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ASTRONOMY, ASTROPHYSICS, AND COSMOLOGY

On the Problem of Constructing Intermediate Trajectories in the Theory of Elastic-Earth Rotation around a Center of Mass

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The development of the exact theory of Earth rotation is a rather complicated mathematical problem and requires the elimination of a number of simplifications in the accepted theory [2]. In spite of the exceptionally high level of modern observations, researchers have failed to attain a complete understanding of such dynamic effects as the free nutation of the Earth-rotation axis and the variation of latitudes, both of which are extremely necessary in constructing a highly accurate theory of rotational motion for the deformable Earth.

We attempted to study certain fine regularities in the theory of Earth rotation around its center of mass from more general positions, namely, translational–rotational movement. As a starting theoretical model, we used the intermediate two-body problem of the Earth–Moon system, which made it possible to take into account the barycentric distance. In this intermediate motion, the Earth uniformly rotates and deforms under the action of centrifugal forces of inertia and the lunar gravitation field. The deformations are considered to proceed quasi-statically (the inertia terms can be ignored). In other words, the motion of the three-axis elastic Earth as a whole around its center of mass can be represented as the motion of a planet with an equilibrium configuration and "frozen" deformations. Furthermore, on the basis of the intermediate model problem, it is of interest to consider the dynamics of evolutionary processes, but already using perturbed motion with allowance for dissipative factors and lunar–solar perturbations.

In our opinion, when developing the theory of the Earth's rotation around its center of mass, an important argument is the fact that the Earth–Moon dynamic system is assigned to the class of systems with a slow evolution in which it is possible to trace multistage dynamic processes with various characteristic times. Thus, it is possible to compare the characteristic times of the Earth's own rotation around its axis, of the precession of the kinetic-moment vector for the deformable Earth, and of the evolution of the rotation-axis inclination to the plane of the ecliptic. In such a system, the natural separation of motions into fast and slow motions takes place and these motions are described by their corresponding parameters. Under certain conditions, it is possible to isolate a set of slow parameters (variables) whose rate of variation is asymptotically slow (with respect to a certain small parameter), and the evolutionary equations describing this variation are separated from the remaining equations of the set [5]. Equations averaged over the fast variables for the translational–rotational motions of the Earth–Moon system in the solar gravitational field are studied independently. In the majority of cases, these equations turn out to be a good approximation to the original equations for a long (in the asymptotic sense) time interval. The equations for the remaining variables form a fast component of the Earth–Moon system and involve the evolutionary-system variables as slowly varying parameters. It should be noted that, from the standpoint of evolutionary processes, the qualitative picture of the fast motions of the system is a background against which the slow evolution of the orbital–rotational motion occurs.

1. The choice of intermediate trajectories for the Earth's motion is based on the spatial variant of the two-body (planet–satellite) problem and, namely, the deformable-Earth–Moon system (the Moon is taken as a mass point) and is analyzed from the positions of a double planet. This automatically presumes the presence of a barycentre and allows for its position in subsequent calculations. The model problem under consideration is formulated as follows: let a deformable planet (the Earth) and its satellite (the Moon) participate in the mutual translational–rotational motion around their common center of mass (barycentre). The satellite orbit is inclined at an arbitrary angle to the planet's equator. The Earth is represented by a two-layer model with a solid core and a viscoelastic mantle [1, 7], which are individually continuous. We introduce the inertial sys-

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tem of coordinates $C\xi_1\xi_2\xi_3$ with the origin in the barycenter of the planetary system. Let **G** be the Earth's own kinetic moment and **Λ** be the orbital angular momentum for the lunar center of mass C_1 and the terrestrial center of mass C_2. The angular momentum $\mathbf{K} = \mathbf{G} + \mathbf{\Lambda}$ of the entire planetary system is immobile in the inertial space and coincides with the axis $C\xi_3$ (see the figure).

