κ to R. Towards a contradiction suppose that a condition $p \in P_\alpha$ forces that τ is a new function. Pick a model N such that in addition to the aforementioned conditions also τ and p are in N. Construct the condition q and observe that for each $\xi < \kappa$ the set

$$D_\xi = \{r \in P_\alpha \mid r \text{ decides the value of } \tau(\xi)\}$$

is a dense and open subset of P_α, and definable from τ and ξ. Hence it is in N. By the construction of q, it decides the value of $\tau(\xi)$ for every $\xi < \kappa$. But this is a contradiction since q extends p.

The condition q constructed above also shows that P'_α is a dense sub-order of P_α. ⊣

CLAIM 3.14. *Suppose $\alpha < \varepsilon$. Then:*

(i) *If G_α is a P_α-generic set, $A \subseteq G_\alpha$ is at most of cardinality κ and $A \in V[G_\alpha]$, then there is a condition $q \in G_\alpha$ with $q \leq p$ for every $p \in A$.*
(ii) $\Vdash_\alpha \dot{P}_{\alpha,\varepsilon}$ *is κ-closed.*
(iii) $\Vdash_\alpha \dot{P}_{\alpha,\varepsilon}$ *is κ-proper.*

PROOF. (i) By 3.13(ii), $A \in V$. By 3.13(iii) we may assume that $A \subseteq P'_\alpha$. We also may assume that $|A| = \kappa$. Let $\{q_i \mid i < \kappa\}$ be an enumeration of A. For each $i < \kappa$ and $\xi \in \operatorname{supt}(q_i)$, let $\langle g^i_\xi, X^i_\xi \rangle = q_i(\xi)$. Define

$$S = \bigcup_{i<\kappa} \operatorname{supt}(q_i),$$
$$\beta = \bigcup\{\sup(\operatorname{dom}(X^i_\xi)) \mid i < \kappa \wedge \xi \in \operatorname{supt}(q_i)\},$$
$$T = \bigcup\{\operatorname{dom}(g^i_\xi) \mid i < \kappa \wedge \xi \in \operatorname{supt}(q_i)\}.$$

If $q \in G_\alpha$ is such that

$$S \subseteq \operatorname{supt}(q), \tag{4}$$

$$\forall \xi \in S(\beta \subseteq \operatorname{dom}(X_\xi)), \tag{5}$$

$$\forall \xi \in S \forall t \in T\big(t \in \operatorname{dom}(g_\xi) \vee q \restriction \xi \Vdash t \notin T(\dot{S}_\xi)\big) \tag{6}$$

where $\langle g_\xi, X_\xi \rangle = q(\xi)$ for $\xi \in \operatorname{supt}(q)$, then $q \leq p$ for every $p \in A$. So, it suffices to prove that the set of conditions satisfying (4) – (6) is dense in P'_α. For each $\xi \in S$, $\gamma < \beta$ and $t \in T$, the sets

$$\begin{aligned} D_\xi &= \{r \in P'_\alpha \mid \xi \in \operatorname{supt}(r)\}, \\ D_{\xi,\gamma} &= \{r \in P'_\alpha \mid \gamma \in \operatorname{dom}(2\mathrm{nd}(r(\xi)))\}, \\ D_{\xi,t} &= \{r \in P'_\alpha \mid t \in \operatorname{dom}(1\mathrm{st}(r(\xi))) \vee r \restriction \xi \Vdash_\xi t \notin T(\dot{S}_\xi)\} \end{aligned}$$

