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CONDITIONALLY DEFINITE MATRICES

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1. Introduction

It is well known how important positivity is in various branches of mathematics. For objects that are positive, one can usually obtain much more complete results than in the general case. For example, in linear algebra, positive and positive-definite matrices are among the most thoroughly studied matrix classes. It is equally important that the results of this study have been well documented: one can learn the properties of the matrix classes above from dozens of textbooks and monographs on matrix theory.

Things are quite different when from the global property of (positive or negative) definiteness one turns to the same property that holds only conditionally, i.e., as long as the argument does not leave a given subspace, orthant, polyhedron, or cone. We combine all these options under the term "conditionally definite" matrices. Matrices of this kind are understood to a lesser extent than the classical positive-definite matrices. Moreover, to the best of our knowledge, no book exposition yet exists of the problem of conditional definiteness. At most, there are some survey papers devoted to the particular types of conditional-definite matrices.

This fact, i.e., that a readily accessible exposition of the field is not available, and the importance of conditional definiteness in a number of applications served as a stimulus for writing this survey. The selection of material for the paper was to a considerable extent guided by what explains our interest in this subject, namely, our wish to develop a collection of computer procedures for checking whether a given matrix possesses a particular type of conditional definiteness. Typically, the matrix has scalar entries, which are then assumed to be integers or rational numbers. It is also admissible, however, that some or even all entries of the matrix contain parameters. In this case, the dependence on parameters must also be expressed by rational functions. Under these assumptions, the procedures must give *exact* answers, not answers that hold up to "round-off error analysis," which are typical of floating-point computations. This predetermines that the procedures to be included in the collection must be finite rational algorithms and their computer implementation must be based on a kind of error-free computation. For matrices with parameters, symbolic computation is used rather than an error-free one.

Let us clarify what was said above by using the ordinary positive definiteness as an example. According to one of the many equivalent definitions of this property, a Hermitian matrix A is positive definite if and only if all its eigenvalues are positive. This statement seems to give a constructive criterion for positive definiteness if one takes into account that well-polished routines are available for computing the eigenvalues of a matrix. These routines are especially efficient and accurate in the case where the matrices are Hermitian. The absolute errors of approximate eigenvalues that are computed by such a routine can be bounded a priori. Hence, when the routine stops, one has a set of "uncertainty intervals" that enclose the spectrum of the matrix under investigation. If none of the intervals contains zero, then authentic inferences are possible concerning the inertia of the matrix. In particular, one can give a certain answer whether the matrix is positive definite. However, if zero belongs to the left-most interval, a precise inference about the definiteness becomes impossible. The transition from the real (or complex) arithmetic to error-free computations does not help here. Indeed, even for an integer matrix, the eigenvalues cannot, in general, be found by a finite (much less by a rational) procedure.

Fortunately, there exist criteria for positive definiteness, say, the classical determinantal Sylvester criterion, that can be implemented by means of error-free or symbolic computations. It turns out that criteria of this kind also exist for conditional definiteness. Our main aim in this survey is to describe them.

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The paper is organized as follows. In Sec. 2, we recall the basic criteria for global definiteness. Some facts from linear algebra are also given, which we shall need in the sequel. Matrices that are definite with respect to a linear subspace are treated in Sec. 3. Copositive matrices are considered in Sec. 4, and $\mathcal{K}$-copositive ones in Sec. 5. Our concluding remarks are given in Sec. 6.

Along with the strong definiteness, we discuss the same property in a weak form (in the classical case, these two species are illustrated by the notions of positive definiteness and positive semidefiniteness, respectively).

Most often, one encounters various kinds of conditional definiteness in mathematical programming where all data typically are real numbers. For this reason, our discussion is limited to real matrices. As was already mentioned, our computer procedures even presuppose that the entries of matrices are integers or rational numbers. At the same time, the procedures can be easily generalized to the case of complex matrices with Gaussian entries.

2. Preliminaries

In this survey, the symbols M_n and S_n stand for the linear space of real $n \times n$ matrices and its subspace consisting of symmetric matrices, respectively.

Definition. A matrix $A \in S_n$ is called *positive semidefinite* if $(Ax, x) \geq 0$ for any vector $x \in \mathbf{R}^n$. If $(Ax, x) > 0$ for any nonzero x, then A is a *positive-definite* matrix.

We denote by $A(i_1, \ldots, i_k)$ the principal submatrix of A lying in rows and columns with indices $i_1, \ldots, i_k$.

Theorem 2.1 (the Sylvester criterion). *A matrix $A \in S_n$ is positive definite if and only if all its leading principal minors are positive, i.e.,*

$$\det A\,(1, \ldots, k) > 0, \qquad k = 1, \ldots, n. \tag{2.1}$$

The property of positive definiteness is invariant under symmetric permutations of rows and columns of a matrix. Therefore, a more general formulation can be given for the Sylvester criterion [6, Theorem 7.2.5].

Theorem 2.2. *All principal minors of a positive-definite matrix are positive. A matrix $A \in S_n$ is positive definite if there exists a nested sequence of n principal minors of A (not just the leading principal minors) that are positive.*

The nonnegativity of *all* principal minors is a necessary and sufficient condition for A to be positive semidefinite, which is implied by the following assertion.

Theorem 2.3. *A matrix $A \in S_n$ is positive semidefinite if and only if the matrix $A + \varepsilon I_n$ is positive definite for any $\varepsilon > 0$.*

Here, one cannot check the signs of only *leading* principal minors, as was the case with the Sylvester criterion. For example [6, p. 404], both leading principal minors of the matrix

$$B = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.2}$$

are zero, i.e., nonnegative. However, the matrix B is not positive semidefinite; rather it is negative semidefinite. The matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

for which all the leading principal minors are again zero, is indefinite.

We denote the number of positive eigenvalues and that of negative eigenvalues of the symmetric matrix A by $\pi(A)$ and $\nu(A)$, respectively, and call them the *positive inertia* and the *negative inertia* of this matrix. The symbol $\delta(A)$ stands for the *defect*, or *rank deficiency*, of A (which is defined as the difference $n - \operatorname{rank} A$). The ordered triple

$$\operatorname{In} A = (\pi(A), \nu(A), \delta(A)) \tag{2.3}$$

is called the *inertia* of A.

The Sylvester criterion is known to be just a particular case of the following signature rule, which is due to Jacobi. Let

$$\begin{aligned} \Delta_0 &= 1, \\ \Delta_k &= \det A\,(1, \ldots, k), \qquad k = 1, \ldots, n. \end{aligned} \tag{2.4}$$

Theorem 2.4. *Assume that all the leading principal minors of a matrix $A \in S_n$ are nonzero. Then the positive inertia of A is equal to the number of sign coincidences in the sequence*

$$\Delta_0,\ \Delta_1,\ \ldots,\ \Delta_n \tag{2.5}$$

and the negative inertia to the number of sign variations in this sequence.

Below, we prove the Jacobi rule. This gives us a good reason to remind the reader of an important extremal characterization of the eigenvalues of a symmetric matrix, which is called the *Courant–Fisher theorem.*

Theorem 2.5. *Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be eigenvalues of a matrix $A \in S_n$. Then*

$$\lambda_k = \max_{L_k} \min_{\substack{x \neq 0 \\ x \in L_k}} \frac{(Ax, x)}{(x, x)}, \tag{2.6}$$

$$\lambda_k = \min_{L_{n-k+1}} \max_{\substack{x \neq 0 \\ x \in L_{n-k+1}}} \frac{(Ax, x)}{(x, x)}. \tag{2.7}$$

In formula (2.6), *the maximum is taken over all k-dimensional subspaces L_k of the space $\mathbf{R}^n$. Similarly, in* (2.7), *L_{n-k+1} is an arbitrary subspace of dimension $n - k + 1$.*

Theorem 2.5 implies the so-called *interlacing inequalities*

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n \tag{2.8}$$

between the eigenvalues of the symmetric matrix A and the eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1}$ of its (arbitrary) principal $(n-1) \times (n-1)$ submatrix.

We prove the Jacobi signature rule by induction on the order n of A. For $n = 1$, the assertion is trivial. Suppose it holds for all $k < n\,(n > 1)$. The truncated sequence

$$\Delta_0,\ \Delta_1,\ \ldots,\ \Delta_{n-1} \tag{2.9}$$

can be regarded as the Jacobi sequence for the leading principal $(n-1) \times (n-1)$ submatrix A_{n-1}. Suppose that there are m coincidences and l variations in sign in sequence (2.9), $m + l = n - 1$. If $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1}$ are eigenvalues of the submatrix A_{n-1}, then, by the inductive assumption, m largest numbers out of the numbers μ_j must be positive, and l smallest numbers must be negative. The interlacing inequalities (2.8) imply that A has at least m positive eigenvalues and l negative ones. Only the sign of the eigenvalue λ_{m+1} has not yet been determined. Dividing the relation

$$\Delta_n = \lambda_1 \cdots \lambda_m\, \lambda_{m+1}\, \lambda_{m+2} \cdots \lambda_n$$

by

$$\Delta_{n-1} = \mu_1 \cdots \mu_m \, \mu_{m+1} \cdots \mu_{n-1},$$

we conclude that the sign of λ_{m+1} coincides with that of the ratio Δ_n/Δ_{n-1}. This proves that the Jacobi rule is valid for the matrix A.

As an illustration, we find the inertia of a quasidefinite matrix [39]. This is the term for a symmetric partitioned matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix}, \tag{2.10}$$

where the square $n_1 \times n_1$ submatrix A_{11} is positive definite, and the $n_2 \times n_2$ submatrix A_{22}, $n_2 = n - n_1$, is negative definite.

Since matrix (2.10) contains the positive-definite submatrix A_{11}, it must have at least n_1 positive eigenvalues. Similarly, the presence of the negative-definite submatrix A_{22} implies that at least n_2 eigenvalues of A must be negative. Since $n_1 + n_2 = n$, the matrix A is nonsingular, and its inertia is

$$\operatorname{In} A = (n_1, n_2, 0).$$

The Jacobi rule and its modifications are very helpful in various root separation problems, i.e., in the class of problems that deal with the location of roots of a (real) polynomial with respect to a given subset of the complex plane. We give two examples of these applications.

Assume that the real polynomial

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n \tag{2.11}$$

has no multiple roots (this can be easily achieved if one divides $f(x)$ by the greatest common divisor of $f(x)$ and its derivative $f'(x)$). We denote by $\alpha_1, \ldots, \alpha_n$ the roots of $f(x)$. The sums

$$s_k = \alpha_1^k + \ldots + \alpha_n^k, \qquad k = 0, 1, 2, \ldots,$$

are known as the Newton sums of the polynomial $f(x)$. Being symmetric functions of the roots $\alpha_1, \ldots, \alpha_n$, the Newton sums can be rationally expressed in terms of the coefficients $a_0, a_1, \ldots, a_n$ of $f(x)$.

Theorem 2.6 (the Borchardt–Jacobi theorem). *The quadratic form*

$$J = \sum_{l,m=0}^{n-1} s_{l+m} \, x_l \, x_m$$

is nondegenerate. If π and ν are the positive and the negative inertia, respectively, of the form J, then the polynomial $f(x)$ has ν pairs of complex-conjugate roots and $\pi - \nu$ real roots.

Our second example is the Routh–Hurwitz–Fujiwara criterion. We set

$$g(x) = f(-x)$$

for polynomial (2.11) and form the Bezout matrix of the polynomials f and g. Recall that the Bezout matrix $B(f, g)$ is the matrix associated with the quadratic form

$$B(w, z) = \frac{f(w)g(z) - f(z)g(w)}{w - z} = \sum_{k,l=1}^{n} b_{kl} \, w^{k-1} \, z^{l-1}.$$

Finally, we transform the matrix $B(f, g) = (b_{ij})$ into a new matrix F according to the rule

$$f_{ij} = (-1)^{i+1} \, b_{ij}, \qquad i, j = 1, \ldots, n.$$

Theorem 2.7 (Routh–Hurwitz–Fujiwara criterion). *If the matrix F is nonsingular, then its positive (negative) inertia gives the number of roots of the polynomial $f(x)$ that have negative (positive) real parts.*

The application of the Jacobi rule in the situations defined by Theorems 2.6 and 2.7 presupposes that all the leading principal minors of the corresponding matrices are nonzero. If this assumption is not valid, it may still be possible to determine the inertia using the extensions of the Jacobi rule by Gundelfinger and Frobenius (see [4, Sec. 8]).

Theorem 2.8. *Assume that in sequence (2.4), the determinant $\Delta_n \neq 0$, but a minor Δ_k, $k < n$, may be zero. In each such occasion (i.e., when $\Delta_k = 0$), assume that $\Delta_{k-1}\Delta_{k+1} \neq 0$. Assign arbitrary signs to the zero minors Δ_k. Then the Jacobi rule holds for the modified sequence (2.4).*

Theorem 2.9. *Assume that in sequence (2.4), the determinant $\Delta_n \neq 0$, but it may be possible that $\Delta_k = \Delta_{k+1} = 0$ when $k < n-1$. In each such occasion, assume that $\Delta_{k-1}\Delta_{k+2} \neq 0$. Assign the same (arbitrary) sign to Δ_k and Δ_{k+1} if $\Delta_{k-1}\Delta_{k+2} < 0$ and different signs (in any one of the two possible ways) if $\Delta_{k-1}\Delta_{k+2} > 0$. Then the Jacobi rule holds for the modified sequence (2.4).*

We reproduce an example from [4], which shows that the further extension of the Jacobi signature rule for the case where (2.4) contains subsequences of three or more successive zeros is impossible if one speaks about general symmetric matrices. Assume that the coefficients α, β, and γ in the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & \alpha \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ \alpha & 0 & 0 & 0 \end{pmatrix}$$

are nonzero. Then sequence (2.4) for this matrix is

$$1,\ 0,\ 0,\ 0,\ -\alpha^2\beta\gamma.$$

The sign of the determinant Δ_4 is determined by that of the product $\beta\gamma$. It is not difficult to see that the inertia of the matrix A is $(3,1,0)$ if $\beta > 0$, $\gamma > 0$, and $(1,3,0)$ if $\beta < 0$, $\gamma < 0$, although in both cases Δ_4 is negative.