The radii vectores for the points C_1 and C_2 in the $C\xi_1\xi_2\xi_3$ coordinate system are given in the form

$$\begin{gathered} \mathbf{R}_i = R_i\mathbf{R}_i^0, \quad R_i = \tilde{c}_i R_{21}, \quad i = 1, 2, \\ \mathbf{R}_i^0 = \Gamma_3(h)\Gamma_1(i)(\cos\vartheta, \sin\vartheta, 0), \quad \mathbf{R}_2^0 = -\mathbf{R}_1^0, \\ \tilde{c}_1 = m_2 m^{-1}, \quad \tilde{c}_2 = m_1 m^{-1}, \quad m = m_1 + m_2. \end{gathered} \tag{1}$$

Here, h, i, and ϑ are the ascending-node longitude, the inclination, and the orbit true anomaly, respectively; $\mathbf{R}_{21} = R_{21}\mathbf{R}_{21}^0$ is the radius vector drawn from the point C_2 to the point C_1, so that $\mathbf{R}_{21} = \mathbf{R}_1^0$. The $C_2x_1x_2x_3$ Cartesian coordinate system is rigidly related to the solid core of the planet. The axes of this system are directed along the principal axes of inertia A, B, and C of the planet. For this coordinate system, we may write out

$$\begin{gathered} O^{-1}(t)\vec{\mathbf{R}}_{21}^0 = (\gamma_1, \gamma_2, \gamma_3), \\ O^{-1}(t) = \Gamma_3^{-1}(\varphi_1)\Gamma_1^{-1}(\delta_2)\Gamma_3^{-1}(\varphi_2)\Gamma_1^{-1}(\delta_1)\Gamma_3^{-1}(\varphi_3), \end{gathered} \tag{2}$$

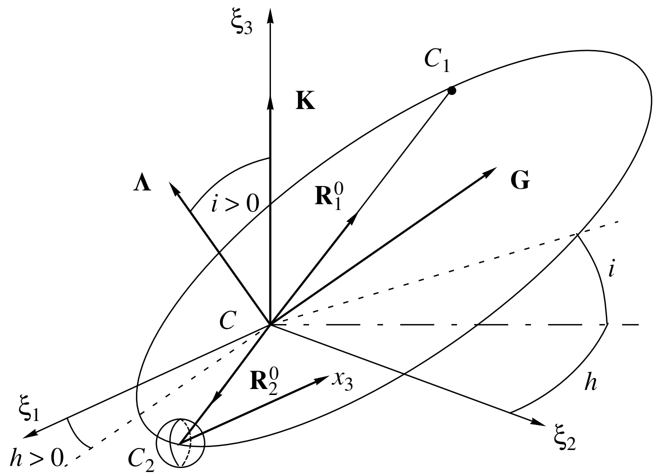
where $O(t)$ is the matrix specifying the passage from the body axes to the inertial axes and is expressed in Andoyer canonical variables: L, $I_2 = |\vec{\mathbf{G}}|$, I_3, φ_1, φ_2, and φ_3; and $\cos\delta_1 = I_3/I_2$, $\cos\delta_2 = L/I_2$ [5, 6].

We describe the mutual orbital motion of mass centers in the Λ, H, ϑ, h Delone canonical variables, where H is the projection of Λ onto the $C\xi_3$-axis, $\Lambda = |\vec{\mathbf{\Lambda}}|$, and $\cos i = H/\Lambda$.

After a number of simple transformations and averaging over the fast variables φ_2 and ϑ, the Routh functional of the intermediate problem is reduced to the form (with an accuracy to an insignificant constant)

$$\mathfrak{R} = \frac{I_2^2 - L^2}{2}\left[\frac{\sin^2\varphi_1}{\tilde{A}} + \frac{\cos^2\varphi_1}{\tilde{B}}\right] + \frac{L^2}{2\tilde{C}} + \text{const}, \tag{3}$$

where $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are the elastic-Earth principal moments of inertia modified under the action of the centrifugal forces induced by the Earth's rotation. With allowance for the barycentric distance, they take the form



Coordinate system and orientation of vectors.

$$\begin{gathered} \tilde{A} = \frac{J_{11}[\vec{\mathbf{u}}]I_2^2}{I_2^2 + 3\mu_1 R_{21}^{-3} d J_{11}^2[\vec{\mathbf{u}}]}, \\ \tilde{B} = \frac{J_{22}[\vec{\mathbf{u}}]I_2^2}{I_2^2 + 3\mu_1 R_{21}^{-3} d J_{22}^2[\vec{\mathbf{u}}]}, \\ \tilde{C} = \frac{J_{33}[\vec{\mathbf{u}}]I_2^2}{I_2^2 + 3\mu_1 R_{21}^{-3} d J_{33}^2[\vec{\mathbf{u}}]}, \\ d = \cos^2\tilde{h} + \cos^2 i \sin^2\tilde{h} \\ + \cos^2\delta_1(\sin^2\tilde{h} + \cos^2 i \cos^2\tilde{h}) + \sin^2 i \sin^2\delta_1, \\ \tilde{h} = \varphi_3 - h, \quad \mu_1 = f m_1. \end{gathered} \tag{4}$$