are dense in P'_α (and in P_α). The denseness of the first two is obvious. To see that the last set is dense in P'_α, let $p \in P'_\alpha$. Suppose first that $p \restriction \xi$ does not force "$t \in T(\dot{S}_\xi)$". Then there is an extension $r \in P'_\alpha$ of p such that $r \restriction \xi$ forces "$t \notin T(\dot{S}_\xi)$". Thus $r \in D_{\xi,t}$. Suppose then that $p \restriction \xi \Vdash_\xi t \in T(\dot{S}_\xi)$. Let $\langle g^p_\xi, X^p_\xi \rangle = p(\xi)$. Suppose $t \notin \operatorname{dom}(g^p_\xi)$ since otherwise we are done. Let G_p be a P'_ξ-generic set with $p \restriction \xi \in G_p$. By Claim 3.11, in $V[G_p]$ there is an extension

$$r = \langle g^r, X^r \rangle \in P(\dot{S}_\xi[G_p], G_p(0))$$

of $p(\xi)$ with $t \in \operatorname{dom}(g^r)$. Let $p' \in G_p$ be an extension of $p \restriction \xi$ that forces

$$r \in P\big(\dot{S}_\xi, \dot{G_p}(0)\big).$$

Define $r' \in P'_\alpha$ as follows: Let $\operatorname{supt}(r') = \operatorname{supt}(p') \cup \operatorname{supt}(p)$ and for each $\zeta \in \operatorname{supt}(r')$, set

$$r'(\zeta) = \begin{cases} p'(\zeta) & \text{if } \zeta < \xi, \\ r & \text{if } \zeta = \xi, \\ p(\zeta) & \text{if } \zeta > \xi. \end{cases}$$

Clearly $r' \leq p$ and $r' \restriction \xi \Vdash_\xi t \in T(\dot{S}_\xi)$.

To complete the proof let $N \prec (H(\chi), \in)$ of cardinality κ be such that

$$\kappa + 1 \subseteq N,\ {}^{<\kappa}N \subseteq N,\ \kappa^+ \cap N = \delta \in \mathrm{Ord},$$
$$p, P'_\alpha, S, \beta, T \in N.$$

Since S, β and T are each at most of cardinality κ, we have $S, \beta, T \subseteq N$. Therefore for all $\xi \in S$, $\gamma < \beta$ and $t \in T$, the sets D_ξ, $D_{\xi,\gamma}$ and $D_{\xi,t}$ are in N. Thus any $\langle N, P'_\alpha \rangle$-generic extension q of p satisfies (4) – (6).

(ii) Suppose G_α is a P'_α-generic set and let $P_{\alpha,\varepsilon} = \dot{P}'_{\alpha,\varepsilon}[G_\alpha]$. Work in $V[G_\alpha]$. Let $\langle p_i \mid i < \gamma \rangle$ be a decreasing sequence of conditions of $P_{\alpha,\varepsilon}$ for some $\gamma < \kappa$. Then, by the definition of the quotient forcing [1, Definition 4.1],

$$\{p_i \restriction \alpha \mid i < \gamma\} \subseteq G_\alpha.$$

By (i), there is $p \in G_\alpha$ with $p \leq p_i \restriction \alpha$ for every $i < \gamma$. Define q such that for each $\xi \in \operatorname{supt}(p) \cup \bigcup_{i<\gamma} \operatorname{supt}(p_i)$,

$$q(\xi) = \begin{cases} p(\xi) & \text{if } \xi < \alpha \\ \bigcup_{i<\gamma} p_i(\xi) & \text{if } \xi \geq \alpha. \end{cases}$$

Since $\gamma < \kappa$, Condition (3.6) is satisfied. Therefore $q \in P'_\varepsilon$. It follows that $q \in P_{\alpha,\varepsilon}$ and $q \leq p_i$ for every $i < \gamma$.