We return to the discussion of the criteria for definiteness.

Theorem 2.10. *A matrix $A \in S_n$ is positive semidefinite if and only if*

$$A = S\,S^T \tag{2.12}$$

for an $n \times m$ matrix S (m may be arbitrary). A matrix A is positive definite if and only if the rank of the matrix S in (2.12) is equal to n.

A positive (semi)definite matrix A can be represented in the form (2.12) in many different ways. The most useful ones are the following.

(1) $S = S^T$. In this case, (2.12) turns into

$$S^2 = A$$

and the matrix S is a square root of A. There exists a unique positive (semi)definite square root of A. It is denoted by $A^{1/2}$.

(2) S is a lower triangular matrix with positive diagonal entries. Usually, this matrix is denoted by L and called the Cholesky factor of the matrix A. The corresponding decomposition of A is the product

$$A = L\,L^T \tag{2.13}$$

of the two triangular matrices, the lower matrix L and the upper one L^T. It is called the Cholesky decomposition of A.

Using relation (2.13) for the entries in position (1,1) yields

$$l_{11} = \sqrt{a_{11}}.$$

Thus, the calculation of the Cholesky factor requires square roots, i.e., the type of operation that we would like to avoid. Meanwhile, there exists another decomposition of a positive-definite matrix that is similar to the Cholesky decomposition but that can be found by employing only arithmetical operations. It is called the LDL^T decomposition:

$$A = LDL^T. \tag{2.14}$$

As opposed to the Cholesky factor, the matrix L in (2.14) has the unit main diagonal. The matrix D is diagonal:

$$D = \operatorname{diag}(d_{11}, \ldots, d_{nn}).$$

It is not difficult to see that

$$\begin{aligned} d_{11} &= a_{11} = \Delta_1, \\ d_{kk} &= \frac{\Delta_k}{\Delta_{k-1}}, \qquad k = 2, 3, \ldots, n. \end{aligned} \tag{2.15}$$

As above, Δ_k is the leading principal $k \times k$ minor of the matrix A.

Note that the LDL^T decomposition exists (and is unique) not only for positive-definite matrices but also for any symmetric matrix A in which all the leading principal minors are nonzero. Moreover, the last of these minors, i.e., $\det A$, may be zero. If A is not positive semidefinite, then D contains diagonal entries of different signs.

The matrix transformation of the form

$$A \to B = PAP^T, \tag{2.16}$$

where P is a nonsingular matrix, is called a *congruence*, and the matrices A and B in (2.16) are referred to as *congruent* matrices. These matrices can be considered to be associated with the same quadratic form but in different bases of the space $\mathbf{R}^n$. As a consequence, congruent matrices have the same inertia. In particular, the diagonal matrix D in the LDL^T decomposition of A indicates the inertia of the latter matrix.

Assume that the $n_1 \times n_1$ submatrix A_{11} in the partitioned matrix (2.10) is nonsingular. Applying to A congruence (2.16) with the matrix

$$P = \begin{pmatrix} I_{n_1} & 0 \\ -A_{12}^T A_{11}^{-1} & I_{n_2} \end{pmatrix} \tag{2.17}$$

yields the block-diagonal matrix

$$B = \begin{pmatrix} A_{11} & 0 \\ 0 & B_{22} \end{pmatrix}. \tag{2.18}$$

The submatrix

$$B_{22} = A_{22} - A_{12}^T A_{11}^{-1} A_{12}$$

is usually denoted by A/A_{11} and is called the *Schur complement of the submatrix* A_{11} *in* A.

One important implication of formula (2.18) is the equality

$$\operatorname{In} A = \operatorname{In} A_{11} + \operatorname{In}(A/A_{11}). \tag{2.19}$$

The inertias here are added entrywise.

Suppose that the inverse matrix $C = A^{-1}$ is partitioned similar to (2.10):

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}.$$

Then

$$C_{22} = (A/A_{11})^{-1}.$$

Hence, the submatrix C_{22} in the inverse matrix C has the same inertia as the Schur complement A/A_{11} in the original matrix A.

3. Positive Definiteness on a Subspace

Suppose that we consider a subspace $\mathcal{L} \subset \mathbf{R}^n$ described by the system of linear equations

$$Bx = 0. \tag{3.1}$$

Here B is a $p \times n$ matrix. Without loss of generality, one can assume that

$$\operatorname{rank} B = p. \tag{3.2}$$

This amounts to removing linearly dependent equations from system (3.1).

Definition. A matrix $A \in S_n$ is called *$\mathcal{L}$-semidefinite* if

$$(A\,x, x) \geq 0 \quad \forall x \in \mathcal{L}. \tag{3.3}$$

If

$$(A\,x, x) > 0 \quad \forall x \in \mathcal{L},\ x \neq 0, \tag{3.4}$$

then A is called an *$\mathcal{L}$-definite* matrix.

Not to complicate terminology, we did not mention positivity in the definitions above. The negative definiteness with respect to a subspace could have been considered with equal reason. However, only positive-definite matrices are generally discussed in this survey.

We mention that the discussion in this section is, to a large extent, based on the review article [9].

The most straightforward approach to checking the $\mathcal{L}$-definiteness of a matrix is to reduce the test to that for ordinary positive definiteness. Let P be a nonsingular $n \times n$ matrix such that

$$BP^T = (0\ I_p). \tag{3.5}$$

We replace x in (3.1), (3.3)–(3.4) by a new variable:

$$x = P^T y. \tag{3.6}$$

Let $y = (y_1, \ldots, y_n)^T$. Then condition (3.1) turns into the set of equalities

$$y_{n-p+1} = 0, \ldots, y_n = 0.$$

Now, instead of (3.3) and (3.4), we arrive at the requirement that the leading principal $(n-p) \times (n-p)$ submatrix of the matrix

$$\widehat{A} = PAP^T \tag{3.7}$$

be positive definite or positive semidefinite, respectively.

The criterion obtained will be restated in an algorithmic form.

Algorithm 1 for checking the $\mathcal{L}$-definiteness of the matrix A

1. Calculate a matrix P satisfying condition (3.5).
2. Form matrix (3.7). In fact, only the leading principal $(n-p) \times (n-p)$ submatrix $\widehat{A}_{n-p}$ of matrix (3.7) can be calculated.
3. Apply to $\widehat{A}_{n-p}$ a criterion for the ordinary positive (semi)definiteness.

To justify our second algorithm, we shall need the following lemma.

Lemma 3.1. *Let $n = 2m$ be an even integer. Assume that a matrix $A \in S_n$ has the block form*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & 0 \end{pmatrix},$$

where all the blocks are of order m, and the submatrix A_{12} is nonsingular. Then the inertia of A is $(m, m, 0)$.

Proof. Obviously, the matrix A is nonsingular, and, hence, $\delta(A) = 0$. Note that A contains a zero principal submatrix of order m. According to the Courant–Fisher theorem, at least m of the eigenvalues of A are nonnegative. Actually, these eigenvalues are positive in view of the nonsingularity of A. In just the same way, A must have no less than m negative eigenvalues. Since $n = 2m$, the assertion of the lemma follows.

Now we form an auxiliary $(n+p) \times (n+p)$ matrix:

$$\mathcal{A} = \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}. \tag{3.8}$$

Let the congruence

$$\widehat{\mathcal{A}} = Q\mathcal{A}Q^T$$

with the transforming matrix

$$Q = \begin{pmatrix} P & 0 \\ 0 & I_p \end{pmatrix}$$

be applied to $\mathcal{A}$, P being a nonsingular matrix from (3.5). The matrix $\widehat{\mathcal{A}}$ thus obtained can be partitioned as follows:

$$\widehat{\mathcal{A}} = \begin{pmatrix} \widehat{A}_{n-p} & \widehat{A}_{n-p,p} & 0 \\ \widehat{A}^T_{n-p,p} & \widehat{A}_p & I_p \\ 0 & I_p & 0 \end{pmatrix}.$$

By Lemma 3.1, the inertia of the submatrix

$$M = \begin{pmatrix} \widehat{A}_p & I_p \\ I_p & 0 \end{pmatrix} \tag{3.9}$$

is

$$\operatorname{In} M = (p, p, 0).$$

Suppose that the matrix M^{-1} is given a partitioned form similar to (3.9). Then it is not difficult to see that block (1,1) in M^{-1} is zero. This implies that

$$\widehat{\mathcal{A}}/M = \widehat{A}_{n-p}.$$

Applying (2.19) to $\widehat{\mathcal{A}}$, one obtains

$$\begin{aligned} \operatorname{In} \mathcal{A} = \operatorname{In} \widehat{\mathcal{A}} = {} & \operatorname{In} M + \operatorname{In} \widehat{A}_{n-p} \\ & = (p, p, 0) + \operatorname{In} \widehat{A}_{n-p}. \end{aligned} \tag{3.10}$$

When deducing Algorithm 1, we have found out that the $\mathcal{L}$-definiteness of the matrix A amounts to positive definiteness of the submatrix $\widehat{A}_{n-p}$. Therefore, the following theorem is valid.

Theorem 3.1. *A matrix $A \in S_n$ is $\mathcal{L}$-definite if and only if, for the corresponding matrix* (3.8), *the positive inertia is equal to n.*

This assertion is immediate from relations (3.10).

In the same way, relations (3.10) imply

Theorem 3.2. *A matrix $A \in S_n$ is $\mathcal{L}$-semidefinite if and only if, for the corresponding matrix* (3.8), *the negative inertia is equal to p.*

Thus, Theorems 3.1 and 3.2 indicate a very simple criterion for $\mathcal{L}$-definiteness.

Algorithm 2 for checking the $\mathcal{L}$-definiteness of the matrix A

1. Form matrix (3.8).

2. Find the inertia of $\mathcal{A}$. If the positive inertia is equal to n, the matrix A is $\mathcal{L}$-definite. If this condition is not fulfilled but the negative inertia of matrix (3.8) is equal to p, then the matrix A is $\mathcal{L}$-semidefinite.

We embed matrix (3.8) into the family of matrices of the form

$$\mathcal{A}_t = \begin{pmatrix} A & B^T \\ B & t\,I_p \end{pmatrix}.$$

If $\pi(\mathcal{A}) = n$, for $\mathcal{A} = \mathcal{A}_0$, then

$$\pi(\mathcal{A}_t) = n \tag{3.11}$$

for any negative value of t that is sufficiently small in modulus. For such a t, the relation

$$\begin{aligned} \mathrm{In}\,(\mathcal{A}_t) &= \mathrm{In}\,(t\,I_p) + \mathrm{In}\,(A - \tfrac{1}{t}\,B^T B) \\ &= (0,\, p,\, 0) + \mathrm{In}\,(A - \tfrac{1}{t}\,B^T B) \end{aligned}$$

in conjunction with (3.11) implies

$$\mathrm{In}\,(A - \frac{1}{t}\,B^T B) = (n,\, 0,\, 0).$$

In this way, an assertion is obtained, which is called the Finsler theorem ([16]; see also [1]).

Theorem 3.3. *A matrix $A \in S_n$ is $\mathcal{L}$-definite if and only if the matrix*

$$A(k) = A + k\,B^T B \tag{3.12}$$

is positive definite for all sufficiently large positive values of k.

By a similar reasoning one can prove

Theorem 3.4. *A matrix $A \in S_n$ is $\mathcal{L}$-semidefinite if and only if, for matrix* (3.12), *the negative inertia is equal to p for all negative values of k that are sufficiently large in modulus.*

Now we shall discuss how the criteria for the $\mathcal{L}$-definiteness contained in Theorems 3.3 and 3.4 can be implemented with the use of computer algebra systems. Assume that the Sylvester criterion is employed to check the positive definiteness of the matrix $A\,(k)$. Any leading principal minor $\Delta_i\,(k)$ of $A\,(k)$ can be regarded as a polynomial in k. The sign of its values, as $k \to +\infty$, is determined by the sign of the leading coefficient. For modest n, one can obtain explicit expansions for the minors $\Delta_i\,(k)$, using a computer algebra system; in fact, only the leading coefficients of these expansions are needed. As a result, we arrive at the following algorithm.

Algorithm 3 for checking the $\mathcal{L}$-definiteness of the matrix A

1. Form matrix (3.12).

2. Determine the signs of the leading coefficients of the polynomials $\Delta_i\,(k)$, which are the leading principal minors of the matrix $A\,(k)$. The matrix A is $\mathcal{L}$-definite if and only if all these signs are positive.

The leading coefficients of the polynomials $\Delta_i\,(k)$ also determine the signs of the values of these polynomials as $k \to -\infty$. Thus, for verifying the $\mathcal{L}$-semidefiniteness of a matrix, one must apply the Jacobi signature rule to the modified sequence of the leading coefficients (i.e., for the polynomial Δ_i, its leading coefficient is multiplied by $(-1)^i$). By virtue of Theorem 3.4, the matrix A is $\mathcal{L}$-semidefinite if the sequence above contains exactly p sign alternations.

