EQUIVALENT VISCOUS DAMPING FOR A BILINEAR HYSTERETIC OSCILLATOR

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ABSTRACT: A bilinear hysteretic model is commonly used to study elastoplastic structures. In this paper, a damped, bilinear hysteretic oscillator is studied under harmonic loading. We show the existence of an equivalent viscous damping for small values of a loading parameter such that the associated linear structure and the hysteretic structure have the same frequency response curves. We use the Kryloff-Bogoliuboff method of averaging to find the equivalent viscous damping as a function of the steady state amplitude. We present a model of a bilinear elastic oscillator which captures the steady-state dynamics of the hysteretic oscillator for low values of the loading parameter. We also study the nature of the dependence of the equivalent viscous damping on the kinematic hardening parameter.

INTRODUCTION

A bilinear hysteretic model is used to study the response of various elastoplastic structures. For fundamental analysis and understanding, a single-degree-of-freedom model often suffices. Much of the literature on studies of elastoplastic structures is based on this model. There have been numerous studies on the forced response of elastoplastic structures (Tanabashi 1956; Caughey 1960; Iwan 1964; Ballio 1970; Pratap et al. 1994; Savi and Pacheco 1997). It is often convenient to study the steady-state dynamics of an elastoplastic oscillator by considering an equivalent linear structure (Jacobsen 1960; Berg 1965; Jennings 1968; Iwan and Gates 1979). Jennings (1968) compares the various ways in which an equivalent linear structure can be defined to get frequency and amplitude matching. He considers an undamped, elasto-perfectly-plastic oscillator under harmonic excitation. Iwan and Gates (1979) compare the accuracy of the various methods for defining equivalent linear systems by considering a damped bilinear hysteretic oscillator subjected to earthquake loading and harmonic excitation.

The resonant amplitude method, dynamic stiffness method, and dynamic mass method are some of the methods used for defining equivalent linear systems. In all the above methods, the frequency shift exhibited by hysteretic systems cannot be taken care of by defining the equivalent viscous damping c_{eq} alone. In this paper, we address the issue of whether the steady-state dynamics of a hysteretic oscillator can be completely captured by an equivalent linear system by just defining a new equivalent viscous damping. We analyze the response of a damped single-degree-of-freedom oscillator, having bilinear hysteresis, to harmonic forcing. For small values of the forcing amplitude and damping, it is shown that the steady-state response of the hysteretic oscillator is represented completely in terms of an equivalent linear structure by finding an equivalent viscous damping. The bilinear hysteretic oscillator and the bilinear elastic oscillator both show soft resonance behavior. We use this behavior to get a better frequency and amplitude match by studying a model of the bilinear nonhysteretic oscillator. The functional relationship between the nondimensional equivalent viscous damping defined in the model and the kinematic hardening parameter is also studied.

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EQUATION OF MOTION

The equation of motion of a forced, damped bilinear hysteretic oscillator is written as

$$m\ddot{y} + c\dot{y} + F(y, \dot{y}) = P \cos \Omega t \quad (1)$$

F is the hysteretic restoring force, c represents the damping coefficient, and y is the displacement. P is the amplitude of the forcing with frequency Ω. The nondimensional equation of motion is given by

$$\ddot{x} + c_0\dot{x} + f(x, \dot{x}) = p \cos \omega\tau \quad (2)$$

where $x = y/y_0$; $\tau = \omega_0 t$; $\omega_0^2 = k/m$; $f = F/F_0$; $p = P/F_0$; $c_0 = c/\sqrt{km}$; $\omega = \Omega/\omega_0$. The force-deflection diagram is shown in Fig. 1. The nondimensional restoring force f is not a single-valued function of the nondimensional displacement x. It depends upon x as well as the sign of $\dot{x}$. For the plastic flow rule, we have assumed kinematic hardening. This essentially means that the total range of displacement in the elastic phase remains the same, irrespective of the net plastic displacement (Mendelson 1968). The parameter η^2 represents the kinematic hardening parameter; $\eta^2 = 0$ would mean a perfectly plastic case, and $\eta^2 = 1$ a perfectly elastic case.

APPROXIMATE FREQUENCY RESPONSE EQUATION

Here we obtain an approximate response of the hysteretic oscillator to the applied harmonic load by using the method of averaging on the harmonic response of the oscillator, ignoring hysteresis. We first assume the solution of the form