(iii) If $\alpha = 0$, then this is 3.13(i). So suppose $\alpha > 0$ and G_α is a P'_α-generic set. Let $P_{\alpha,\varepsilon} = P'_{\alpha,\varepsilon}[G_\alpha]$. Work in $V[G_\alpha]$. Let $N \prec (H(\chi), \in)$ be such that

$$|N| = \kappa,\ \kappa + 1 \subseteq N,\ {}^{<\kappa}N \subseteq N,$$
$$\kappa^+ \cap N = \delta \text{ is an ordinal,}$$
$$P'_{\alpha,\varepsilon} \in N.$$

Let $F = \bigcup\{p(0) \mid p \in G_\alpha\}$ and $t = F(\delta)$ (this is defined as $\operatorname{cf}(\delta) = \kappa$). Let ρ be the least ordinal with $t \restriction \rho \notin N$. Then $\rho \leq \delta$. Let $\langle D_\zeta \mid \zeta < \kappa\rangle$ be an enumeration of dense subsets of $P'_{\alpha,\varepsilon}$ that are in N. Let $p \in P'_{\alpha,\varepsilon} \cap N$. Next we choose a decreasing sequence $\bar{p} = \langle p_\zeta \mid \zeta < \kappa\rangle$ of conditions of $P_{\alpha,\varepsilon}$ such that $p_0 = p$ and for each $\zeta < \kappa$,

$$p_{\zeta+1} \in D_\zeta \cap N.$$

If $\rho < \delta$, then there are no other requirements on how the sequence is chosen. In the case that $\rho = \delta$ the sequence $\bar{p}$ is chosen as follows: Let

$$\pi : \kappa \to (\varepsilon \backslash \alpha) \cap N$$

be a bijection. If ζ is a limit ordinal, then let $p_\zeta \in P'_{\alpha,\varepsilon}$ be any condition that extends p_β for each $\beta < \zeta$. This is possible by (ii).

Suppose $\zeta = \beta + 1$. First pick $p'_\zeta \in D_\beta \cap N$ with $p'_\zeta \leq p_\beta$. Let

$$i = \pi(\beta),$$
$$\langle g'_i, X'_i\rangle = p'_\zeta(i),$$
$$\gamma_i = \max(\sup(\operatorname{dom}(X'_i)), o(g'_i)).$$

(Note that it is possible that $g'_i = \emptyset$ or $X'_i = \emptyset$.) Since $g'_i, X'_i \in N$ and both are of cardinality at most κ, we have $\gamma_i < \delta$. Define p_ζ such that $\operatorname{supt}(p_\zeta) = \operatorname{supt}(p'_\zeta)$ and for each $\xi \in \operatorname{supt}(p_\zeta)$, set

$$p_\zeta(\xi) = \begin{cases} p'_\zeta(\xi) & \text{if } \xi \neq i, \\ \langle g'_i, X'_i \cup \{\langle \gamma_i + 1, t \restriction (\gamma_i + 1)\rangle\}\rangle & \text{if } \xi = i. \end{cases}$$

Clearly $p_\zeta \in P_{\alpha,\varepsilon}$. This ends the description how the sequence $\bar{p}$ is chosen.

By (i), let $q' \in G_\alpha$ be such that it extends each $p_\zeta \restriction \alpha$. For each $\zeta < \kappa$ and $\xi \in \operatorname{supt}(p_\zeta)$, let $\langle g^\zeta_\xi, X^\zeta_\xi\rangle = p_\zeta(\xi)$, and let $\langle g_\xi, X_\xi\rangle = \langle \bigcup_{\zeta<\kappa} g^\zeta_\xi, \bigcup_{\zeta<\kappa} X^\zeta_\xi\rangle$. Define q by

$$q(\xi) = \begin{cases} q'(\xi) & \text{if } \xi < \alpha, \\ \langle g_\xi, X_\xi\rangle & \text{if } \xi \geq \alpha \end{cases}$$

for each $\xi \in \operatorname{supt}(q') \cup \bigcup_{\zeta<\kappa} \operatorname{supt}(p_\zeta)$.

We show by induction on $\zeta \in [\alpha, \varepsilon]$ that $q\restriction\zeta$ is in P'_ζ. Suppose $\zeta = \alpha$. Then $q\restriction\zeta = q' \in G_\alpha$. Suppose ζ is a limit ordinal. It follows from the definition of the κ^+-support iterated forcing that $q\restriction\zeta \in P'_\zeta$.