The criterion for the $\mathcal{L}$-definiteness given by Theorem 3.1 can be converted into a set of determinantal inequalities similar to the Sylvester criterion. Assume that a nonzero minor of the maximal order is contained in the first p columns of B. This can always be achieved by a proper permutation of the columns of B and a (symmetrical) permutation of the rows and columns of A. Along with matrix (3.8), consider its principal submatrices of the form

$$\mathcal{A}_r = \begin{pmatrix} A_r & B_r^T \\ B_r & 0 \end{pmatrix}, \qquad r = p+1, \ldots, n. \tag{3.13}$$

Here A_r is the leading principal $r \times r$ submatrix of A, and the $p \times r$ matrix B_r has been obtained by deleting the last $n-r$ columns of B. Obviously, $\operatorname{rank} B_r = p$. Thus, the arguments used in the proof of Theorem 3.1 are applicable to the matrix $\mathcal{A}_r$ as well. As a consequence, one has

$$\operatorname{In} \mathcal{A}_r = (p,\, p,\, 0) + \operatorname{In} \widehat{A}_{r-p}.$$

If A is $\mathcal{L}$-definite, the submatrices $\widehat{A}_{r-p}$, $r = p+1, \ldots, n$, must be positive definite (see Algorithm 1). Hence, for all matrices (3.13), the determinants have the same sign, namely, $(-1)^p$. In other words,

$$(-1)^p \det\ \mathcal{A}_r > 0, \qquad r = p+1, \ldots, n. \tag{3.14}$$

Conversely, assume that inequalities (3.14) hold. We define the matrix $\mathcal{A}_p$ by analogy with (3.13). Note that B_p corresponds to a nonzero minor of B and, hence, is nonsingular. Applying Lemma 3.1, we have

$$\operatorname{sign} \det\ \mathcal{A}_p = (-1)^p.$$

According to (3.14), for all matrices $\mathcal{A}_r$, $r = p+1, \ldots, n$, the determinants have the same sign, which coincides with the sign of $\det \mathcal{A}_p$.

Observe that the matrices $\mathcal{A}_p, \mathcal{A}_{p+1}, \ldots, \mathcal{A}_{n-1}$ become the *leading* principal submatrices of $\mathcal{A} = \mathcal{A}_n$ when the rows and columns of the latter are properly (and symmetrically) reordered. By the Jacobi signature rule the coincidence of signs of their determinants means that the positive inertia of $\mathcal{A}$ is at least

$$\pi(\mathcal{A}_p) + (n-p) = n$$

and the negative inertia of $\mathcal{A}$ is at least $\nu(\mathcal{A}_p) = p$. However, the order of $\mathcal{A}$ is $n+p$, and, hence its inertia is equal to $(n,\, p,\, 0)$. By Theorem 3.1, this amounts to the $\mathcal{L}$-definiteness of A.

The determinantal inequalities (3.14) were found in [13, 35]. They lead to one more criterion for $\mathcal{L}$-definiteness.

Algorithm 4 for checking the $\mathcal{L}$-definiteness of the matrix A

1. Find a nonzero minor of the maximal order in B. By permuting the columns of B, place this minor into the first p columns. Perform the corresponding permutation of the rows and columns of A.

2. For the sequence of matrices (3.13), check whether all inequalities (3.14) hold. If they do, then the matrix A is $\mathcal{L}$-definite.

Suppose that the Jacobi rule is employed for computing the inertia of A in Algorithm 2. Then its stage 2 actually differs from stage 2 of Algorithm 4 only by the choice of a different sequence of principal minors. However, the principal minors in Algorithm 4 are known to be nonzero, which cannot be guaranteed for Algorithm 2. The price of this guarantee is that one has to carry out additional calculations at stage 1.

The determinantal conditions for the $\mathcal{L}$-semidefiniteness can be established in a similar way, but they are more intricate. Let R be a subset of the index set $\{1, 2, \ldots, n\}$. We denote by A_R the principal submatrix of A which is defined by the choice of R. The symbol B_R stands for the matrix that is obtained by deleting the columns of B whose indices do not belong to R. By analogy with (3.13), let

$$\mathcal{A}_R = \begin{pmatrix} A_R & B_R^T \\ B_R & 0 \end{pmatrix}.$$

The precondition according to which the leftmost $p \times p$ submatrix of B is nonsingular remains valid. Then the following assertion holds.

Theorem 3.5. *A matrix $A \in S_n$ is $\mathcal{L}$-semidefinite if and only if*

$$(-1)^p \det \mathcal{A}_R \geq 0 \tag{3.15}$$

for any index subset R such that

$$R \supset \{1, 2, \ldots, p\}.$$

One can compare inequalities (3.15) with the determinantal criterion for the ordinary positive semidefiniteness. This criterion requires that all principal minors (and not only the leading ones) be nonnegative.

In [9], three other assertions are given that indicate constructive techniques for checking the $\mathcal{L}$-definiteness. Although obviously impractical, these techniques are described below just for the sake of completeness.

Let

$$\mathcal{M}_t = \begin{pmatrix} A + t\,I_n & B^T \\ B & 0 \end{pmatrix}. \tag{3.16}$$

We will be interested in the roots of the equation

$$\det \mathcal{M}_t = 0. \tag{3.17}$$

This is an algebraic equation in t of degree less than n. Another option is to consider (3.17) as a generalized eigenvalue problem with t being the spectral parameter:

$$\det\,(\mathcal{A} + t\,C) = 0. \tag{3.18}$$

Here $\mathcal{A}$ is matrix (3.8) and

$$C = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}.$$

Both matrices are symmetric; in addition, C is semidefinite. Therefore, all the roots of Eq. (3.17) are real.

Theorem 3.6. *A matrix $A \in S_n$ is $\mathcal{L}$-definite if and only if all the roots of Eq.* (3.17) *are negative.*

Proof. If A is $\mathcal{L}$-definite, then the matrix $\mathcal{A}$ is nonsingular. Note that the way in which the matrix $\mathcal{M}_t$ is obtained from the pair of matrices $(A + t\,I_n,\ B)$ is similar to that in which $\mathcal{A}$ is generated from the pair $(A,\ B)$. Also note that, along with A, the matrix $A + t\,I_n$ is $\mathcal{L}$-definite for any positive value of t. Hence, M_t is nonsingular on the whole half-line $t \geq 0$. This proves the necessity part of the theorem.

Conversely, assume that, for given matrices A and B, all the roots of Eq. (3.17) are negative. Consider the eigenvalues $\lambda_1, \ldots, \lambda_{n+p}$ of the matrix $\mathcal{M}_t$ as functions of the parameter t. Then none of these functions can vanish on the half-line $t \geq 0$. Thus, all the matrices $\mathcal{M}_t$ $(t \geq 0)$ have the same inertia. Since they contain a zero $p \times p$ submatrix, the matrices must have at least p negative eigenvalues (none can be zero since $\mathcal{M}_t$ is nonsingular). When t is positive and sufficiently large, the submatrix $A + t\,I_n$ is positive definite, which implies that at least n eigenvalues of $\mathcal{M}_t$ are positive. Therefore, one must have

$$\pi(\mathcal{M}_t) = n, \quad \nu(\mathcal{M}_t) = p$$

for any $t \geq 0$. In particular, $\pi(\mathcal{A}) = n$. By Theorem 3.1, this ensures the $\mathcal{L}$-definiteness of the matrix A.

The corresponding criterion for $\mathcal{L}$-semidefiniteness is stated as follows.

Theorem 3.7. *A matrix $A \in S_n$ is $\mathcal{L}$-semidefinite if and only if all roots of Eq.* (3.17) *are nonpositive.*

Let $f(x) = 0$ be a given algebraic equation. It was mentioned in Sec. 2 that, by counting the inertia of the symmetric matrices appropriately generated from the coefficients of f, one can solve such problems as determining the number of real roots of the equation or the number of roots in the left half-plane of the complex plane. In principle, the criterion contained in Theorem 3.6 amounts to checking that all the roots of the algebraic equation (3.17) are real and, also, belong to the left half-plane. In the case under consideration, the additional difficulty encountered in these standard tests is that we do not have an explicit expansion of the polynomial $\varphi(t) = \det \mathcal{M}_t$. One can argue that we encountered a similar difficulty in Algorithm 3. Here, however, the situation is more complicated. When and if the coefficients of $\varphi(t)$ have been found, one still has to generate from them two new matrices and then compute the inertia of these new matrices.

Our next criterion is given by

Theorem 3.8. *A matrix $A \in S_n$ is $\mathcal{L}$-definite if and only if the matrix $\mathcal{A}$ in (3.8) is nonsingular and the leading principal $n \times n$ submatrix in the inverse matrix $\mathcal{A}^{-1}$ is positive semidefinite.*

Proof. For this assertion, it is the sufficiency part that is easier to prove. Being nonsigular and having a zero $p \times p$ block, the matrix $\mathcal{A}$ must have at least p negative eigenvalues. The positive semidefiniteness of the $n \times n$ submatrix implies that $\pi(\mathcal{A}^{-1}) \geq n$. Hence, $\pi(\mathcal{A}) = \pi(\mathcal{A}^{-1}) = n$, and the matrix A is $\mathcal{L}$-definite by Theorem 3.1.

Conversely, let A be $\mathcal{L}$-definite. According to Theorem 3.1, the matrix $\mathcal{A}$ is nonsingular. Let $\mathcal{A}^{-1}$ be partitioned:

$$\mathcal{A}^{-1} = \begin{pmatrix} K & L \\ L^T & M \end{pmatrix},$$

where K is an $n \times n$ submatrix. We show that any nonzero eigenvalue μ of K must be positive. We denote by x the corresponding eigenvector, $Kx = \mu\, x$, and let

$$u = \frac{1}{\mu} L^T x.$$

Then the vector

$$w = \begin{pmatrix} x \\ u \end{pmatrix}$$

satisfies the equation

$$w = \frac{1}{\mu} \begin{pmatrix} K & 0 \\ L^T & 0 \end{pmatrix} w$$

or, which is the same, the equation

$$\mathcal{M}_{-\frac{1}{\mu}} w = 0.$$

By Theorem 3.6, the number $-\frac{1}{\mu}$ must be negative. This proves the necessity part of the theorem.

If the test for the $\mathcal{L}$-definiteness is to be based on Theorem 3.8, then it should provide for inverting the matrix $\mathcal{A}$ and then counting the inertia of the block K in the inverse matrix $\mathcal{A}^{-1}$ (for which, say, the Jacobi rule can be employed). It is clear that such an approach is necessarily less efficient than Algorithm 2.

Our last assertion modifies Theorems 3.1 and 3.2 for the case where the matrix A is nonsingular. In this case, the Schur complement $\mathcal{A}/A$ is well defined, and the following equality holds:

$$\operatorname{In} \mathcal{A} = \operatorname{In} A + \operatorname{In} (\mathcal{A}/A) = \operatorname{In} A + \operatorname{In} (-BA^{-1}B^T).$$

Theorem 3.9. *A nonsingular matrix $A \in S_n$ is $\mathcal{L}$-definite if and only if*

$$\pi(A) + \nu(BA^{-1}B^T) = n.$$

For the $\mathcal{L}$-semidefiniteness, the necessary and sufficient condition is the equality

$$\nu(A) + \pi(BA^{-1}B^T) = p.$$

The advantage of this formulation is that it involves only matrices of orders n and p and does not involve matrices of order $n+p$ as in Algorithms 2 and 4. On the other hand, the nonsingularity of A is required as a precondition, and one must compute the inverse matrix A^{-1} (or the product $BA^{-1}B^T$ without explicitly forming A^{-1}).

In conclusion, we shall show some applications of the notion of $\mathcal{L}$-definiteness.

The extremal problem

$$\min_{x \in \mathbf{R}^n} \quad f(x)$$

under the linear constraint (3.1) gives the most obvious example of a situation where the property of a matrix to be $\mathcal{L}$-definite is crucial. The matrix A that is important here is the matrix of the second differential of f at a stationary point $x_0 \in \mathcal{L}$. For f to have a local minimum at x_0, it is necessary that A be $\mathcal{L}$-semidefinite. The point x_0 does supply a local minimum to f if A is $\mathcal{L}$-definite.

Euclidean distance matrices give one more example of the conditional definiteness.

Definition. A matrix $A \in S_n$ with zero diagonal entries and nonnegative off-diagonal ones is called a *distance matrix.* The matrix A is called a *Euclidean distance matrix* if there exist points $x_1, \ldots x_n \in \mathbf{R}^r$ $(r \leq n)$ such that

$$a_{ij} = ||x_i - x_j||_2^2 \qquad (1 \leq i, j \leq n). \tag{3.19}$$

If relations (3.19) hold for a set of points in $\mathbf{R}^r$ but not in $\mathbf{R}^{r-1}$, then A is said to be *irreducibly embeddable* in $\mathbf{R}^r$.

As early as in the thirties, various characterizations were proposed that distinguish the Euclidean distance matrices in the class of the general distance matrices. The following assertion due to Schoenberg [17, 36] is of the most interest to us.

Theorem 3.10. *Let*

$$e = (1, 1, \ldots, 1)^T. \tag{3.20}$$

Then the distance $n \times n$ matrix A is a Euclidean distance matrix if and only if it is negative $\mathcal{L}$-semidefinite with respect to the subspace

$$e^T x = 0. \tag{3.21}$$

If

$$P = I_n - \frac{1}{n} e\, e^T$$

is the orthoprojector on the subspace $\mathcal{L}$, and

$$r = \operatorname{rank}(PAP),$$

then A is irreducibly embeddable in $\mathbf{R}^r$.

Applying Theorem 3.2 to this particular situation, we can restate criterion 3.10 as follows.

Theorem 3.11. *The distance $n \times n$ matrix A is a Euclidean distance matrix if and only if the bordered matrix*

$$\mathcal{A} = \begin{pmatrix} -A & e \\ e^T & 0 \end{pmatrix}$$

has negative inertia 1. Furthermore, A is irreducibly embeddable in the space $\mathbf{R}^r$, where

$$r = n - 1 - \delta(\mathcal{A}).$$

The criterion is given in this form in [20]. We mention that Euclidean distance matrices are used in conformation calculations to represent the squares of distances between the atoms in a molecular structure [19].

4. Copositive Matrices

The nonnegative orthant of the n-dimensional space $\mathbf{R}^n$ will be denoted by $\mathbf{R}^n_+$:

$$\mathbf{R}^n_+ = \{\, x \mid x \geq 0 \} .$$

Inequalities like $x \geq y$, where x and y are vectors in $\mathbf{R}^n$, are to be interpreted elementwise.