$$x(\tau) = x_s \cos(\omega\tau + \phi) \quad (3)$$

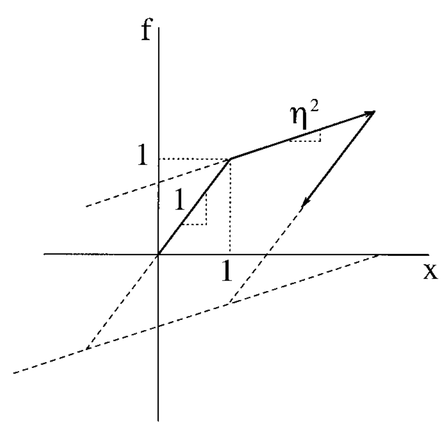

FIG. 1. Force Deflection Diagram for Bilinear Hysteresis

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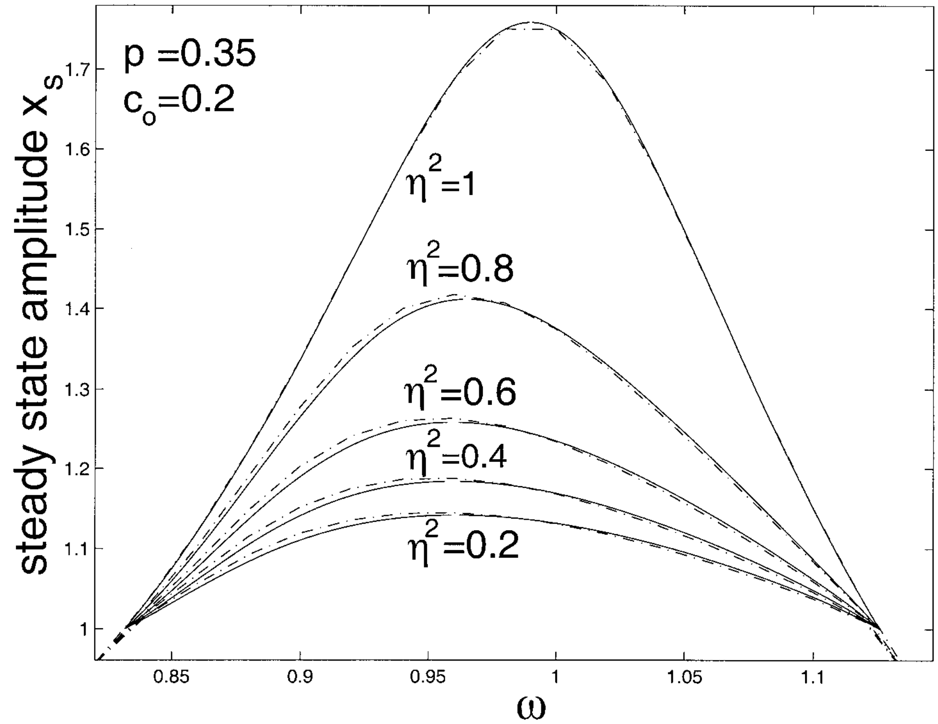

FIG. 2. Comparison of Approximate Solution (Solid Curve) against Numerical Solution (Dotted-Dashed Curve)

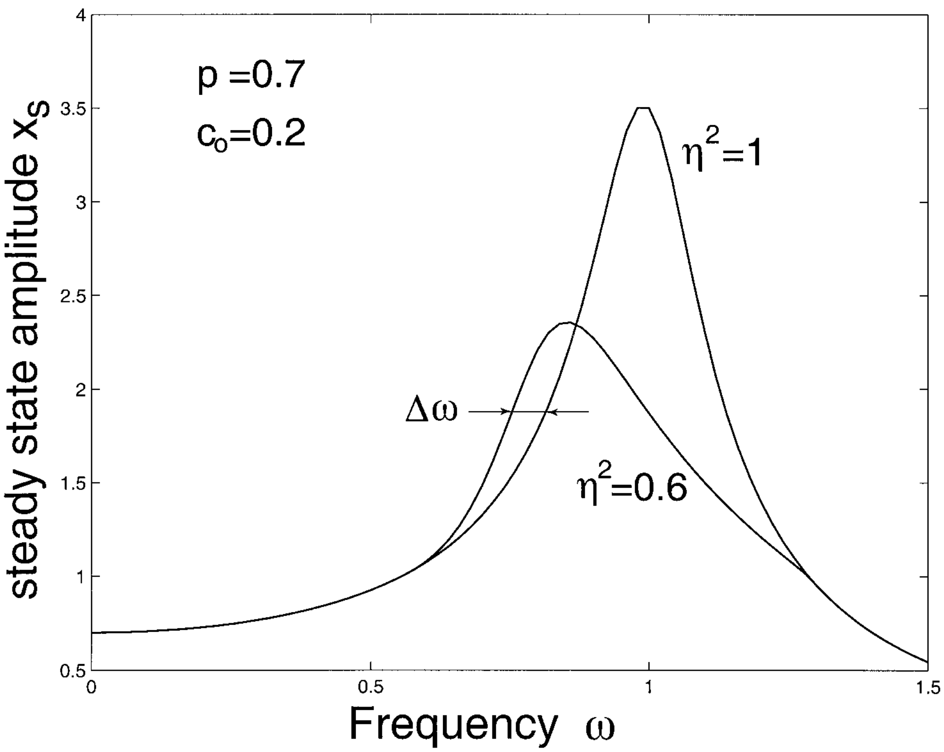

FIG. 3. Typical Frequency Response Curve

where x_s and ϕ are slowly varying parameters of τ. Using the method of slowly varying parameters, and for $\omega \approx 1$, we get (see Appendix I for details of the calculation)

$$\omega^2 = \frac{(2\beta - c_0^2) \pm \sqrt{(2\beta - c_0^2)^2 - 4(\alpha^2 + \beta^2 - \gamma^2 - 2c_0\alpha)}}{2} \tag{4}$$

The variables γ, β, and α are functions of the amplitude of motion x_s (see Appendix I). Hence, for constant c_0, p, and η^2, (4) can be written as

$$\omega^2 = G(x_s) \tag{5}$$

where $G(x_s)$ represents the functional dependence of ω^2 on x_s. Fig. 2 shows a comparison between the exact (numerical) solution and the approximate solution. It is seen that the method of averaging approximates the exact solution quite well, even for values of ω away from unity.