Suppose $\zeta = \beta + 1$. If $\zeta \notin \operatorname{supt}(q)$, then $q\restriction\zeta = q\restriction\beta$ and it follows that $q\restriction\zeta \in P'_\zeta$. Suppose then that $\zeta \in \operatorname{supt}(q)$. The proof proceeds as the proof of 3.13(i). The only difference is that in case (C) we have to distinguish the cases $\rho < \delta$ and $\rho = \delta$. If $\rho < \delta$, then we have an initial segment of t that is not in N, and therefore we can proceed as in (C). If every initial segment of t is in N, i.e., $\rho = \delta$, then we have, by the choice of the sequence $\bar{p}$, $X_\beta(\gamma_\beta + 1) = t\restriction(\gamma_\beta + 1)$. Therefore the image of a sequence reaching up to level δ in $\mathcal{T}(G(0))$ must diverge from t, which is the only branch cut at level δ in $\mathcal{T}(G(0))$. Hence the union of the image is in $\mathcal{T}(G(0))$.

Thus $q \in P'_\varepsilon$. Since $q\restriction\alpha = q' \in G_\alpha$, $q \in P_{\alpha,\varepsilon}$. So, q is the required $\langle N, P_{\alpha,\varepsilon}\rangle$-generic extension of p. ⊣

CLAIM 3.15. *Suppose G is P_ε-generic set. Then in $V[G]$ there are no κ^+-branches in $\mathcal{T}(G(0))$.*

PROOF. Suppose that $\dot{b}$ is name for a function in ${}^{\kappa^+}\kappa^+$. It is enough to show that the set

$$D = \{q \in P_\varepsilon \mid q \Vdash_\varepsilon \exists\alpha < \kappa^+ (\dot{b}\restriction\alpha \notin \mathcal{T}(\dot{G}(0)))\}$$

is dense in P_ε. Let $p \in P_\varepsilon$. Pick a model $N \prec (H(\chi), \in)$, where χ is large enough, such that

$$|N| = \kappa,\ \kappa + 1 \subseteq N,\ {}^{<\kappa}N \subseteq N,$$
$$p, P_\varepsilon, \dot{b} \in N,$$
$$N \cap \kappa^+ = \delta \text{ is an ordinal.}$$

As ${}^{<\kappa}N \subseteq N$, $\operatorname{cf}(\delta) = \kappa$. Let $\{D_\zeta \mid \zeta < \kappa\}$ enumerate the dense subsets of P_ε that are in N. Choose a descending sequence $\langle p_\zeta \mid \zeta < \kappa\rangle$ of conditions such that

$$p_0 = p,$$
$$p_{\zeta+1} \in D_\zeta \cap N.$$

This is possible as P_ε is κ-closed and ${}^{<\kappa}N \subseteq N$. For every $\alpha < \delta$, the set

$$\{q \in P_\varepsilon \mid q \text{ decides the value of } \dot{b}(\alpha)\}$$

is dense in P_ε and definable from α and $\dot{b}$. Hence it is in N. Therefore the sequence $\langle p_\zeta \mid \zeta < \kappa\rangle$ determines the value of $\dot{b}\restriction\delta$. Let this value be t. By Claim 3.13, it is in V. By the same argument as in the proof of Claim 3.13 we find $g^\zeta_\xi \in V$, $X^\zeta_\xi \in V$ and $\xi' < \kappa$ for every $\xi < \kappa$ and $\zeta \in \operatorname{supt}(p_\xi)$ such that

$$p_{\xi'}\restriction\zeta \Vdash_\varepsilon p_\xi(\zeta) = \langle g^\zeta_\xi, X^\zeta_\xi\rangle.$$