Definition. A matrix $A \in S_n$ is called *copositive* if

$$(Ax, x) \geq 0 \quad \forall\, x \in \mathbf{R}^n_+, \tag{4.1}$$

and *strictly copositive* if

$$(Ax, x) > 0 \quad \forall\, x \in \mathbf{R}^n_+, \quad x \neq 0. \tag{4.2}$$

Remark. Sometimes an intermediate class of *copositive-plus* matrices is also considered (for example, see [37]). These matrices are defined as copositive matrices with the additional property that

$$(Ax, x) = 0, \quad x \in \mathbf{R}^n_+ \implies Ax = 0.$$

However, in this survey the discussion is limited to matrix classes (4.1) and (4.2).

Obviously, if A is (strictly) copositive, the same is true of any one of its principal submatrices. In particular, we have

Lemma 4.1. *All the diagonal entries of the copositive matrix A are nonnegative. If A is strictly copositive, then all its diagonal entries are positive.*

Let P be a nonsingular *nonnegative* matrix.

Lemma 4.2. *If a matrix $A \in S_n$ is (strictly) copositive, then $B = P^T AP$ is also (strictly) copositive.*

Corollary 4.1. *Suppose that B is obtained from a (strictly) copositive matrix A by a symmetrical permutation of rows and columns. Then B is also (strictly) copositive.*

We denote by $\mathcal{C}_n$ the set of $n \times n$ copositive matrices. Obviously, $\mathcal{C}_n$ is a cone in the space M_n (or S_n). (Here and in what follows, a "cone" always means a "convex cone.") The two subsets of the cone $\mathcal{C}_n$ are well known.

Lemma 4.3. *Any nonnegative matrix $A \in S_n$ is copositive. If all the diagonal elements a_{ii} in the nonnegative matrix A are positive, then A is strictly copositive.*

Proof. The first assertion of the lemma is obvious. Indeed, for $x \geq 0$, the inner product (Ax, x) is the sum of n^2 nonnegative terms. If the main diagonal of A is positive and the component x_i of the nonnegative vector x is positive, then

$$(Ax, x) \geq a_{ii} x_i^2 > 0.$$

Thus, the cone $\mathcal{N}_n$ of symmetric nonnegative $n \times n$ matrices is a part of the cone $\mathcal{C}_n$. Positive semidefinite $n \times n$ matrices constitute one more subset of this cone, which we denote by $\mathcal{PSD}_n$. Obviously, $\mathcal{PSD}_n$ is also a cone.

It turns out that, for $n = 2$, the following relation holds:

$$\mathcal{C}_2 = \mathcal{N}_2 \cup \mathcal{PSD}_2. \tag{4.3}$$

Indeed, let the 2×2 matrix A be copositive. Then

$$a_{11} \geq 0, \quad a_{22} \geq 0.$$

If we also have $a_{12} \geq 0$, then $A \in \mathcal{N}_2$. In the case $a_{12} < 0$, we consider the inner product (Ax, x) for the vectors $x = (x_1, x_2)^T$ with a nonzero first component x_1. Setting $t = x_2/x_1$ yields

$$\begin{aligned}(Ax, x) &= a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 \\ &= x_1^2(a_{11} + 2a_{12}t + a_{22}t^2).\end{aligned}$$

For the polynomial

$$f(t) = a_{11} + 2a_{12}t + a_{22}t^2$$

to be nonnegative for any $t \geq 0$, it is necessary, first of all, that the inequality $a_{22} > 0$ hold. Also, the value $f(t_0)$ that the polynomial assumes at the point of minimum $t_0 = -\frac{a_{12}}{a_{22}}$ must be nonnegative, i.e.,

$$a_{11} - \frac{a_{12}^2}{a_{22}} \geq 0. \tag{4.4}$$

The inequality $a_{22} > 0$ combined with (4.4) ensures the positive semidefiniteness of the matrix A.

Corollary 4.2. *The conditions that are necessary and sufficient for the symmetric 2×2 matrix A to be copositive can be described by the set of inequalities*

$$\begin{aligned}&a_{11} \geq 0, \quad a_{22} \geq 0, \\ &a_{12} + \sqrt{a_{11}a_{22}} \geq 0.\end{aligned} \tag{4.5}$$

Formally, conditions (4.5) involve radicals. However, using relation (4.3), one can easily avoid computing radicals when verifying the copositivity of a matrix.

Corollary 4.3. *Since any principal 2×2 submatrix of the matrix $A \in \mathcal{C}_n$ is also copositive, one must have*

$$a_{ij} + \sqrt{a_{ii}a_{jj}} \geq 0 \quad \forall\, i, j, \quad i \neq j. \tag{4.6}$$

Among other things, this implies that if $a_{ii} = 0$ for the copositive matrix A, then $a_{ij} = a_{ji} \geq 0$ for any j.

Remark. In the same way as equality (4.3) was justified, one can prove the following assertion. Any strictly copositive 2×2 matrix is either a nonnegative matrix with positive diagonal entries or a positive-definite matrix.

Familiarity with the conjugate cone $\mathcal{C}_n^*$ allows one to better understand how the cone $\mathcal{C}_n$ itself is organized. Recall that, for the cone $\mathcal{K}$ in the Euclidean space E, the conjugate (or dual) cone $\mathcal{K}^*$ is defined by the formula

$$\mathcal{K}^* = \{\, y \mid (x, y) \geq 0 \quad \forall\, x \in \mathcal{K}\}.$$

If $\mathcal{K}^* = \mathcal{K}$, then $\mathcal{K}$ is a self-conjugate cone.

For the matrix spaces M_n and S_n, the most natural inner product is

$$(A, B) = \operatorname{tr}(AB^T) = \sum_{i,j=1}^{n} a_{ij}b_{ij}. \tag{4.7}$$

Obviously, $\operatorname{tr}(AB^T) \geq 0$ for any $A, B \in \mathcal{N}_n$. Hence,

$$\mathcal{N}_n \subset \mathcal{N}_n^*. \tag{4.8}$$

We denote by E_{ij} the $n \times n$ matrix whose only nonzero entry is equal to 1 and placed in the position (i, j). It is easily seen that $(A, E_{ij}) = \operatorname{tr}(AE_{ji}) = a_{ij}$.

For the time being, we assume M_n to be an underlying Euclidean space. Therefore, $\mathcal{N}_n$ will be regarded as the cone of *all* nonnegative matrices (i.e., including nonsymmetric ones). Note that the condition $(A, B) \geq 0 \ \forall\, B \in \mathcal{N}_n$ implies, in particular, that

$$(A, E_{ij}) \geq 0 \quad \forall\, i, j.$$

Hence, any matrix $A \in \mathcal{N}_n^*$ must be nonnegative. Comparing this fact to (4.8) shows that $\mathcal{N}_n$ is a self-conjugate cone.

This result (i.e., that the cone $\mathcal{N}_n$ is self-conjugate) also holds in the case where $\mathcal{N}_n$ is interpreted as a subset of S_n; one must only replace the matrix E_{ij} in the argument above by the symmetric matrix $E_{ij} + E_{ji}$.

Using the spectral decomposition of a symmetric matrix, it is easy to prove the following proposition: the cone $\mathcal{PSD}_n$ of positive-semidefinite $n \times n$ matrices is self-conjugate as a subset of the space S_n.

Let us now find out what the cone $\mathcal{C}_n^*$ is, which is conjugate to the cone of copositive matrices with respect to the inner product (4.7). The definition below will be helpful.

Definition. A quadratic form $Q = (Bx, x)$ is called *completely positive* if Q can be expressed in the form

$$Q = \sum_{i=1}^{N} L_i^2. \tag{4.9}$$

Here L_i, $i = 1, \ldots, N$, are linear forms with nonnegative coefficients.

The matrix of a completely positive quadratic form is also called completely positive. If B is such a matrix, then it can be expressed in the form

$$B = \sum_{i=1}^{N} l_i l_i^T, \quad l_i \in \mathbf{R}_+^n, \quad i = 1, \ldots, N, \tag{4.10}$$

which corresponds to representation (4.9).

Remark. One can show that the classical definition of a completely positive matrix [2, Chap. XIII, Sec. 8] restricted to the case of square symmetric matrices is equivalent to the definition given above.

Theorem 4.1. *For the cone $\mathcal{C}_n$ of copositive $n \times n$ matrices, the conjugate cone (with respect to the inner product (4.7)) coincides with the cone $\mathcal{B}_n$ of completely positive $n \times n$ matrices:*

$$\mathcal{B}_n = \mathcal{C}_n^*. \tag{4.11}$$

Proof. The fact that $\mathcal{B}_n$ is really a cone is quite obvious from (4.10). Suppose that $A \in \mathcal{B}_n^*$. Then

$$\operatorname{tr}(AB) \geq 0 \quad \forall\, B \in \mathcal{B}_n. \tag{4.12}$$

In particular, setting here

$$B = xx^T, \quad x \in \mathbf{R}_+^n,$$

one has

$$\operatorname{tr}(Ax, x^T) = x^T A x = (Ax, x) \geq 0.$$

Since this inequality holds for any $x \in \mathbf{R}_+^n$, the matrix A is copositive. Thus, the inclusion

$$\mathcal{B}_n^* \subset \mathcal{C}_n \tag{4.13}$$

is established.

Assume now that $A \in \mathcal{C}_n$ and that B is a completely positive matrix. Using representation (4.10) for B, one finds

$$\operatorname{tr}(AB) = \sum_{i=1}^{N} \operatorname{tr}(Al_i, l_i^T) = \sum_{i=1}^{N} (Al_i, l_i) \geq 0.$$

Hence, $A \in \mathcal{B}_n^*$, and

$$\mathcal{C}_n \subset \mathcal{B}_n^*.$$

Combining this with (4.13) yields the equality

$$\mathcal{B}_n^* = \mathcal{C}_n,$$

which is equivalent to (4.11).

It is shown in [14] that the relation

$$\mathcal{C}_n = \mathcal{PSD}_n + \mathcal{N}_n \tag{4.14}$$

holds for $n = 3, 4$. Recall that, for subsets X and Y of the vector space V, the sum $X + Y$ is defined to be the set

$$\{x + y \mid x \in X,\, y \in Y\}.$$

M. Hall proved in [5, Chap. 16, Sec. 2] that equality (4.14) is false already for $n = 5$. Below we reproduce his arguments.

By conjugating both sides of (4.14), the equivalent relation is obtained, namely,

$$\mathcal{B}_n = \mathcal{C}_n^* = \mathcal{PSD}_n^* \cap \mathcal{N}_n^* = \mathcal{PSD}_n \cap \mathcal{N}_n. \tag{4.15}$$

Thus, in order to disprove (4.14), it suffices to produce a matrix that would be nonnegative and positive semidefinite but, at the same time, not completely positive. For $n = 5$, all these requirements are satisfied by the matrix B of the quadratic form

$$Q(x_1, \ldots, x_5) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_1x_2 + x_1x_5 + x_2x_3 + \frac{3}{2}x_3x_4 + x_4x_5$$

$$= (x_2 + \frac{1}{2}x_1 + \frac{1}{2}x_3)^2 + (x_5 + \frac{1}{2}x_1 + \frac{1}{2}x_4)^2 + \frac{1}{2}(x_1 - \frac{1}{2}x_3 - \frac{1}{2}x_4)^2 + \frac{5}{8}(x_3 + x_4)^2. \tag{4.16}$$

It is obvious from the first representation of the form Q that B is nonnegative, and from the second, that B is positive semidefinite. Suppose that B is completely positive. Then Q, being completely positive, admits the representation

$$Q = L_1^2 + \cdots + L_r^2 + L_{r+1}^2 + \cdots + L_N^2.$$

All linear forms L_i, $i = 1, \ldots, N$, have nonnegative coefficients and are numbered so that $L_1, \ldots, L_r$ are those in which x_3 and x_4 have positive coefficients. Since the form Q has no terms with the products x_1x_3, x_2x_4, and x_3x_5, the forms $L_1, \ldots, L_r$ must have zero coefficients for x_1, x_2, and x_5, or

$$L_i = l_3^{(i)}x_3 + l_4^{(i)}x_4, \quad l_3^{(i)} > 0,\ l_4^{(i)} > 0,\ i = 1, \ldots, r.$$

Hence,

$$L_1^2 + \cdots + L_r^2 = ax_3^2 + \frac{3}{2}x_3x_4 + bx_4^2, \quad a > 0, b > 0. \tag{4.17}$$

Let us set

$$Q_1 = Q_1(x_1, \ldots, x_5) = L_{r+1}^2 + \cdots + L_N^2.$$

Then

$$Q = ax_3^2 + \frac{3}{2}x_3x_4 + bx_4^2 + Q_1(x_1, \ldots, x_5). \tag{4.18}$$

Consider both sides of (4.18) when the unknowns x_3 and x_4 assume arbitrary values and the other unknowns are expressed in terms of x_3 and x_4 by the formulas

$$x_1 = \frac{x_3 + x_4}{2}, \quad x_2 = -\frac{3x_3 + x_4}{4}, \quad x_5 = -\frac{x_3 + 3x_4}{4}.$$

In this case (see (4.16)),

$$Q = \frac{5}{8}(x_3 + x_4)^2,$$

and (4.18) assumes the form

$$\frac{5}{8}(x_3 + x_4)^2 = ax_3^2 + \frac{3}{2}x_3x_4 + bx_4^2 + Q_1.$$

Since the quadratic form Q_1 is positive semidefinite, the inequalities

$$0 \le a \le \frac{5}{8}, \quad 0 \le b \le \frac{5}{8}$$

must hold. Now we set $x_3 = 1$, $x_4 = -1$ in (4.17) and have

$$(l_3^{(1)} - l_4^{(1)})^2 + \cdots + (l_3^{(r)} - l_4^{(r)})^2 = a + b - \frac{3}{2} \le \frac{5}{8} + \frac{5}{8} - \frac{3}{2} = -\frac{1}{4},$$

which is impossible. The source of the contradiction is the assumption that the form Q is completely positive. Therefore, Q and its matrix B cannot be completely positive.