EQUIVALENT VISCOUS DAMPING

Fig. 3 shows a typical frequency response curve for a bilinear hysteretic oscillator, which exhibits soft resonance. To find

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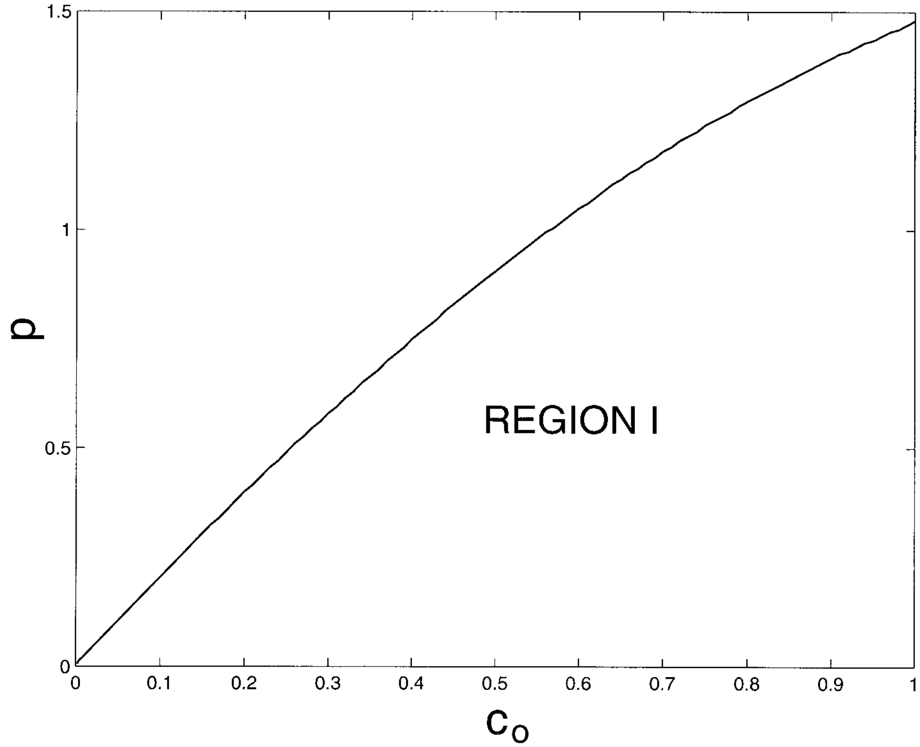

FIG. 4. Region I Giving Values of p and c_0 for Which Curves for $\eta^2 \in [0, 1)$ Fall within Curve for $\eta^2 = 1$

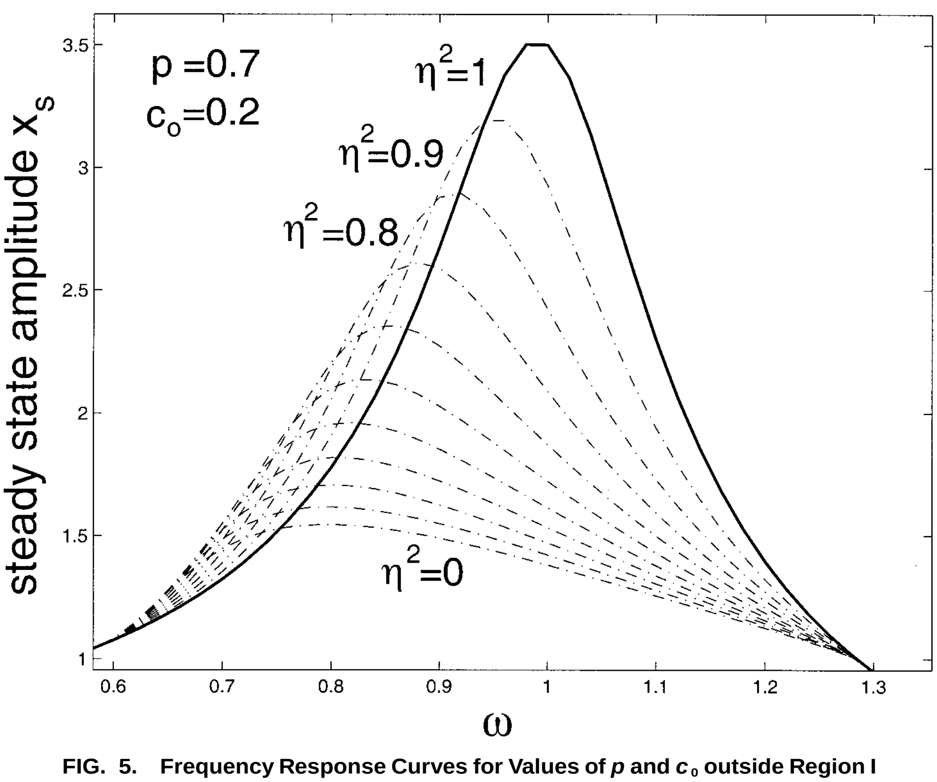

FIG. 5. Frequency Response Curves for Values of p and c_0 outside Region I

the extent of frequency shift, we can find out the difference between the frequency ω_{η^2} for a given $\eta^2 \in [0, 1)$ and ω_1 for $\eta^2 = 1$ for different values of x_s. A positive value of $(\omega_{\eta^2} - \omega_1) = \Delta\omega$ will mean that the curve for a given η^2 falls inside the curve for $\eta^2 = 1$, and a negative value will mean that the curve falls outside the curve for $\eta^2 = 1$. We can find the values of p and c_0 for which the frequency response curves for all $\eta^2 \in [0, 1)$ fall within the curve for $\eta^2 = 1$. See Fig. 4. For all p and c_0 within Region I, we get frequency response curves for all $\eta^2 \in [0, 1)$ within the curve for $\eta^2 = 1$. For small values of c_0, the line separating Region I is almost linear and has a slope equal to 2. Fig. 5 shows the case for values of p and c_0 outside Region I. The above observations lead to the following propositions.