Again, let $g_\zeta = \bigcup_{\xi<\kappa} g^\zeta_\xi$ and $X_\zeta = \bigcup_{\xi<\kappa} X^\zeta_\xi$, and define q by

$$q(\zeta) = \begin{cases} \bigcup_{\xi<\kappa} p_\xi(0) \cup \{\langle\delta, t\rangle\} & \text{if } \zeta = 0 \\ \langle g_\zeta, X_\zeta\rangle & \text{otherwise.} \end{cases}$$

We show by induction on ζ that $q\restriction\zeta$ is a condition. The case $\zeta = 1$ is clear since $\operatorname{cf}(\delta) = \kappa$. Suppose that ζ is a limit ordinal and for all $\zeta' < \zeta$, $q\restriction\zeta'$ is a condition.

It follows from the definition of the κ^+-support iterated forcing notion that $q\restriction\zeta$ is a condition.

Suppose then that $\zeta = \beta + 1$. If $\zeta \notin \operatorname{supt}(q)$, then $q\restriction\zeta = q\restriction\beta$. Suppose that $\zeta \in \operatorname{supt}(q)$. Since $\operatorname{supt}(q) \subseteq N$, $\zeta \in N$. Clearly Conditions (3.1) – (3.5) are satisfied. So we verify only Condition (3.6). Let $\langle t_\xi \mid \xi < \kappa\rangle$ be an increasing sequence of elements of $\operatorname{dom}(g_\beta)$. Let $u = \bigcup_{\xi<\kappa} g_\beta(t_\xi)$ and let $\gamma = \operatorname{dom}(u)$. Then $\operatorname{cf}(\gamma) = \kappa$.

(A) If there is ξ^* such that every t_ξ is in $\operatorname{dom}(g^\beta_{\xi^*})$, then $\bigcup_{\xi<\kappa} g^\beta_{\xi^*}(t_\xi) = u$ and $q\restriction\beta$ forces "$u \in \mathscr{T}(\dot{G}(0))$" as required.

(B) Suppose $\gamma < \delta$. Then there are $\xi^*, \xi' < \kappa$ with

$$o(g^\beta_{\xi^*}) > \gamma,\ \operatorname{dom}(g^\beta_{\xi^*}(t_{\xi'})) > \operatorname{dom}(X^\beta_{\xi^*}(\gamma))$$

from which it follows that $X_\beta(\gamma) \nsubseteq u$. Hence $q\restriction\beta$ forces "$u \in \mathscr{T}(\dot{G}(0))$".

(C) Suppose $\gamma = \delta$. Clearly

(7) $$\Vdash_\varepsilon \{\alpha < \kappa^+ \mid \dot{b}\restriction\alpha \in {}^\alpha\alpha \wedge \operatorname{cf}(\alpha) = \kappa\} \text{ is a } \kappa\text{-cub in } \kappa^+.$$

By 3.14(iii), we have

(8) $$\Vdash_\varepsilon \dot{S}_\beta \text{ is a } \kappa\text{-bistationary subset of } \kappa^+.$$

By combining the above two observations and using the maximal principle we find a P_ε-name $\dot{\alpha}$ for an ordinal with

(9) $$\Vdash_\varepsilon \dot{\alpha} < \kappa^+ \wedge \dot{b}\restriction\dot{\alpha} \in {}^{\dot{\alpha}}\dot{\alpha} \wedge \dot{\alpha} \notin \dot{S}_\beta \wedge \operatorname{cf}(\dot{\alpha}) = \kappa.$$

As N is an elementary submodel of $H(\chi)$ and χ is large enough, the observations (7) – (9) hold also in N. Clearly the set

$$D_{\dot{\alpha}} = \{r \in P_\varepsilon \mid \exists\alpha < \kappa^+(r \Vdash_\varepsilon \dot{\alpha} = \alpha)\}$$

is dense in P_ε and in N (we may assume that the name $\dot{\alpha}$ is in N). Therefore for some $\xi < \kappa$ and $\alpha < \delta$,

$$p_\xi \Vdash_\varepsilon\ \dot{\alpha} = \alpha \wedge t\restriction\alpha \text{ is not an } \dot{S}_\beta\text{-node.}$$