Note now a remarkable spectral property of copositive matrices that makes them similar, to some extent, to nonnegative and positive-semidefinite matrices.

Definition. A square matrix A is said to have the *Perron property* if its spectral radius $\rho(A)$ is an eigenvalue of A.

Evidently, the Perron property is inherent in any positive-semidefinite matrix. The fact that all nonnegative matrices (including nonsymmetric ones) have the Perron property is the substance of the famous Perron–Frobenius theorem.

It turns out that the Perron property is also valid for copositive matrices [21].

Theorem 4.2. *The spectral radius $\rho(A)$ of the matrix $A \in \mathcal{C}_n$ is an eigenvalue of A.*

Proof. Suppose that $\rho = \rho(A)$ is not an eigenvalue of A. Then A must have $-\rho$ as an eigenvalue. Let x be a corresponding unit eigenvector:

$$Ax = -\rho x, \quad \| x \|_2 = 1.$$

We represent x as

$$x = y - z, \quad y \ge 0, \; z \ge 0, \; (y, z) = 0,$$

and set

$$u = y + z.$$

Obviously, $u \ge 0$, $\| u \|_2 = 1$. Furthermore,

$$(Ax, x) + (Au, u) = -\rho + (Au, u) = 2\,[(Ay, y) + (Az, z)] \ge 0.$$

Hence,

$$(Au, u) \ge \rho. \tag{4.19}$$

However, the values that the quadratic form (Av, v) assumes on unit vectors v cannot exceed the maximal eigenvalue λ_1 of the matrix A (see (2.6)). Thus, (4.19) implies

$$(Au, u) = \lambda_1 = \rho,$$

which contradicts the original assumption.

Remark. In fact, a more general result than Theorem 4.2 is proved in [21]. Let $\mathcal{K}$ be an arbitrary cone in $\mathbf{R}^n$. A matrix $A \in S_n$ is said to be copositive with respect to $\mathcal{K}$ if $(Ax, x) \ge 0$ for any $x \in \mathcal{K}$.

Theorem 4.3 [21]. *A matrix $A \in S_n$ has the Perron property if and only if A is copositive with respect to a self-conjugate cone $\mathcal{K} \subset \mathbf{R}^n$.*

It was pointed out at the beginning of this section that any principal submatrix of the copositive matrix A is also copositive. This explains why most criteria for copositivity are based on a sequential analysis of principal submatrices, arranged in increasing order from 1 to n. Thus, if the inspection of the main diagonal shows that some of its elements are negative, then A cannot be copositive. If all the diagonal entries are nonnegative, then A is still not copositive if at least one of inequalities (4.6) is violated, and so on.

If we use the inductive approach indicated above, then it is of major importance to examine the situation where, for a certain k, all principal $k \times k$ submatrices have already been tested and proved to be copositive, and one has to pass to the analysis of $(k+1) \times (k+1)$ submatrices. In this analysis, the terminology given below will be useful.

Definition. Let m be a positive integer, $1 \leq m \leq n$. A matrix $A \in S_n$ is called *(strictly) copositive of order* m if any principal $m \times m$ submatrix of A is (strictly) copositive. If A is a (strictly) copositive matrix of order m, but not of order $m+1$, then m is said to be the *exact order* or *index* of (strict) copositivity of A.

In these terms, we have to verify whether the matrix $A \in S_n$ is (strictly) copositive if it is already known that A is (strictly) copositive of order $n-1$. Following [18], we introduce one more auxiliary definition.

Definition. Let $A, B \in S_n$, and B be strictly copositive. The pair (A, B) is called *codefinite* if

$$Ax = \lambda Bx, \quad x > 0, \tag{4.20}$$

implies that $\lambda \geq 0$. If (4.20) implies that $\lambda > 0$, then the pair (A, B) is said to be *strictly codefinite.*

Let us consider the reasoning that is used in the proofs of several subsequent assertions. Suppose that $A \in S_n$ is a copositive matrix of order $n-1$; at the same time, A is not copositive. Then, for some vectors in the nonnegative orthant $\mathbf{R}_+^n$, the values of the Rayleigh ratio

$$\varphi(x) = \frac{(Ax, x)}{(x, x)} \tag{4.21}$$

are negative. Since all the principal $(n-1)\times(n-1)$ submatrices are copositive, functional (4.21) is nonnegative on the boundary of $\mathbf{R}_+^n$. Therefore, for the Rayleigh ratio, the minimum in $\mathbf{R}_+^n$ is furnished by an interior vector $x_0 > 0$.

It is well known that the gradient of functional (4.21) at the point x_0 is expressed by the formula

$$\operatorname{grad} \varphi(x_0) = \frac{2}{\| x_0 \|^2}(Ax_0 - \varphi(x_0)x_0).$$

If x_0 delivers a local minimum to the Rayleigh functional, then x_0 is an eigenvector of the matrix A, and $\varphi(x_0)$ is the corresponding eigenvalue. In the case under consideration, the matrix A must have a *positive* eigenvector associated with a *negative* eigenvalue.

Instead of the eigenvalues and eigenvectors of the matrix A, one can examine those of the problem $Ax = \lambda Bx$, where $B \in S_n$ is a strictly copositive matrix. Assume that A, not being copositive, has the index of copositivity $n-1$. Repeating the argument above with obvious alterations, one arrives at the following conclusion: there exists a positive eigenvector x_0 for the pair (A, B) that corresponds to the negative eigenvalue λ_0. In particular, taking as B the rank-1 matrix

$$B = b\,b^T,$$

where b is a positive vector, we find that

$$Ax_0 = \lambda_0 b\, b^T x_0 = \mu\, b, \tag{4.22}$$

the scalar factor $\mu = (x_0, b)\lambda_0$ being negative.

Theorem 4.4 [18]. *Let $A \in S_n$ be a (strictly) copositive matrix of order $n-1$. Then the following statements are equivalent.*

(1) *A is (strictly) copositive.*

(2) *There exists a strictly copositive matrix $B \in S_n$ such that the pair (A, B) is (strictly) codefinite.*

(3) *For any strictly copositive matrix $B \in S_n$ the pair (A, B) is strictly copositive.*

Proof. Assume (1), and let $B \in S_n$ be a strictly copositive matrix. Suppose that the vector x in

$$Ax = \lambda B x$$

is positive. Then the number

$$\lambda = \frac{(Ax, x)}{(Bx, x)}$$

is nonnegative (positive, respectively) since A is copositive (strictly copositive). Thus, the pair (A, B) is codefinite (strictly codefinite).

The implication (3) $\longrightarrow$ (2) is obvious. It remains to show that (2) $\longrightarrow$ (1). Suppose that the matrix A is not copositive. Then the pair (A, B) admits a positive eigenvector x_0 associated with the negative eigenvalue λ_0. This contradicts the fact that (A, B) is a codefinite pair. The case where the pair (A, B) is strictly codefinite can be analyzed in a similar way.

Theorem 4.5 [18]. *Let $A \in S_n$ be a copositive matrix of order $n-1$. Then the following statements are equivalent.*

(1) *A is not copositive.*

(2) *For any positive vector b, there exists a vector $x > 0$ such that $Ax = \mu b$, $\mu < 0$.*

(3) *The matrix A^{-1} exists and is nonpositive.*

(4) $\det A < 0$, *and the adjoint matrix $\widehat{A} = \operatorname{adj} A$ is nonnegative.*

Proof. The implication (1) $\longrightarrow$ (2) is essentially established above (see (4.22)). Conversely, it follows from (2) that the pair (A, bb^T) is not codefinite; hence, A is not copositive. Thus, statements (1) and (2) are equivalent.

Now assume (2) and suppose that $Ay = 0$. For any $b > 0$, the equation $Ax = b$ admits a solution x which implies

$$y^T b = y^T A x = 0.$$

Since b is an arbitrary positive vector, the vector y must be zero. Therefore, A is nonsingular. Moreover, it follows from (2) that $A^{-1}b < 0$ when $b > 0$. Regarding the coordinate vectors e_i as the limits of sequences of positive vectors, we infer that the vectors $A^{-1}e_i$, i.e., the columns of the inverse matrix A^{-1}, must be nonpositive. As for the reverse implication (3) $\longrightarrow$ (2), it is obvious.

The implication (4) $\longrightarrow$ (3) is equally obvious. Thus, the theorem will be proved completely if we prove the implication (1) $\longrightarrow$ (4).

According to the reasoning that precedes Theorem 4.4, the matrix A has a negative eigenvalue λ with an associated positive eigenvector x. Suppose that another eigenvalue μ of A is also negative and that y is the corresponding eigenvector. One can assume that $(x, y) = 0$. Clearly, the vector y has positive as well as negative entries. Hence, we can construct a linear combination

$$z = x + \alpha y \tag{4.23}$$

which satisfies the following conditions: (a) the vector z is nonnegative, and (b) at least one of its entries is zero.

Vector (4.23) belongs to the boundary of the orthant $\mathbf{R}^n_+$. Since A is a copositive matrix of order $n-1$, the inequality

$$(Az, z) \geq 0$$

holds. On the other hand,

$$(A(x + \alpha y), x + \alpha y) = \lambda(x, x) + \mu\alpha^2(y, y) < 0.$$

This contradiction proves that the matrix A has only one negative eigenvalue; thus, $\det A < 0$. Since (1) and (3) are equivalent, the nonnegativity of the adjoint matrix follows immediately.

The "strictly copositive version" of Theorem 4.5 can be proved in a similar way.

Theorem 4.6 [18]. *Let $A \in S_n$ be a strictly copositive matrix of order $n-1$. Then the following statements are equivalent:*

(1) *A is not strictly copositive.*

(2) *For any positive vector b, there exists a vector $x > 0$ such that $Ax = \mu b$, $\mu \leq 0$.*

(3) $\det A \leq 0$, *and the adjoint matrix* $\operatorname{adj} A$ *is positive.*

Remark. We use Theorems 4.5 and 4.6 to give a different proof of the criteria for copositivity and strict copositivity in the case of 2×2 matrices. For $n = 2$, the copositivity of order $n-1$ is equivalent to the relations

$$a_{11} \geq 0, \quad a_{22} \geq 0.$$

According to statement (4) of Theorem 4.5, A *is not* copositive if and only if

$$\det A = a_{11}a_{22} - a_{12}^2 < 0$$

and

$$-a_{12} = (\operatorname{adj} A)_{21} > 0.$$

On the other hand, inequalities (4.5) are equivalent to A being copositive.

In the same way, one can deduce the criterion of strict copositivity,

$$a_{11} > 0, \quad a_{22} > 0, \quad a_{12} + \sqrt{a_{11}a_{22}} > 0$$

from statement (3) of Theorem 4.6.

Remark. In [18], Theorems 4.5 and 4.6 are used to produce criteria for copositivity and strict copositivity in the case of 3×3 matrices. The copositivity of order $n-1$ means here that, first, the main diagonal of A must be nonnegative and, second, the inequalities

$$a_{12} + \sqrt{a_{11}a_{22}} \geq 0, \quad a_{13} + \sqrt{a_{11}a_{33}} \geq 0, \quad a_{23} + \sqrt{a_{22}a_{33}} \geq 0 \tag{4.24}$$

must be satisfied (see (4.6)). If these requirements are met, A will be copositive if and only if at least one of the conditions below holds:

$$\det A \geq 0 \tag{4.25}$$

and

$$a_{12}\sqrt{a_{33}} + a_{13}\sqrt{a_{22}} + a_{23}\sqrt{a_{11}} + \sqrt{a_{11}a_{22}a_{33}} \geq 0. \tag{4.26}$$

To obtain a criterion of strict copositivity, one must require that, first, the main diagonal of A be positive, second, inequalities (4.24) be satisfied, with the sign $\geq$ replaced by $>$, and, third, one of the conditions (4.25) and (4.26) hold; in the third case, the sign $\geq$ in (4.25) must also be replaced by $>$.

The bad feature of these criteria is that they involve radicals. Later in this section, rational criteria for copositivity will be described that are applicable to low-order matrices. For the time being, we continue discussing the case of an arbitrary n.

In [37], Theorems 4.5 and 4.6 are supplemented by the following assertion.

Theorem 4.7. *Let $A \in S_n$ be a matrix with the index of copositivity $n-1$. Then*

(1) *$\operatorname{In} A = (n-1, 1, 0)$.*

(2) *A is positive semidefinite of order $n-1$.*

(3) *If A is strictly copositive of order $n-1$, then it is positive definite of order $n-1$, and the inverse matrix A^{-1} is negative.*

[We mention that the notions of positive definiteness and positive semidefiniteness of order m are introduced by a complete analogy with the above definitions for the copositivity and strict copositivity of order m.]

Proof. The first statement of this theorem was already proved when the implication (1) $\longrightarrow$ (4) was justified in Theorem 4.5. Let A_{n-1} be an arbitrary principal $(n-1) \times (n-1)$ submatrix of the matrix A. Then (1) and interlacing inequalities (2.8) imply that all the eigenvalues μ_i of A_{n-1}, except, perhaps, for the smallest one, are positive. However, observe that $\det A_{n-1}$ is nonnegative since its value is a diagonal entry of the adjoint matrix $\operatorname{adj} A$. Hence, for the smallest eigenvalue μ_{n-1} one obtains

$$\mu_{n-1} \geq 0. \tag{4.27}$$

This proves statement (2) of the theorem. By Theorem 4.6, if A is strictly copositive of order $n-1$, then the adjoint matrix is positive; therefore we have $\mu_{n-1} > 0$ instead of (4.27). Since $\det A < 0$, the inverse matrix A^{-1} must be negative.