Proposition 1. In all steady-state motions of the form $x_s \cos(\omega\tau + \phi)$ involving plastic cycles, for small values of c_0 and for $(p/c_0) < 2$, there exists $c_e > 0$ for all $\eta^2 \in [0, 1)$, where $c_e = c_e(x_s, \eta^2)$, such that the following linear equation

$$\ddot{x} + (c_0 + c_e)\dot{x} + x = p \cos \omega\tau \tag{6}$$

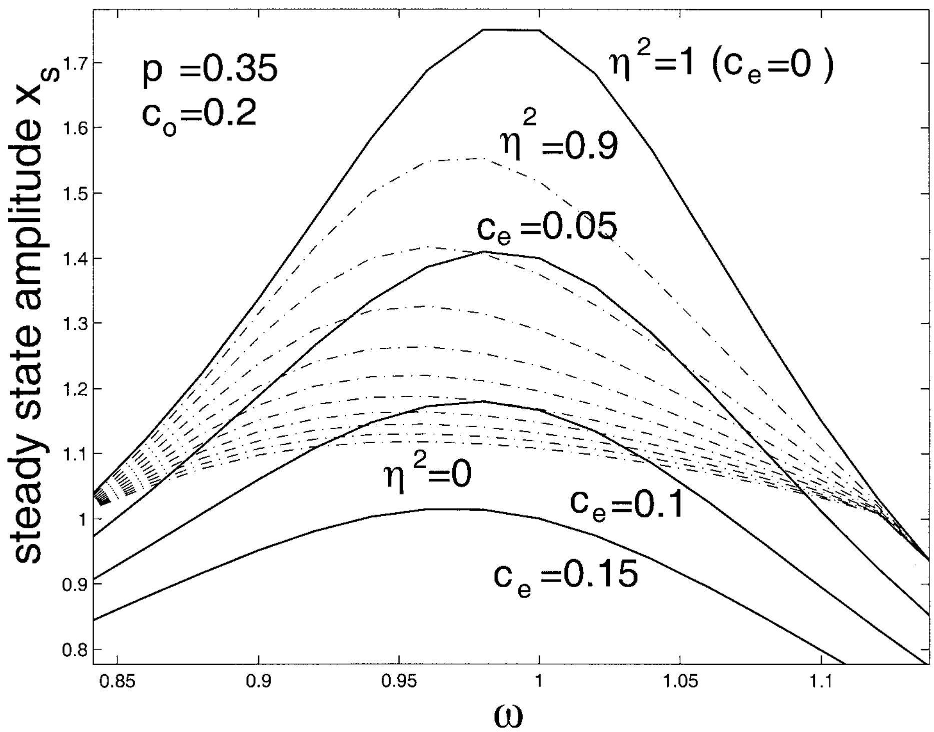

FIG. 6. Frequency Response Curves for Different Values of η^2 (Dotted-Dashed Curve) and c_e (Solid Curve) for Same p and c_0

and (2) are equivalent, that is, the frequency response curves are the same for a given p and c_0. We define c_e as the equivalent viscous damping.

Proof: The proof is based upon the nature of the frequency response curves for different values of $\eta^2 \in [0, 1)$ for values of $(p/c_0) < 2$ (Fig. 6). It is seen that all curves fall within the curve for $\eta^2 = 1$. We also know that the frequency response curves of the associated linear system given by (6) with increasing values of c_e lie within the curve for $\eta^2 = 1$ (or $c_e = 0$). Since the hysteretic curves and the curves for increasing constant values of c_e start and end at different points respectively, the intersection of these curves is always transversal, except for the case discussed in the next proposition. This means that every point on the curve with a particular η^2 is intersected by a curve with a particular constant value of c_e. Since there is a continuous one-to-one mapping between the curves with different constant values of c_e, every point on a curve with a particular η^2 has a unique c_e such that the amplitude is the same at the same frequency. This shows that the frequency response for a particular η^2 can be exactly represented by (3) with a variable c_e that depends on x_s.

Proposition 2. For small values of c_0 and $(p/c_0 < 2$, there exists a constant c_e for a particular $\eta^2 \in [0, 1)$, such that the frequency response curve with c_e is tangent to the frequency response curve with η^2 for a given p and c_0.

Proof: To prove Proposition 2, it is enough to note that if a particular curve with a constant c_e intersects a curve for a particular η^2, it does so in at most two points. Now there is a continuous one-to-one mapping between the curve with a constant c_e that intersects with the curve for η^2 and one with a different value of c_e that does not intersect. This implies that there exists one curve between these two curves that is tangent to the η^2 curve (Fig. 6).