On the other hand, by the definition of the condition q, we have for all $\gamma < \delta$,

$$q \Vdash_\varepsilon u\restriction\gamma \text{ is an } \dot{S}_\beta\text{-node.}$$

Combining the above observations with the fact that $q \leq p_\xi$ yields

$$q \Vdash_\varepsilon u \neq t.$$

As q forces "t is the only branch cut at the level δ", it forces "$u \in \mathscr{T}(\dot{G}(0))$", too. Hence q is a condition. Clearly it extends p and it forces "$\dot{b}\restriction\delta = t \wedge t \notin \mathscr{T}(\dot{G}(0))$". Thus it is in D. ⊣

Claim 3.16. *For each $\alpha \leq \varepsilon$ the identity mapping from P'_α to P_α is a dense embedding.*

Proof. This follows immediately from Claim 3.13(iii). ⊣

Claim 3.17. *For each $\alpha < \varepsilon$, $|P'_\alpha| \leq \kappa^+$.*

Proof. This follows from GCH. ⊣

CLAIM 3.18. *For all $\alpha \leq \varepsilon$, P'_α and P_α have the ε-c.c.*

PROOF. For $\alpha < \varepsilon$, by the previous claim, P'_α has the ε-c.c.

Let $\{p_\zeta \mid \zeta < \varepsilon\} \subseteq P'_\varepsilon$ be of cardinality ε. By the Δ-system lemma, there is $I \subseteq \varepsilon$ of cardinality ε such that the set $\{\mathrm{supt}(p_\zeta) \mid \zeta \in I\}$ forms a Δ-system with root d. As d is of cardinality at most κ, $\sup(d) = \xi < \varepsilon$. Since $|P'_\xi| \leq \kappa^+$ there are ζ_0 and ζ_1 with $p_{\zeta_0}\restriction d = p_{\zeta_1}\restriction d$. But then p_{ζ_0} and p_{ζ_1} are compatible.

By Claim 3.16, also P_α has the ε-c.c. for every $\alpha \leq \varepsilon$. ⊣

CLAIM 3.19. *For all $\alpha < \varepsilon$ there are at most ε nice P'_α-names for a subset of $(\kappa^+)\check{}$.*

PROOF. By Claim 3.18, P'_α has the ε-c.c. Hence the number of nice P'_α-names for a subset of $(\kappa^+)\check{}$ is

$$\left(|P'_\alpha|^{\kappa^+}\right)^{\kappa^+} = \left(\kappa^{+\kappa^+}\right)^{\kappa^+}.$$

It follows from GCH that $\left(\kappa^{+\kappa^+}\right)^{\kappa^+} = \kappa^{++} = \varepsilon$. ⊣

CLAIM 3.20. *For every $\alpha < \varepsilon$ and nice P_α-name $\dot{S}$ for a subset of $(\kappa^+)\check{}$ there is a nice P'_α-name $\dot{S}'$ for a subset of $(\kappa^+)\check{}$ with*

$$\Vdash_{P_\alpha} \dot{S} = \dot{S}'.$$

PROOF. Suppose that $\dot{S}$ is a nice P_α-name for a subset of $(\kappa^+)\check{}$. Let $\nu = |\dot{S}|$ and let $\{\langle \tau_\zeta, p_\zeta\rangle \mid \zeta < \nu\}$ enumerate $\dot{S}$. For each $\zeta < \nu$, choose an antichain $A_\zeta \subseteq P'_\alpha$ such that A_ζ is a maximal antichain below p_ζ and contains only conditions stronger than p_ζ. (If $p_\zeta \in P'_\alpha$ then let $A_\zeta = \{p_\zeta\}$.) This is possible by Claim 3.16. Let

$$\dot{S}' = \bigcup_{\zeta<\nu}(\{\tau_\zeta\} \times A_\zeta).$$

Clearly $\Vdash_{P_\alpha} \dot{S} = \dot{S}'$. ⊣

Now we continue the proof of Theorem 3.10. By Claim 3.13, forcing with P_ε preserves all cardinals less than or equal to κ^+. By Claim 3.18, it does not collapse ε.