In [37], one can also find the following extension of Theorem 4.7.

Theorem 4.8. *Let $A \in S_n$ be a matrix with the index of copositivity $n-1$. Then*

(1) *A is positive definite of order $n-2$.*

(2) *In the inverse matrix A^{-1}, all the principal minors, with the possible exception of diagonal entries, are negative.*

(3) *A^{-1} is nonpositive, and its off-diagonal entries are negative.*

Proof. Assume that A contains a singular principal submatrix A_{n-2}. No generality will be lost in considering A_{n-2} as the leading principal submatrix. We denote by r the rank of A_{n-2}. By permuting symmetrically the first $n-2$ rows and columns in A, one can always achieve that the leading principal $r \times r$ submatrix A_r will be nonsingular. In the Schur complement $C = A/A_r$, the diagonal entry c_{11} is zero; however, some off-diagonal entries of the first column must be nonzero. Otherwise, $\det A = \det A_r \det C = 0$, which contradicts statement (4) of Theorem 4.5.

Suppose that the entry $c_{k1}, k > 1$, is nonzero. Consider the principal $(r+2) \times (r+2)$ submatrix

$$A(1, \ldots, r, r+1, r+k).$$

Note that its order $r+2$ does not exceed $n-1$. This submatrix is congruent to the direct sum

$$A_r \oplus \begin{pmatrix} 0 & c_{k1} \\ c_{k1} & c_{kk} \end{pmatrix}. \tag{4.28}$$

Since the second term in (4.28) has inertia $(1,1,0)$, the submatrix under consideration is not positive semidefinite, contrary to the second statement of Theorem 4.7.

Statement (2) is immediate from (1) if we bear in mind the classical relationship between the minors of the matrix A and those of its inverse $B = A^{-1}$ [2, Chap. I, Sec. 4]. Being applied to the principal minors, this relation has the form

$$\det B(i_1', \ldots, i_{n-k}') = \frac{\det A(i_1, \ldots, i_k)}{\det A}.$$

Here $i_1', \ldots, i_{n-k}'$ is a subset of the index set $1, 2, \ldots, n$, complementing the subset $i_1, \ldots, i_k$.

Assume that the off-diagonal entry b_{ij} in $B = A^{-1}$ is zero. Then, for the corresponding principal 2×2 minor, one has

$$\begin{vmatrix} b_{ii} & b_{ij} \\ b_{ij} & b_{jj} \end{vmatrix} \geq 0,$$

in contradiction to statement (2).

Using the results above, one can give a complete description of matrices with the index of copositivity $n-1$.

Theorem 4.9. *Any of the two sets of conditions below is necessary and sufficient for a matrix $A \in S_n$ to have the index of copositivity $n-1$:*

(1) $\operatorname{In} A = (n-1, 1, 0)$ *and A^{-1} is nonpositive;*

(2) $\det A < 0$, *A^{-1} is nonpositive, and all the principal minors of A up to the order $n-2$ are positive.*

Proof. We shall first show that both sets of conditions are equivalent. In the implication (1) $\longrightarrow$ (2), the only nontrivial assertion is that of the positivity of the principal minors. Assume that the leading principal minor Δ_k, $k \leq n-2$, is nonpositive. Consider first the case where $\Delta_k < 0$. Then at least one of the eigenvalues of the leading principal $k \times k$ submatrix A_k is negative. A similar assertion is true for the principal submatrix A_{n-1}. The conditions $\det A < 0$, $A^{-1} \leq 0$ imply specifically that all the principal minors of order $n-1$ in A are nonnegative; hence, $\Delta_{n-1} \geq 0$. In such an event, aside from the smallest eigenvalue μ_{n-1}, which is negative, the submatrix A_{n-1} must have another nonpositive eigenvalue. As a consequence, at least two eigenvalues of A are negative, which contradicts (1).

Now assume that $\Delta_k = 0$. Let r be the rank of the principal submatrix A_k; obviously, $r < n-2$. Applying the same reasoning as in the proof of the first statement of Theorem 4.8, we can show that a principal submatrix of order $\leq n-1$ exists that has a negative eigenvalue. Then the leading principal submatrix A_{n-1} must also have a negative eigenvalue. The rest of the proof is similar to the analysis of the case $\Delta_k < 0$.

Next we prove the reverse implication (2) $\longrightarrow$ (1). Denoting by Δ_{n-1} the determinant of the leading principal submatrix A_{n-1}, we again use the inequality $\Delta_{n-1} \geq 0$, which follows from the conditions $\det A < 0$, $A^{-1} \leq 0$. In conjunction with the positivity of the leading principal minors of smaller orders, this inequality shows that the submatrix A_{n-1} is positive semidefinite. Therefore, the eigenvalue λ_{n-1} of the matrix A must be nonnegative. Since $\det A < 0$, we have, in fact, $\lambda_{n-1} > 0$, $\lambda_n < 0$, and

$$\operatorname{In} A = (n-1, 1, 0).$$

The fact that both sets of conditions (1) and (2) are necessary is established by Theorems 4.7 and 4.8. Now we shall prove that they are sufficient. Since set (1) is invariant under symmetric reorderings of rows and columns in A, the same must be true for set (2). In the preceding paragraph, the positive semidefiniteness of the leading principal submatrix A_{n-1} was derived from set (2). The invariance pointed out above implies that in reality *all* the principal $(n-1) \times (n-1)$ submatrices are positive semidefinite and, hence, copositive. Thus, the matrix A is copositive of order $n-1$.

To prove that A is not copositive, one can use the "anti-Perron" property of the inverse matrix $B = A^{-1}$. By the hypothesis, $B \leq 0$; hence, the negative number $-\rho(B)$ is an eigenvalue of B. It is associated with the "Perron" eigenvector x all of whose nonzero components are positive. Recall that B and A have identical eigenvectors. Thus, A has a nonnegative eigenvector x with the associated negative eigenvalue

$$-\frac{1}{\rho(B)} = \frac{(Ax, x)}{(x, x)}.$$

However, the inequalities $x \geq 0$, $(Ax, x) < 0$ are incompatible with copositivity. We infer that A has the index of copositivity $n-1$, which completes the proof of the theorem.

The next two assertions describe a special subset of copositive matrices. They are proved in the same way as the theorems above [37].

Theorem 4.10. *Let $A \in S_n$ be a copositive matrix with index of strict copositivity $n-1$. Then*

(1) $\operatorname{In} A = (n-1, 0, 1)$, *and the zero eigenvalue of A is associated with a positive eigenvector*;

(2) *A is positive semidefinite of rank $n-1$*;

(3) *A is positive definite of order $n-1$.*

Theorem 4.11. *The set of conditions below is necessary and sufficient for the matrix $A \in S_n$ to be copositive with index of strict copositivity $n-1$:* (a) $\det A = 0$, (b) *the leading principal minors of A up to the order $n-1$ are positive, and* (c) *A has a positive eigenvector associated with the zero eigenvalue.*

Now that the special cases of copositivity have been described, one can obtain a complete characterization of it. The two auxiliary assertions below will be helpful.

Lemma 4.4. *Let $A \in S_n$ be a copositive matrix. Then the relations $x_0 \geq 0$ and $x_0^T A x_0 = 0$ imply the inequality $Ax_0 \geq 0$.*

Proof. One can deduce from the hypothesis of the lemma that the vector x_0 supplies a minimum in the nonnegative orthant to the quadratic form

$$\psi(x) = (Ax, x). \tag{4.29}$$

Therefore, for any coordinate vector e_i, we have

$$\left.\frac{\partial\psi}{\partial x_i}\right|_{x_0} = \lim_{\alpha\longrightarrow+0}\frac{\psi(x_0+\alpha e_i)-\psi(x_0)}{\alpha} = \lim_{\alpha\longrightarrow+0}\frac{(A(x_0+\alpha e_i), x_0+\alpha e_i)}{\alpha} \geq 0.$$

Thus,

$$\operatorname{grad}\psi = \left(\frac{\partial\psi}{\partial x_1},\ldots,\frac{\partial\psi}{\partial x_n}\right)^T \geq 0.$$

Now the assertion of the lemma is immediate from the well-known formula

$$\operatorname{grad}\psi = 2A\,x.$$

Lemma 4.5. *Let $A \in S_n$ be a nonsingular copositive matrix. Then no column of the inverse matrix $B = A^{-1}$ can be nonpositive.*

Proof. Assume the contrary, i.e., that the ith column b_i of B is nonpositive. Let $x = -b_i$. Then $x \geq 0$, and

$$y = A\,x = -A\,b_i = -e_i \leq 0. \tag{4.30}$$

This yields

$$b_{ii} = (e_i, b_i) = (y, x) = (Ax, x) \leq 0.$$

Since A is copositive, we must, in fact, have the equality $(Ax, x) = 0$. This equality, combined with (4.30) and the assumption $x \geq 0$, contradicts the previous lemma.

Now we can state a criterion for copositivity. To be more exact, this is a criterion for the opposite property, i.e., the lack of copositivity.

Theorem 4.12 [37]. *A matrix $A \in S_n$ is not copositive if and only if it contains a nonsingular principal submatrix D such that a certain column of the inverse matrix D^{-1} is nonpositive.*

Proof. The sufficiency part of the theorem is almost obvious. According to Lemma 4.5, the submatrix D cannot be copositive, but then the whole matrix A is not copositive either.

To prove the necessity part, assume that the index of copositivity of A is k, $k \leq n-1$ (including the case $k = 0$). Then A contains a principal $(k+1) \times (k+1)$ submatrix, say, D which is not copositive. By Theorem 4.5, the inverse matrix D^{-1} exists and is nonpositive.

Theorem 4.12 essentially coincides with the determinantal criterion of copositivity due do E. Keller.

Theorem 4.13 [23]. *A matrix $A \in S_n$ is not copositive if and only if it contains a principal submatrix D with $\det D < 0$ for which all the cofactors of the last column are nonnegative.*

Proof. If we replace the word "last" in this formulation by the word "certain," then the identity of both statements, the present one and that of Theorem 4.12, will be obvious. However, the mention of the last column does not have any real significance. Indeed, it was found in the proof of necessity that the whole matrix D^{-1} is nonpositive, i.e., *any* column of the adjoint matrix is nonnegative.

The criteria of strict copositivity below are justified in a similar way.

Theorem 4.14 [37]. *A matrix $A \in S_n$ is not strictly copositive if and only if at least one of the following conditions is satisfied:*

(a) *A contains a nonsingular principal submatrix D such that a certain column of D^{-1} is nonpositive;*

(b) *A contains a singular positive semidefinite principal submatrix with a nonnegative eigenvector attached to the zero eigenvalue.*

Theorem 4.15. *A matrix $A \in S_n$ is not strictly copositive if and only if it contains a principal submatrix D with $\det D \leq 0$ for which all the cofactors of the last column are positive.*

This determinantal criterion for strict copositivity was found by Motzkin in 1967 [31]. It is expedient to clarify the relationship between condition (b) of Theorem 4.14 and the Motzkin condition for the case where $\det D = 0$. Suppose that A has the index of strict copositivity k. Then D is a singular principal $(k+1) \times (k+1)$ submatrix that is not strictly copositive. According to statement (3) of Theorem 4.6, the adjoint matrix $\widehat{D} = \operatorname{adj} D$ is positive. If, for a singular D, the matrix $\widehat{D}$ is not zero, then $\operatorname{rank} D = k$. In this case, $\operatorname{rank} \widehat{D} = 1$, and any column of $\widehat{D}$ is a solution to the homogeneous linear system $Dx = 0$. In other words, any column of the adjoint matrix is a positive eigenvector of D associated with the zero eigenvalue.

Recall that the use of criteria of this kind presupposes a sequential analysis of the principal submatrices arranged in increasing order. The procedure terminates as soon as a noncopositive submatrix is found. Indeed, in this case the matrix A itself cannot be copositive. The positive answer for copositivity can only be obtained when *all* the principal submatrices are inspected. Therefore, in general, the amount of computational work in these criteria grows exponentially with the order n of a matrix.

This rapid growth is, to a certain extent, unavoidable. It was proved in [32] that the problem of verifying whether a given square integer matrix is copositive or not is NP-complete. This explains why the situations that make it possible to significantly reduce the inspection of principal submatrices are so important. Two situations of this kind are discussed in [37].

Theorem 4.16. *Suppose that a matrix $A \in S_n$ has p positive eigenvalues, $p < n$. Then A is (strictly) copositive if and only if it is (strictly) copositive of order $p+1$.*

Proof. For definiteness, we shall consider only the statement relating to copositive matrices. The strictly copositive case can be proved similarly.

The necessity part of the theorem is obvious. To prove the sufficiency part, assume that A is not copositive and has index of copositivity l. Then A contains a principal $(l+1) \times (l+1)$ submatrix $\widehat{A}$ that is not copositive. According to Theorem 4.7, the submatrix $\widehat{A}$ must have l positive eigenvalues. Then A has at least l positive eigenvalues, which implies that $l \leq p$. However, this means that A contains a noncopositive principal submatrix of order $\leq p+1$, contrary to the hypothesis of the theorem.

Theorem 4.17. *Suppose that a matrix $A \in S_n$ is singular of rank r. Then A is copositive if and only if it is copositive of order r.*

Proof. Only the sufficiency part needs proving. If $r = \pi(A)$, then A is positive semidefinite and, hence, copositive. For $r > \pi(A)$, the preceding theorem can be applied. Indeed, the copositivity of A follows from the fact that it is copositive of order $\pi(A) + 1 \leq r$.

The strictly copositive version of Theorem 4.17 is stated as follows.

Theorem 4.18. *Suppose that a matrix $A \in S_n$ is singular of rank r. Then A is strictly copositive if and only if it is strictly copositive of order $r+1$.*

Thus, if the rank or the positive inertia of a matrix $A \in S_n$ is considerably smaller than n, then, in order to get a positive answer to the question concerning its copositivity or strict copositivity, one can terminate the inspection of the principal submatrices much earlier than in the general case.