APPROXIMATE EXPRESSION FOR THE EQUIVALENT VISCOUS DAMPING

In case of the linear oscillator given by (6), we can express c_e as a function of x_s and ω as

$$c_e = \sqrt{\frac{\gamma^2 - (1 - \omega^2)^2}{\omega^2}} - c_0 \quad (7)$$

Substituting for ω^2 from (5), we have

$$c_e = \sqrt{\frac{\gamma^2 - (1 - G(x_s))^2}{G(x_s)}} - c_0 \quad (8)$$

As η^2 decreases, the range of values that c_e takes increases because the amount of plastic damping increases.

BILINEAR ELASTIC OSCILLATOR AND EQUIVALENT VISCOUS DAMPING

The frequency response curves of both the bilinear nonhysteretic (elastic) oscillator and the bilinear hysteretic oscillator show a leftward shift. This behavior is typical of soft springs (Nayfeh and Mook 1979). Because the linear system does not exhibit soft spring behavior, we do not generally get frequency match when the amplitudes are matched by defining the equivalent viscous damping only. However, since the bilinear hysteretic and the nonhysteretic oscillator show the same behavior as far as frequency response curves are concerned (for low loading parameters), we can expect a better frequency match when the amplitudes are matched in the equivalent bilinear nonhysteretic oscillator.

Hysteretic damping comes into play when the oscillator goes into the plastic regime ($|x| > 1$ in the nondimensional model). So, the only way this can be captured in the bilinear nonhysteretic oscillator is by introducing additional damping in the branch with slope η^2. The equivalent viscous damping, which would be a function of the loading parameter p and viscous damping c_0, would be just that value which gives the same resonant amplitude for a given value of η^2. The value of c_0 together with p decides the values of ω, for which we have $|x| > 1$. Since we are introducing damping partly, the range of these values of ω remains unaffected, which is not the case when we introduce additional damping throughout.

The bilinear nonhysteretic model studied is described by the following equations:

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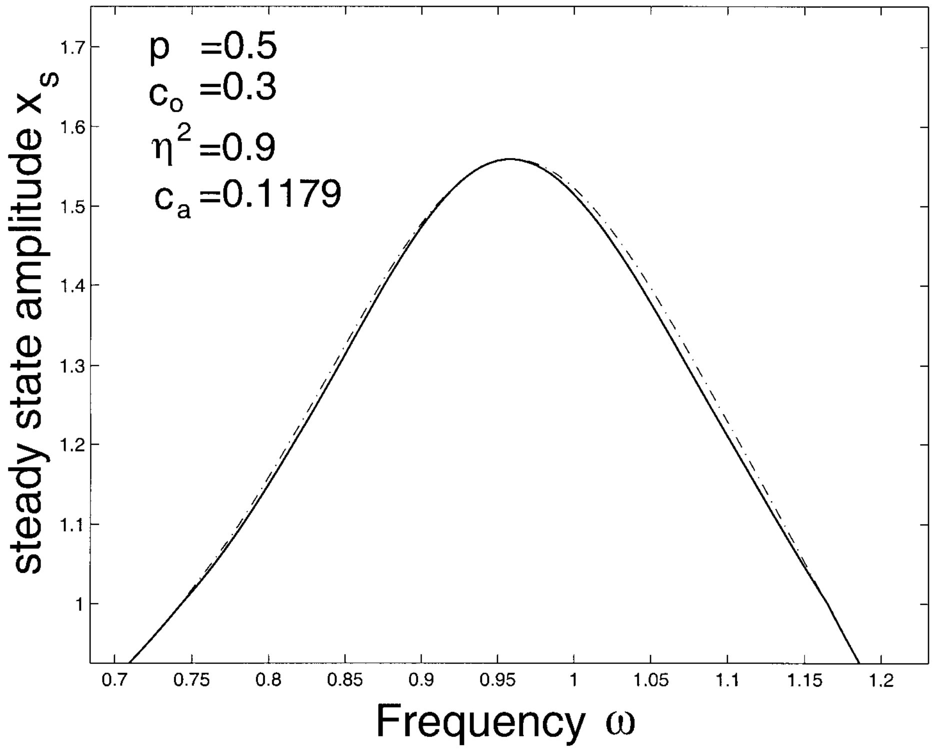

FIG. 7. Frequency Response Curves of Bilinear Hysteretic Oscillator (Solid Curve) and Bilinear Nonhysteretic Oscillator (Dotted-Dashed Curve) with Corresponding Value of c_a for Resonant Amplitude Match