Here, f is the gravitation constant; $J_{ii}[\mathbf{u}] = \operatorname{diag}\{J_{11}[\mathbf{u}], J_{22}[\mathbf{u}], J_{33}[\mathbf{u}]\}$ is the inertia-tensor components dependent on the vector $\mathbf{u}$ of the elastic translation for the deformed planet in the planetary coordinate system; and $J[0] = \operatorname{diag}\{A, B, C\}$ for $\mathbf{u} = 0$. Structural form (3) coincides exactly with the traditional expression for the Routh function of a perfectly rigid body in the Andoyer variables and is a basis for introducing the action–angle variables I_i and w_i ($i = 1, 2, 3$). It should be noted that the action–angle variables compose the Hamiltonian part of the variables, whereas the generalized coordinates (modal variables) describing the deformations of the shell compose the Lagrangian part of the variables.

Thus, we introduce the following principal dynamic parameters:

$$\kappa^2 = \frac{\tilde{C}(\tilde{A} - \tilde{B})}{\tilde{A}(\tilde{B} - \tilde{C})}, \quad \lambda^2 = \kappa^2\frac{2E\tilde{C} - I_2^2}{I_2^2 - 2E\tilde{A}}, \tag{5}$$

where E is the energy constant.

The relation between the action–angle variables and the Andoyer variables is given by the following formulas [3, 4] (for brevity, we restrict ourselves by the case $0 \le \lambda < 1$):

$$
\begin{gathered}
I_1 = \frac{2I_2}{\pi\kappa}\sqrt{\frac{1+\kappa^2}{\lambda^2+\kappa^2}}\left[(\lambda^2+\kappa^2)\Pi\left(\frac{\pi}{2}, \kappa^2, \lambda\right) - \lambda^2 K(\lambda)\right],\\
I_2 = I_2, \quad I_3 = I_3,\\
w_1 = \pm\frac{\pi}{2}\left[\frac{F(\xi, \lambda)}{K(\lambda)}\right], \quad \xi = \pm am(u, \lambda), \quad \mathbf{u} = \frac{2K(\lambda)}{\pi}w_1,\\
w_2 = \varphi_2 \pm \left\{\frac{1}{\kappa}\sqrt{(1+\kappa^2)(\kappa^2+\lambda^2)}\right.\\
\left.\times\left[\Pi(\xi, \kappa^2, \lambda) - \Pi\left(\frac{\pi}{2}, \kappa^2, \lambda\right)\frac{F(\xi, \lambda)}{K(\lambda)}\right]\right\},\\
\cot\varphi_1 = -\sqrt{1+\kappa^2}\tan\xi, \quad w_3 = \varphi_3.
\end{gathered} \tag{6}
$$

Here, $K(\lambda)$, $\Pi(\pi/2, \kappa^2, \lambda)$ are the complete elliptic integrals of the first and third kinds; and $F(\xi, \lambda)$, $\Pi(\xi, \kappa^2, \lambda)$ are the elliptic integrals of the first and third kinds.

In this case, Routh functional (3) can be written out as

$$
\mathfrak{R} = \frac{I_2^2}{2\tilde{A}}\left(1 - \frac{\tilde{C}-\tilde{A}}{\tilde{C}}\frac{\kappa^2}{(\lambda^2+\kappa^2)}\right). \tag{7}
$$

The general solution to the intermediate problem with functional (7) has the form

$$
\begin{gathered}
I_i = I_i(0), \quad i = 1, 2, 3, \quad w_1 = n_{w_1}t + w_{10},\\
w_2 = n_{w_2}t + w_{20},\\
n_{w_1} = \frac{I_2(\tilde{A}-\tilde{C})}{2\tilde{A}\tilde{C}}\frac{\pi\kappa}{\sqrt{(1+\kappa^2)(\kappa^2+\lambda^2)}K(\lambda)},\\
n_{w_2} = \frac{I_2}{\tilde{C}}\left(1 - \frac{\tilde{A}-\tilde{C}}{\tilde{A}}\frac{\Pi(\pi/2, \kappa^2, \lambda)}{K(\lambda)}\right),
\end{gathered} \tag{8}
$$

where w_{10} and w_{20} are the initial values of the angular variables.