Criteria of the Motzkin or Keller type are called *inner* criteria in [37], because when analyzing a principal submatrix D, they do not use information concerning the part of the matrix A that is exterior to D. There exist criteria of a different type, which are called *outer* criteria; the first of them were constructed in [37]. We give one of these criteria without proof. However, some preliminary nomenclature must be introduced simply for formulating it.

Assume that a leading principal submatix A_{11} in a matrix $A \in S_n$ is nonsingular. We partition A as in (2.10) and form the $n \times n$ matrix

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, \tag{4.31}$$

where

$$C_{11} = A_{11}^{-1}, \quad C_{12} = -A_{11}^{-1}A_{12}, \quad C_{21} = A_{12}^{T}A_{11}^{-1}, \tag{4.32}$$

and $C_{22} = A/A_{11}$ is the Schur complement of the submatrix A_{11} in A. In contrast to A, the matrix C is generally not symmetric since $C_{21} = -C_{12}^T$. The transition from A to C is called the *principal block pivotal operation with pivot* A_{11}.

The principal block pivoting motivates a remarkable representation of the quadratic form (4.29). Let

$$y = Ax \tag{4.33}$$

and partition the vectors x and y in accordance with the partitioning of A:

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Since the block A_{11} is nonsingular, the subvector x_1 can be expressed in terms of y_1 and x_2 (see (4.33)),

$$x_1 = A_{11}^{-1}y_1 - A_{11}^{-1}A_{12}x_2. \tag{4.34}$$

Substituting this relation into the second block equality (4.33) yields

$$y_2 = A_{12}^{T}A_{11}^{-1} + (A_{22} - A_{12}^{T}A_{11}^{-1}A_{12})x_2. \tag{4.35}$$

Hence, the matrix of Eqs. (4.34)–(4.35) is precisely matrix the (4.31)–(4.32).

If, in the quadratic form $\psi(x) = (Ax, x)$, the original vector of the unknowns x is replaced by the new vector

$$z = \begin{pmatrix} y_1 \\ x_2 \end{pmatrix},$$

then one gets

$$\psi(x) = (C_{11}y_1, y_1) + (C_{22}x_2, x_2).$$

Thus, the block elimination in the matrix A is associated with the block decomposition of the corresponding quadratic form.

In outer criteria for copositivity, the inspection of a current principal submatrix D is connected with an analysis of the corresponding block pivotal operation. By a symmetric permutation of rows and columns of A, one can place D in the position of the block A_{11} in (2.10). Then the principal block pivotal operation is described by matrix (4.31). If the submatrix A_{11} is of order k, then let $l = n - k$ be the order of the block C_{22}.

Definition. Let A_{11} be a given principal submatrix. We say that situation I occurs if, for a certain i, $i = 1, \dots, l$, the diagonal entry c_{ii} of the block C_{22} and row i of the block C_{21} are *nonpositive*. If, for the index i above, there exists a *negative* entry in row i of the block C_{21}, then we say that situation II occurs.

Theorem 4.19 [37]. *A matrix $A \in S_n$ is not (strictly) copositive if and only if situation* II (*situation* I) *occurs for a positive-definite principal submatrix D of A.*

Remark. A case is possible where the submatrix D in Theorem 4.19 is vacuous. The vacuous square matrix is considered to be positive definite, and, in this case, the matrix A itself must be interpreted as the Schur complement C_{22}.

It is claimed in [37] that, from the computational standpoint, the outer criteria are much more efficient than the inner ones because the former require only the inspection of *positive-definite* principal submatrices. However, no numerical experiments are reported that would support this claim. As for the argument presented above, one must say that, being purely speculative, it is rather weak since something akin to it can be said of criteria of the Motzkin–Keller type, namely, they involve only principal submatrices with *nonpositive determinants.* It is not clear a priori what principal submatrices are larger in number in the given matrix

A: those that are important for an outer criterion or those that are accounted for by an inner criterion. For example, a positive-definite matrix A does not contain any principal submatrices with nonpositive determinants; on the contrary, all the principal submatrices are positive definite. It should be added that a step of the outer criterion that amounts to a principal block pivotal operation is considerably more labor-consuming than one of the inner criterion.

Both inner and outer criteria organize the inspection of principal submatrices in the "bottom-up" direction, i.e., from the smallest order to the largest one. However, the opposite direction should not be neglected. In some situations, the testing of an $n \times n$ matrix A for copositivity can easily be reduced to a similar test for one or several matrices of a smaller order. This approach was recently pursued in [7, 24].

The simplest situation of this kind is described by the lemma below.

Lemma 4.6. *Let the matrix $A \in S_n$ be partitioned:*

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1m} \\ A_{12}^T & A_{22} & \dots & A_{2m} \\ \dots & \dots & \dots & \dots \\ A_{1m}^T & A_{2m}^T & \dots & A_{mm} \end{pmatrix},$$

where all off-diagonal submatrices A_{ij}, $i \neq j$, are nonnegative. Then the (strict) copositivity of A amounts to the (strict) copositivity of all its principal submatrices $A_{11}, \dots, A_{mm}$.

Corollary 4.4. *Assume that the matrix $A \in S_n$ is partitioned:*

$$A = \begin{pmatrix} a_{11} & a^T \\ a & A_{n-1} \end{pmatrix}, \tag{4.36}$$

where the entry a_{11} and the vector a are nonnegative. Then the copositivity of A amounts to the copositivity of the principal submatrix A_{n-1}.

Remark. For convenience, here and in later formulations we consider only partitioning (4.36) of A, with the first row and the first column singled out. However, similar assertions clearly hold for other rows and columns of A.

The situation where the vector a in (4.36) is nonpositive is the next in order of complexity. To analyze it, we need two assertions from [24].

Lemma 4.7. *Assume that in the partitioned matrix (2.10) the block A_{11} is of order 2. For a real parameter t, we define the $(n-1) \times (n-1)$ matrix*

$$B(t) = \begin{pmatrix} b_{11}(t) & b_{12}(t) \\ b_{12}^T(t) & A_{22} \end{pmatrix} \tag{4.37}$$

by the formulas

$$b_{11}(t) = (A_{11}u, u), \quad b_{12}(t) = u^T A_{12},$$

where

$$u = (t, 1-t)^T.$$

Then the matrix A is (strictly) copositive if and only if the matrix $B(t)$ is (strictly) copositive for any $t \in [0, 1]$.

Proof. Any vector $x \in \mathbf{R}_+^n$ with the first or second entry nonzero can be written as

$$x = \alpha \begin{pmatrix} t \\ 1-t \\ y \end{pmatrix}, \quad \alpha > 0, \quad y \in \mathbf{R}_+^{n-2}, \quad t \in [0, 1]. \tag{4.38}$$

It follows that the (strict) copositivity of the matrix A is equivalent to the following two requirements: first, the submatrix A_{22} must be (strictly) copositive, and, second, the inner product (Ax, x) must be nonnegative (positive) for any vector x of type (4.38). The last requirement amounts to that the scalar product (Bv, v), where

$$v = \begin{pmatrix} 1 \\ y \end{pmatrix} \in \mathbf{R}_+^{n-1},$$

be nonnegative or positive respectively. Combining this with the (strict) copositivity of the submatrix A_{22}, we conclude that $B(t)$ must be (strictly) copositive for any $t \in [0, 1]$.

Lemma 4.8. *Assume that a matrix $A \in S_n$ is partitioned as in (4.36). Then A is (strictly) copositive if and only if the three conditions below are met:*

(1) $a_{11} \geq 0 \quad (a_{11} > 0)$;

(2) *the submatrix A_{22} is (strictly) copositive;*

(3) *for any vector y such that*

$$y \in \mathbf{R}_+^{n-1}, \quad (a, y) \leq 0,$$

the inequality

$$((a_{11}A_{22} - aa^T)y, y) \geq 0 \quad (> 0, \text{ respectively}) \tag{4.39}$$

holds.

Proof. For definiteness, we shall prove the assertion relating to copositive matrices. The original definition of copositivity

$$(Ax, x) \geq 0 \quad \forall x \in \mathbf{R}_+^n \tag{4.40}$$

can be recast as the requirement that the 2×2 submatrix

$$\begin{pmatrix} a_{11} & (a, y) \\ (a, y) & (A_{22}y, y) \end{pmatrix} \tag{4.41}$$

be copositive for any vector $y \in \mathbf{R}_+^{n-1}$. This becomes clear if one writes the vector x in (4.40) as

$$x = \begin{pmatrix} t \\ sy \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} t \\ s \end{pmatrix},$$

with $(t, s)^T \in \mathbf{R}_+^2$.

If $(a, y) \geq 0$, then the copositivity of matrix (4.41) is immediate from conditions (1) and (2). For the case where $(a, y) < 0$, all three conditions (4.5) are needed to ensure copositivity. The last of them assumes the form of inequality (4.39).

Theorem 4.20. *Assume that the vector a in matrix (4.36) is nonpositive. Then A is (strictly) copositive if and only if the two conditions below are fulfilled:*

(1) $a_{11} \geq 0$ (> 0);

(2) *the $(n-1) \times (n-1)$ matrices A_{22} and $a_{11}A_{22} - aa^T$ are (strictly) copositive.*

Proof. Here we also restrict ourselves to considering only the copositive case. Since $a \leq 0$, the vector y in condition (3) of Lemma 4.8 is an arbitrary vector from $\mathbf{R}_+^{n-1}$. But then inequality (4.39) turns into the requirement that the matrix $a_{11}A_{22} - aa^T$ be copositive.

Corollary 4.5. *Assume that the vector a in matrix (4.36) is nonpositive and $a_{11} > 0$. Then A is (strictly) copositive if and only if the (only) $(n-1) \times (n-1)$ matrix $a_{11}A_{22} - aa^T$ is (strictly) copositive.*

Proof. If the matrix $B = a_{11}A_{22} - aa^T$ is (strictly) copositive, the same must be true for the matrix $B + aa^T = a_{11}A_{22}$. Since $a_{11} > 0$, the submatrix A_{22} is also (strictly) copositive. Thus, the hypothesis of Theorem 4.20 relating to A_{22} is fulfilled automatically.

Remark. Under the hypothesis of the corollary above, the matrix B differs from the Schur complement A/a_{11} only by a positive scalar factor a_{11}.

Another remarkable consequence of Theorem 4.20 is

Theorem 4.21. *Assume that all the off-diagonal entries of a matrix $A \in S_n$ are nonpositive. Then A is (strictly) copositive if and only if it is positive semidefinite (definite).*

Proof. The sufficiency part of the theorem is obvious. To prove the necessity, we use induction. Suppose that the assertion of the theorem is valid for any $(n-1) \times (n-1)$ matrix. If $a_{11} = 0$, then the vector a must be zero (see the remark after inequalities (4.6)). By the inductive hypothesis, the copositive submatrix A_{22} is positive semidefinite. Hence, the whole matrix A is positive semidefinite.

Now assume that $a_{11} > 0$. According to Theorem 4.20, the matrix $B = a_{11}A_{22} - aa^T$ is copositive and, hence, positive semidefinite. Writing a vector $x \in \mathbf{R}^n$ as

$$x = \begin{pmatrix} \alpha \\ y \end{pmatrix}, \quad \alpha \in \mathbf{R}, \quad y \in \mathbf{R}^{n-1},$$

one deduces the positive semidefiniteness of A from the identity

$$a_{11}(Ax, x) = (By, y) + [a_{11}\alpha + (a, y)]^2.$$

The strictly copositive case can be proved similarly.

In [24], Theorem 4.20 is used for constructing a *rational* criterion for copositivity of the 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}. \tag{4.42}$$

Theorem 4.22. *Assume that all the diagonal entries in matrix (4.42) are nonnegative (positive). Then*

(1) *If all the off-diagonal entries are nonnegative, then A is (strictly) copositive.*

(2) *If exactly one off-diagonal entry, say, a_{ij}, is negative, then A is (strictly) copositive if and only if the submatrix*

$$\begin{pmatrix} a_{ii} & a_{ij} \\ a_{ij} & a_{jj} \end{pmatrix} \tag{4.43}$$

is (strictly) copositive.

(3) *If the entries a_{ij} and a_{ik} are negative, then A is (strictly) copositive if and only if the 2×2 matrices*

$$\begin{pmatrix} a_{jj} & a_{jk} \\ a_{jk} & a_{kk} \end{pmatrix}, \quad \begin{pmatrix} a_{ii}a_{jj} - a_{ij}^2 & a_{ii}a_{jk} - a_{ij}a_{ik} \\ a_{ii}a_{jk} - a_{ij}a_{ik} & a_{ii}a_{kk} - a_{ik}^2 \end{pmatrix} \tag{4.44}$$

are (strictly) copositive.

Proof. The first assertion of the theorem follows from Lemma 4.6, the second follows from Corollary 4.5, and the third from Theorem 4.20.

Remark. For 2×2 matrices (4.43) and (4.44), the copositivity is tested by means of inequality (4.5). It was already noted that this inequality can be given a rational form.

The main result of [24] is a rational criterion for copositivity of the 4×4 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix}. \tag{4.45}$$

Suppose that the main diagonal of matrix (4.45) has already been inspected and all the diagonal entries a_{ii} proved to be nonnegative (positive). In the criterion below, eight cases are distinguished that correspond to distinct sign distributions for the off-diagonal entries a_{ij}. In each case, we either state that A is (strictly) copositive or claim that the (strict) copositivity problem for A is equivalent to the same problem for matrices of a smaller order, their number being one, two, or three. Note that, in the descriptions of the cases, the numbers i, j, k, and l are indices of *distinct* rows (or columns) of the matrix A.

Case 1. All the off-diagonal entries a_{ij} are nonnegative.

In this case, A is (strictly) copositive by Lemma 4.6.