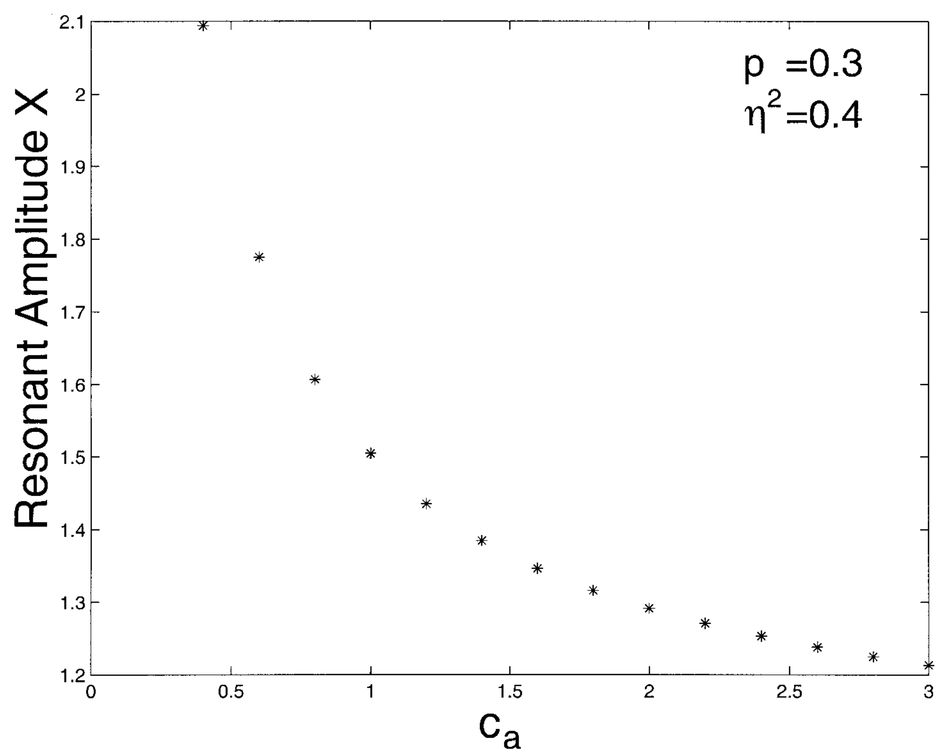

FIG. 8. Variation of X against c_a for Given Values of p and η^2 for Bilinear Nonhysteretic Oscillator

$$\ddot{x} + d(\dot{x}) + s(x) = p\cos\omega t \tag{9}$$

where

$$d(\dot{x}) = \begin{cases} c_0\dot{x} & |x| \le 1 \\ (c_0 + c_a)\dot{x} & |x| > 1 \end{cases}$$

$$s(x) = \begin{cases} x & |x| \le 1 \\ \eta^2(x - 1) + \text{sign}(x) & |x| > 1 \end{cases}$$

and

$$\text{sign}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

The value of c_a, which gives the same resonant amplitude for a given η^2, is defined as the new nondimensional equivalent viscous damping C_e for a given p and c_0. Fig. 7 shows that we get a resonant frequency match when the resonant amplitudes are matched.

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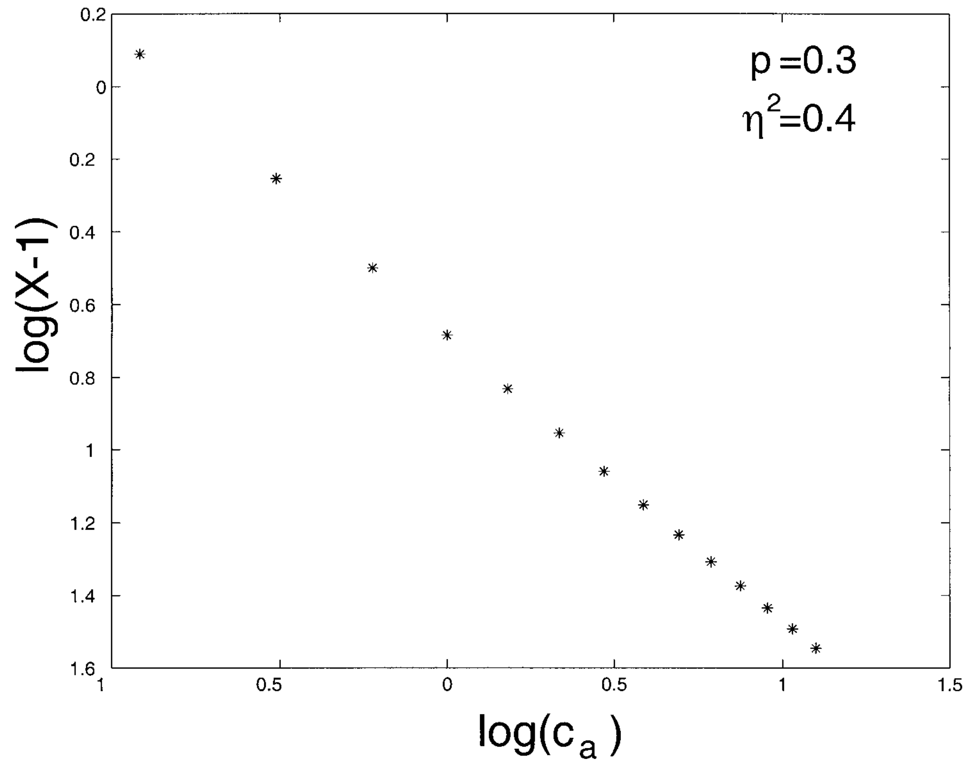

FIG. 9. Logarithmic Plot of $X - 1$ against c_a for Bilinear Nonhysteretic Oscillator