Solving the first equation of set (6) with respect to λ with an accuracy to λ^2 inclusively, we obtain

$$
\lambda^2 = \frac{2\kappa^2(I_2 - I_1)}{I_2\sqrt{1+\kappa^2}}, \quad 0 \le \lambda < 1. \tag{9}
$$

Using (9), we can represent the frequencies of intermediate motion (8) in the form ($0 \le \lambda < 1$)

$$
\begin{gathered}
n_{w_1} = \frac{I_2}{\kappa_1}\frac{\tilde{A}-\tilde{C}}{\tilde{A}\tilde{C}}\left(1 - \frac{2+\kappa^2}{2\kappa_1}\frac{I_2-I_1}{I_2}\right), \quad \kappa_1 = \sqrt{1+\kappa^2},\\
n_{w_2} = \frac{I_2}{\tilde{C}}\left[1 + \frac{\tilde{C}-\tilde{A}}{\kappa_1\tilde{A}}\left(1 + \frac{I_2-I_1}{2\kappa_1 I_2}(2\kappa_1 - 2 - \kappa^2)\right)\right].
\end{gathered} \tag{10}
$$

2. As an example, we consider the Chandler wobble of the Earth's poles. The wobble is determined as a motion of the rotation axis with respect to the figure axis [8]:

$$
\begin{gathered}
x_p = \frac{\omega_x}{\omega} = \frac{I_2\lambda}{\tilde{A}\omega\sqrt{\kappa^2+\lambda^2}}\mathrm{cn}\mathbf{u},\\
y_p = -\frac{\omega_y}{\omega} = -\frac{\lambda I_2\sqrt{1+\kappa^2}}{\tilde{B}\omega\sqrt{\kappa^2+\lambda^2}}\mathrm{sn}\mathbf{u},\\
\omega = \frac{I_2}{\tilde{A}\sqrt{\kappa^2+\lambda^2}}\sqrt{\lambda^2 + \frac{\tilde{A}^2}{\tilde{C}^2}\kappa^2 - \lambda^2\left(1 + \frac{\tilde{A}^2}{\tilde{C}^2}\kappa^2\right)\mathrm{sn}^2\mathbf{u}}.
\end{gathered} \tag{11}
$$

Here, $\mathrm{sn}\mathbf{u}$ and $\mathrm{cn}\mathbf{u}$ are the Jacobi elliptic functions, which are represented by the expansions in the Fourier trigonometric series [3, 4]. In (11), according to the accepted convention [8], y_p is directed along the 90° meridian of the western longitude.

The period of the Chandler wobble is determined by the following expression:

$$
T = \frac{2\pi}{n_{\mathrm{W}_1}}. \tag{12}
$$

Formula (12) relates the period of the Chandler wobble with the principal dynamic parameters introduced by the authors: the energy constant, the kinetic-moment modulus, and the Earth's principal moments of inertia.

The analytical expressions obtained in this study for the moments of inertia of the Earth with allowance for its elasticity and the translational type of its motion is a basis for both constructing a highly accurate theory of the Chandler wobble and for a comparison of this theory with long-term astronomical observations and estimates for the period of this vibration. It should be especially emphasized that, on the basis of the derived averaged differential equations, we can trace the evolution of both the pole vibrational motion and the moments of inertia for the Earth.

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REFERENCES

1. Yu. G. Markov, L. V. Rykhlova, and I. V. Skorobogatykh, Dokl. Akad. Nauk **370**, 613 (2000) [Dokl. Phys. **45**, 70 (2000)].
2. N. Moritz and A. Mueller, *Earth Rotation: Theory and Observation* (Ungar, New York, 1987; Naukova Dumka, Kiev, 1992).
3. Yu. A. Sadov, Prikl. Mat. Mekh. **34**, 962 (1970).
4. Yu. V. Barkin, Astron. Astrophys. Trans. **17**, 179 (1998).
5. V. G. Vil’ke, *Analytical and Qualitative Methods in Mechanics of Systems with an Infinite Number of Degrees of Freedom* (Mosk. Gos. Univ., Moscow, 1986).
6. Yu. G. Markov, Astron. Zh. **73**, 748 (1996) [Astron. Rep. **40**, 681 (1996)].
7. V. V. Bondarenko, Yu. G. Markov, and I. V. Skorobogatykh, Astron. Vestn. **32**, 340 (1998).
8. H. Kinoshita, Celest. Mech. **15**, 277 (1977).

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