We have to ensure that we have added an order preserving function for every κ-bistationary subset of κ^+ in a P_ε-generic extension. To that end choose in the definition of $\bar{Q}$ the enumeration $\{\dot{S}_\beta \mid \beta < \varepsilon\}$ so that for each $\alpha < \varepsilon$ every nice P'_α-name for a κ-bistationary subset of $(\kappa^+)\check{}$ appears in the enumeration with an index greater than α. By Claim 3.19, this is possible. Suppose that G is a P_ε-generic set and S is a κ-bistationary subset of κ^+ in $V[G]$. Let $\dot{S}$ be a nice P_ε-name for a κ-bistationary subset of κ^+ with $S = \dot{S}[G]$. Since P_ε has the ε-c.c. there is $\eta < \varepsilon$ such that $\dot{S}$ is P_η-name. By Claim 3.20, an equivalent P'_η-name appears in the enumeration $\{\dot{S}_\beta \mid \beta < \varepsilon\}$. Hence there is an order preserving function from $T(S)$ to $\mathcal{T}(G(0))$ in $V[G]$.

By Claim 3.13 forcing with P_ε does not add new subsets of κ. Hence all cardinals up to κ are preserved and GCH holds up to κ. Since P_ε is κ-proper, it preserves κ^+ (this follows also from Claim 3.13(ii)). All cardinals above κ^+ are preserved as P_ε has the κ^{++}-c.c. Suppose that $\lambda \geq \kappa^+$. Then $2^\lambda \leq \left(((|P'_\varepsilon|^{\kappa^+})^\lambda)\right)^V$ in $V[G \cap P'_\varepsilon] = V[G]$. But $|P'_\varepsilon| = \kappa^{++}$ which yields $2^\lambda \leq \left((\kappa^{++})^\lambda\right)^V$. Using GCH and the assumption that $\lambda \geq \kappa^+$, we get $2^\lambda = \lambda^+$ in $V[G]$. ⊣

§4. A forcing extension where there are no κ-canary trees. The proof of the following theorem is essentially from [3], but for the sake of completeness we give it.

THEOREM 4.1. *Assume* GCH. *Suppose κ is a regular cardinal. Then there is a partial order P such that there are no κ-canary trees in any P-generic extension, but all cardinals are preserved and* GCH *still holds.*

PROOF. We claim that in every $\mathrm{Fn}(\kappa^{++}, 2, \kappa^+)$-generic extension there are no κ-canary trees. Towards a contradiction assume that G is an $\mathrm{Fn}(\kappa^{++}, 2, \kappa^+)$-generic set and T is a κ-canary tree in $V[G]$. Since we assumed GCH in the ground model, the partial order $\mathrm{Fn}(\kappa^{++}, 2, \kappa^+)$ has the κ^{++}-c.c.. It is also κ^+-closed. Hence it preserves all cardinals. It also follows that GCH holds in $V[G]$.

We can regard $\mathrm{Fn}(\kappa^{++}, 2, \kappa^+)$ as a product of length κ^{++} of partial order $\mathrm{Fn}(\kappa^+, 2, \kappa^+)$ with support less than κ^+, i.e., the partial order

$$P_{\kappa^{++}} = \left\{ f \in \prod_{\xi \in \kappa^{++}} \mathrm{Fn}(\kappa^+, 2, \kappa^+) \mid |\mathrm{supt}(f)| < \kappa^+ \right\}$$

ordered coordinatewise is isomorphic to $\mathrm{Fn}(\kappa^{++}, 2, \kappa^+)$. Thus, we regard G as a $P_{\kappa^{++}}$-generic set.