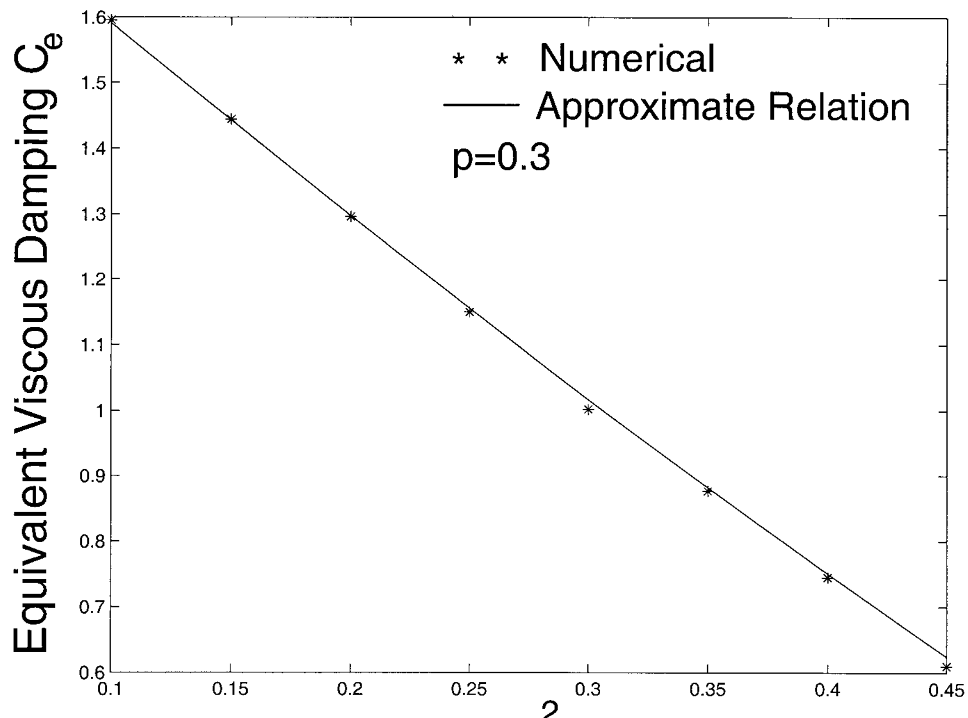

FIG. 10. Comparison of Approximate Relation between C_e and η^2 with Values Obtained Numerically

FUNCTIONAL RELATIONSHIP BETWEEN C_e AND η^2: UNDAMPED CASE

Consider an undamped ($c_0 = 0$) hysteretic oscillator to seek a relationship between the equivalent viscous damping C_e and the kinematic hardening parameter η^2. For the undamped hysteretic oscillator, we have a simple relation between the resonant amplitude X and η^2 for a given p (Caughey 1960).

$$X = \frac{4(1 - \eta^2)}{4(1 - \eta^2) - \pi p} \tag{10}$$

If we can get a relation between the resonant amplitude X and c_a for the bilinear nonhysteretic oscillator, for a range of values of η^2 and some given p, then equating this relation with (10) would give us the desired relationship between C_e and η^2. It is clear that as c_a takes large values, X_s will tend to unity.

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See Fig. 8, which suggests an exponential relationship between X and c_a for a given η^2 and p. This would mean that we have unity on the right hand side of the probable relation between X and c_a. A logarithmic plot of $X - 1$ versus c_a gives a straight line (Fig. 9).

It is clear that the relation between X and c_a for the range of chosen values of η^2 is of the form

$$X = Bc_a^A + 1 \tag{11}$$

The values of η^2 are chosen away from the value that gives unbounded resonance ($\eta^2 = 1 - \pi p/4$) for a given p. Also the values of c_a start away from zero to circumvent the difficulty of finding the resonant amplitude numerically. A is the slope of the straight line obtained from the logarithmic plot, and B is equal to the exponential of the vertical intercept. A and B depend upon η^2 for a given p. It has been observed that the variation of A and B with respect to η^2 is very slight. A and B are calculated using average values of slope and the vertical intercept over the range of values of η^2 used. $A = -0.8$ and $B = 0.5142$ in our case for $p = 0.3$. Equating (10) and (11), and noting that the value of c_a that gives the same resonant amplitude is C_e, we have

$$C_e = \left(\frac{\pi p}{B(4(1 - \eta^2) - \pi p)}\right)^{1/A} \tag{12}$$

It can be seen from Fig. 10 that the approximate relationship given by (12) compares well with the values obtained numerically.

The relation in (12) is a quick way to obtain a measure of the dissipation in a structure that shows hysteresis. Engineers are primarily concerned with the resonant amplitude and resonant frequency of a structure. The preceding discussion shows that it is possible to study the steady-state dynamics of a somewhat involved model of a hysteretic structure by studying a simpler nonhysteretic model with equivalent damping.

CONCLUSIONS

It is shown that for certain values of the loading parameter and viscous damping that is present throughout, the frequency response of a single-degree-of-freedom damped bilinear hysteretic oscillator under harmonic loading can be represented exactly by the frequency response of a linear oscillator under the same loading by introducing a variable equivalent viscous damping. An approximate implicit relation between the equivalent viscous damping and the steady-state amplitude is derived using the Kryloff-Bogoliuboff method of averaging. The ''soft'' type of resonance exhibited by the damped bilinear hysteretic oscillator and the bilinear nonhysteretic oscillator is used to suggest a model of the bilinear nonhysteretic oscillator with extra damping introduced to capture the steady-state dynamics of the hysteretic oscillator. Resonant amplitudes are matched, and the corresponding extra damping is termed the equivalent viscous damping. It is observed that the resonant frequencies are nearly the same when the resonant amplitudes are matched. An approximate relation between the equivalent viscous damping in the case of the bilinear nonhysteretic oscillator and the kinematic hardening parameter is derived for a given value of the loading parameter.