Since T is of cardinality κ^+ in $V[G]$ and $P_{\kappa^{++}}$ has the κ^{++}-c.c., there are $A \subseteq \kappa^{++}$ of cardinality κ^+ and a P_A-name $\dot{T}$ with $T = \dot{T}[G]$. Let $\alpha = \min(\kappa^{++} \setminus A)$, and $B = \kappa^{++} \setminus \{\alpha\}$. Then $T \in V[G_B]$ and $P_{\kappa^{++}} \cong P_B \times P_{\{\alpha\}}$. Working in $V[G_B]$, let $\dot{S}$ be a $P_{\{\alpha\}}$-name for the κ-bistationary set $\{\zeta < \kappa^+ \mid (\bigcup G_{\{\alpha\}})(\zeta) = 1\}$ and $\dot{Q}$ a $P_{\{\alpha\}}$-name for the partial order $T(\dot{S})$. By GCH in $V[G]$, $I[\kappa^+]$ is improper in $V[G]$. Thus, forcing with Q adds a κ-cub into the κ-stationary set S without adding subsets of cardinality at most κ [2]. Therefore in every $P_{\{\alpha\}} * \dot{Q}$-generic extension over $V[G_B]$ there is a κ^+-branch in T as T is assumed to be a κ-canary tree. Let τ be a $P_{\{\alpha\}} * \dot{Q}$-name for this branch. First we note that $\dot{Q}[G_{\{\alpha\}}] \subseteq V$ since $P_{\kappa^{++}}$ is κ^+-closed. Choose recursively conditions $\langle p_i, s_i \rangle \in P_{\{\alpha\}} * \dot{Q}$ and ordinals β_i for $i < \kappa^+$ as follows: First let $\langle p_0, s_0 \rangle = 1$ and $\beta_0 = 0$, and then for each $i < \kappa^+$, choose an extension $\langle p'_{i+1}, \dot{s}_{i+1} \rangle$ of $\langle p_i, s_i \rangle$ which decides the value of τ at level i. Then choose an extension p''_{i+1} of p'_{i+1} that decides the value of $\dot{s}_{i+1}$. Let this value be s'_{i+1}, and set

$$\begin{aligned}
\beta_{i+1} &= \bigcup \left(\mathrm{dom}(p''_{i+1}) \cup \mathrm{ran}(s'_{i+1}) \right) + 1, \\
p_{i+1} &= p''_{i+1} \cup \{\langle \beta_{i+1}, 1 \rangle\}, \\
s_{i+1} &= s'_{i+1} \frown \langle \beta_{i+1} \rangle,
\end{aligned}$$

and for every limit $\delta < \kappa^+$,

$$\begin{aligned}
\beta_\delta &= \bigcup \{ \sup(\mathrm{dom}(p_i)) \mid i < \delta \}, \\
p_\delta &= \bigcup_{i<\delta} p_i \cup \{\langle \beta_\delta, 1 \rangle\}, \\
s_\delta &= \bigcup_{i<\delta} s_i \frown \langle \beta_\delta \rangle.
\end{aligned}$$

Note that for a limit ordinal $\delta < \kappa^+$,

$$\bigcup \{\sup(\mathrm{dom}(p_i)) \mid i < \delta\} = \bigcup \{\sup(\mathrm{ran}(s_i)) \mid i < \delta\}.$$

This ensures that p_δ forces "$s_\delta \in \dot{Q}$". For successor case, since p''_{i+1} decides the value of $\dot{s}_{i+1}$, we have p_{i+1} forces "$s_{i+1} \in \dot{Q}$". Thus, $\langle\langle p_i, s_i\rangle \mid i < \kappa^+\rangle$ is a descending sequence of conditions in $V[G_B]$ such that $\langle p_i, s_i\rangle$ decides the value of τ at level i. Thus there is a κ^+-branch in T in $V[G_B]$. This contradicts the assumption that T is a κ-canary tree in $V[G]$. ⊣

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