APPENDIX I. APPROXIMATE FREQUENCY RESPONSE EQUATION

Let $\mu = 1 - \eta^2$. Hence, for the case $\mu = 0$, we have

$$x(\tau) = x_s \cos(\omega\tau + \phi) \tag{13}$$

$$\dot{x}(\tau) = -\omega x_s \sin(\omega\tau + \phi) \tag{14}$$

where x_s and ϕ are constants. For nonzero but small μ, assume a solution of the form

$$x(\tau) = x_s \cos(\omega\tau + \phi) \tag{15}$$

where x_s and ϕ are slowly varying parameters of τ. Therefore the velocity is given by

$$\dot{x}(\tau) = -\omega x_s \sin\theta + \dot{x}_s \cos\theta - x_s\dot{\phi} \sin\theta \tag{16}$$

where $\theta = \omega\tau + \phi$ for simplicity. Now, from (14), we get

$$\dot{x}(\tau) = -\omega x_s \sin(\theta) \tag{17}$$

Comparing (16) and (17) we have

$$\dot{x}_s \cos\theta - x_s\dot{\phi} \sin\theta = 0 \tag{18}$$

Similarly

$$\ddot{x}(\tau) = -\omega^2 x_s \cos\theta - \omega\dot{x}_s \sin\theta - \omega x_s\dot{\phi} \cos\theta \tag{19}$$

The method of averaging (Caughey 1960) is used to find the frequency ω as a function of the steady-state amplitude x_s, for $\omega \approx 1$, as

$$\omega^2 = \frac{(2\beta - c_0^2) \pm \sqrt{(2\beta - c_0^2)^2 - 4(\alpha^2 + \beta^2 - \gamma^2 - 2c_0\alpha)}}{2} \tag{20}$$

For a bilinear hysteretic oscillator, β and α can be found to be

$$\alpha = -\frac{\mu}{\pi}[\sin^2\theta^*] \tag{21a}$$

$$\beta = \frac{1}{\pi}\left[\mu\theta^* + (1 - \mu)\pi - \frac{\mu}{2}\sin 2\theta^*\right]; \quad \gamma = p/x_s \tag{21b,c}$$

where $\theta^* = \cos^{-1}[1 - (2/x_s)]$.

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APPENDIX II. REFERENCES

Ballio, G. (1970). ''On the dynamic behavior of an elastoplastic oscillator (experimental and theoretical investigation).'' *Meccanica*, 5, 87–97.

Berg, G. V. (1965). ''A study of the earthquake response of inelastic systems.'' *Proc., 34th Convention of Struct. Engrs. Assoc. of California*, Coronado, 63–67.

Caughey, T. K. (1960). ''Sinusoidal excitation of a system with bilinear hysteresis.'' *J. Appl. Mech.*, 27, 640–643.

Iwan, W. D. (1964). ''The dynamic response of the one degree of freedom bilinear hysteretic system.'' *Proc., 3rd World Conf. on Earthquake Engrg.*, 783–796.

Iwan, W. D., and Gates, N. C. (1979). ''Estimating earthquake response of simple hysteretic systems.'' *J. Engrg. Mech. Div.*, ASCE, 105(3), 391–405.

Jacobsen, L. S. (1960). ''Damping in composite structures.'' *Proc., 3rd World Conf. on Earthquake Engrg.*, Vol. II, 1029–1044.

Jennings, P. C. (1968). ''Equivalent viscous damping for yielding structures.'' *J. Engrg. Mech. Div.*, ASCE, 94, 103–116.

Mendelson, A. (1968). *Plasticity*, McMillan, New York.

Nayfeh, A. H., and Mook, D. T. (1979). *Non-linear oscillations*, Wiley, New York.

Pratap, R., Mukheejee, S., and Moon, F. C. (1994). ''Dynamic behavior of a bilinear hysteretic elastoplastic oscillator, Part II: Oscillations under periodic impulsive loading.'' *J. Sound and Vibration*, 172(3), 339–358.

Savi, M. A., and Pacheco, P. M. C. L. (1997). ''Non-linear dynamics of an elastoplastic oscillator with kinematic and isotropic hardening.'' *J. Sound and Vibration*, 207(2), 207–226.

Tanabashi, R. (1956). ''Studies on nonlinear vibration of structures subjected to destructive earthquakes.'' *Proc., World Conf. on Earthquake Engrg.*, University of California, Berkeley, Calif., 6-1–6-7.

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APPENDIX III. NOTATION

The following symbols are used in this paper:

C_e = equivalent viscous damping in case of bilinear nonhysteretic oscillator;
c = coefficient of viscous damping;
c_a = extra damping in case of bilinear nonhysteretic oscillator;
c_e = equivalent viscous damping in case of linear oscillator;
c_0 = nondimensional coefficient of viscous damping;
F = restoring force for bilinear hysteretic oscillator;
f = nondimensional restoring force for bilinear hysteretic oscillator;
G = function defining relation between frequency ratio and steady state amplitude;
k = stiffness of elastic branch of hysteretic oscillator;
m = mass of hysteretic oscillator;
P = forcing amplitude;
p = nondimensional forcing amplitude;
t = time;
X = resonant amplitude;
x = nondimensional displacement;
x_s = steady state amplitude;
y = displacement of hysteretic oscillator;
η^2 = kinematic hardening parameter;
τ = nondimensional time;
ϕ = slowly varying phase;
Ω = frequency of harmonic forcing; and
ω = nondimensional frequency.

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