Case 2. There is only one negative off-diagonal entry, say, a_{ij}.

The matrix A is (strictly) copositive if and only if

$$a_{ii}a_{jj} - a_{ij}^2 \geq 0 \quad (> 0). \tag{4.46}$$

Indeed, inequality (4.46) ensures the (strict) copositivity of submatrix (4.43). Consider the transition from this submatrix to A as a twofold augmentation by nonnegative rows and columns. Then it is clear that A must preserve the (strict) copositivity property.

Case 3. There are exactly two negative off-diagonal entries a_{ij} and a_{kl} belonging to distinct rows and columns of A.

The matrix A is (strictly) copositive if and only if the following inequalities hold:

$$a_{ii}a_{jj} - a_{ij}^2 \geq 0 \quad (> 0), \quad a_{kk}a_{ll} - a_{kl}^2 \geq 0 \quad (> 0). \tag{4.47}$$

Assume, for simplicity, that $i = 1, j = 2, k = 3$, and $l = 4$. Then inequalities (4.47) ensure the (strict) copositivity of the block diagonal matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \oplus \begin{pmatrix} a_{33} & a_{34} \\ a_{34} & a_{44} \end{pmatrix}.$$

One can obtain A from this matrix by adjoining nonnegative elements in the off-diagonal blocks. By Lemma 4.6, A preserves the (strict) copositivity property.

Case 4. There are exactly two negative off-diagonal entries a_{ij} and a_{ik} belonging to the same row i of A.

The matrix A is (strictly) copositive if and only if its principal submatrix corresponding to the indices i, j, and k is (strictly) copositive.

If we again set $i = 1, j = 2$, and $k = 3$, then, to obtain A, the leading principal 3×3 submatrix must be augmented by the fourth row and column, which are nonnegative. This augmentation preserves the (strict) copositivity property.

Case 5. There are exactly three negative off-diagonal entries a_{ij}, a_{jk}, and a_{ik} belonging to the same principal 3×3 submatrix of A.

The matrix A is (strictly) copositive if and only if the matrix

$$\begin{pmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ij} & a_{jj} & a_{jk} \\ a_{ik} & a_{jk} & a_{kk} \end{pmatrix} \tag{4.48}$$

is positive semidefinite (definite).

By Theorem 4.21, matrix (4.48) is (strictly) copositive if and only if it is positive semidefinite (definite). The transition from (4.48) to A can again be carried out by adjoining nonnegative numbers to the former matrix.

Case 6. There are exactly three negative off-diagonal entries a_{ij}, a_{ik}, and a_{il} belonging to the same row i of A.

The matrix A is (strictly) copositive if and only if the 3×3 matrix below is (strictly) copositive:

$$\begin{pmatrix} a_{ii}a_{jj} - a_{ij}^2 & a_{ii}a_{jk} - a_{ij}a_{ik} & a_{ii}a_{jl} - a_{ij}a_{il} \\ a_{ii}a_{jk} - a_{ij}a_{ik} & a_{ii}a_{kk} - a_{ik}^2 & a_{ii}a_{kl} - a_{ik}a_{il} \\ a_{ii}a_{jl} - a_{ij}a_{il} & a_{ii}a_{kl} - a_{ik}a_{il} & a_{ii}a_{ll} - a_{il}^2 \end{pmatrix}.$$

This claim is based on Corollary 4.5. Note that since the off-diagonal entries of the row i are negative, one must have $a_{ii} > 0$ (otherwise, A cannot be copositive; see inequality (4.6)).

For the two remaining cases, the analysis is much more complicated. Therefore, the claims below are not supported by explanations. We only point out that their justification in [24] uses Lemma 4.7.

Case 7. There are exactly three negative off-diagonal entries a_{ij}, a_{jk}, and a_{kl} (they are not contained in the same row or in the same principal submatrix of A).

The matrix A is (strictly) copositive if and only if inequality (4.46) holds and the 3×3 matrix B below is (strictly) copositive. The entries of the matrix B are given by the formulas

$$\begin{aligned}
b_{11} &= a_{kk}(a_{jj}a_{ik}^2 - 2a_{ik}a_{jk}a_{ij} + a_{ii}a_{jk}^2),\\
b_{22} &= a_{jj}a_{kk} - a_{jk}^2,\\
b_{33} &= a_{kk}a_{ll} - a_{kl}^2,\\
b_{12} &= a_{kk}(a_{jj}a_{ik} - a_{ij}a_{jk}),\\
b_{13} &= a_{kk}(a_{ik}a_{jl} - a_{jk}a_{il}),\\
b_{23} &= a_{kk}a_{jl} - a_{jk}a_{kl}.
\end{aligned} \tag{4.49}$$

We give an illustration of Case 7 taken from [24]. For the matrix

$$A = \begin{pmatrix} 2 & -2 & -1 & 2 \\ -2 & 3 & 2 & -3 \\ -1 & 2 & 1 & 1 \\ 2 & -3 & 1 & 4 \end{pmatrix} \tag{4.50}$$

the conditions above are met with $i = 3, j = 1, k = 2$. Calculating matrix (4.49), one finds

$$B = \begin{pmatrix} 12 & 6 & 18 \\ 6 & 2 & 0 \\ 18 & 0 & 3 \end{pmatrix}.$$

Being a nonnegative matrix with a positive main diagonal, B is strictly copositive. Since

$$a_{33}a_{11} - a_{13}^2 = 1,$$

the matrix A is strictly copositive as well.

Case 8. There are exactly four negative off-diagonal entries a_{ij}, a_{jk}, a_{kl}, and a_{il}.

The matrix A is (strictly) copositive if and only if inequality (4.46) holds and the two 3×3 matrices are (strictly) copositive, namely, matrix (4.49) and the matrix C with the entries

$$\begin{aligned}
c_{11} &= a_{ll}(a_{ii}a_{jl}^2 - 2a_{jl}a_{il}a_{ij} + a_{jj}a_{il}^2),\\
c_{22} &= a_{ii}a_{ll} - a_{il}^2,\\
c_{33} &= a_{kk}a_{ll} - a_{kl}^2,\\
c_{12} &= a_{ll}(a_{ii}a_{jl} - a_{ij}a_{il}),\\
c_{13} &= a_{ll}(a_{ik}a_{jl} - a_{il}a_{jk}),\\
c_{23} &= a_{ll}a_{ik} - a_{il}a_{kl}.
\end{aligned} \tag{4.51}$$

This case is also illustrated by an example in [24]. For the matrix

$$A = \begin{pmatrix} 3 & 2 & -2 & -2 \\ 2 & 8 & -3 & -3 \\ -2 & -3 & 2 & 2.5 \\ -2 & -3 & 2.5 & 2 \end{pmatrix},$$

the conditions defining the case hold with $i = 1, j = 3, k = 2, l = 4$. Inequality (4.46) is valid and matrices (4.49) and (4.51) are as follows:

$$B = \begin{pmatrix} 88 & -16 & -8 \\ -16 & 7 & 11 \\ -8 & 11 & 7 \end{pmatrix}, \quad C = \begin{pmatrix} 13.5 & 7 & -2 \\ 7 & 2 & -2 \\ -2 & -2 & 7 \end{pmatrix}.$$

We use Theorem 4.22 to show that both matrices are strictly copositive. In the matrix B, the off-diagonal entries of the first row are negative. The matrix

$$88B_2 - \begin{pmatrix} 16 \\ 8 \end{pmatrix} (16 \ \ 8)$$

has positive entries and, hence, is strictly copositive. It follows that the submatrix

$$B_2 = \begin{pmatrix} 7 & 11 \\ 11 & 7 \end{pmatrix}$$

is also strictly copositive. This proves that B is strictly copositive.

For the matrix C, the strict copositivity can be verified in much the same way. One only has to consider the third row instead of the first one. As a consequence, the whole matrix A is strictly copositive.

Rational algorithms for testing copositivity, similar in the approach to the criteria above, are given in [7]. They are also meant for matrices of low orders (up to order five inclusive). These algorithms take into account the sign distribution in a single fixed row rather than in the whole matrix; therefore, the number of distinct cases here is smaller than in [24]. No generality will be lost if one fixes the first row for the analysis below.

To describe the algorithms, the following notation will be needed:

$\bar{V}^{i,j}$	is a row vector of length $n-1$, with the entries $\{\bar{V}^{i,j}\}_l = \begin{cases} a_{1,j+1}, & l = i, \\ -a_{1,i+1}, & l = j, \\ 0, & \text{otherwise} \end{cases}$
$\operatorname{mat}(b_1, \ldots, b_k)$	is a square matrix with the rows $b_1, \ldots, b_k \in \mathbf{R}^k$
$e_i,\ i = 1, \ldots,$	are coordinate row vectors in the arithmetic space under consideration.

Assume that A is partitioned as

$$A = \begin{pmatrix} a_{11} & a^T \\ a & A_{22} \end{pmatrix}.$$

Let $B = a_{11}A_{22} - aa^T$. Suppose that $a_{11} \geq 0$ (otherwise, A is obviously not copositive). The values of the indices i, j, k (i, j, k, l) in the descriptions below constitute a permutation of $\{1, 2, 3\}$ $(\{1, 2, 3, 4\})$.

For the 4×4 matrix A, the following distinct cases are possible.

Case 1. All the off-diagonal entries of the first row are nonnegative.

The matrix A is copositive if and only if its submatrix A_{22} is copositive.

Case 2. All the off-diagonal entries of the first row are nonpositive.

The matrix A is copositive if and only if the two 3×3 matrices A_{22} and B are copositive.

Case 3. There is exactly one negative off-diagonal entry $a_{1,i+1}$ in the first row.

The matrix A is copositive if and only if the two 3×3 matrices A_{22} and $W(i)BW(i)^T$ are copositve. Here

$$W(i) = \operatorname{mat}(e_i,\ \bar{V}^{i,j},\ \bar{V}^{i,k}).$$

Case 4. There are exactly two negative off-diagonal entries $a_{1,i+1}$ and $a_{1,j+1}$ in the first row.

The matrix A is copositive if and only if the three 3×3 matrices A_{22}, $W_1BW_1^T$, and $W_2BW_2^T$ are copositive. Here

$$W_1 = \operatorname{mat}(e_i,\ e_j,\ \bar{V}^{ik}), \quad W_2 = \operatorname{mat}(e_j,\ \bar{V}^{i,k},\ \bar{V}^{j,k}).$$

We illustrate this algorithm by an example borrowed from [7]. Suppose that we again check whether matrix (4.50) is copositive. For this matrix, the first row and the second satisfy the conditions of case 4 and the two remaining rows satisfy those of case 3. Technically, case 3 is simpler (one must form and analyze two

3×3 matrices rather than three). Therefore, we choose, say, the fourth row as a pivotal one. For convenience, we interchange rows and columns 1 and 4. The new matrix A is

$$A = \begin{pmatrix} 4 & -3 & 1 & 2 \\ -3 & 3 & 2 & -2 \\ 1 & 2 & 1 & -1 \\ 2 & -2 & -1 & 2 \end{pmatrix}.$$

Both off-diagonal entries of the third row in the submatrix A_{22} are negative. Hence, the copositivity of A_{22} depends on whether the 2×2 matrix

$$2\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix}(2 \ \ 1) = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$$

is copositive. This matrix is even positive; thus, A_{22} is copositive.

Calculating the matrix B yields

$$B = \begin{pmatrix} 3 & 11 & -2 \\ 11 & 3 & -6 \\ -2 & -6 & 4 \end{pmatrix}.$$

Taking the vector $x = (0, 1, 1)^T$, one has

$$(Bx, x) = -5 < 0,$$

i.e., B is not copositive. However, the description of case 3 says that it is not the matrix B that is important. Rather one needs that the matrix $C = W(1)BW(1)^T$ be copositive, where

$$W(1) = \begin{pmatrix} 1 & 0 & 0 \\ a_{13} & -a_{12} & 0 \\ a_{14} & 0 & -a_{12} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 2 & 0 & 3 \end{pmatrix}.$$

The matrix C turns out to be nonnegative:

$$C = \begin{pmatrix} 3 & 36 & 0 \\ 36 & 96 & 12 \\ 0 & 12 & 24 \end{pmatrix}.$$

Hence, A is copositive.

Now consider a 5×5 matrix A. Here, five cases will be distinguished. Cases 1 and 2 are defined and analyzed just as for $n = 4$. The description of case 3 is also preserved, with the only alteration that $W(i)$ is now the following 4×4 matrix:

$$W(i) = \operatorname{mat}(e_i,\ \bar{V}^{i,j},\ \bar{V}^{i,k},\ \bar{V}^{i,l}).$$

Case 4. There are exactly two negative off-diagonal entries $a_{1,i+1}$ and $a_{1,j+1}$ in the first row.

The matrix A is copositive if and only if the four 4×4 matrices A_{22}, $P_1BP_1^T$, $P_2BP_2^T$, and $P_3BP_3^T$ are copositive. Here

$$P_1 = \operatorname{mat}(e_i,\ e_j,\ \bar{V}^{i,k},\ \bar{V}^{i,l}),$$

$$P_2 = \operatorname{mat}(\bar{V}^{j,l},\ e_j,\ \bar{V}^{i,k},\ \bar{V}^{i,l}),$$

$$P_3 = \operatorname{mat}(\bar{V}^{j,l},\ e_j,\ \bar{V}^{i,k},\ \bar{V}^{j,k}).$$

Case 5. There are exactly three negative off-diagonal entries $a_{1,i+1}, a_{1,j+1}$, and $a_{1,k+1}$ in the first row.

The matrix A is copositive if and only if the four 4×4 matrices A_{22}, $Q_1BQ_1^T$, $Q_2BQ_2^T$, and $Q_3BQ_3^T$ are copositive. Here

$$Q_1 = \operatorname{mat}(e_i,\ e_j,\ e_k,\ \bar{V}^{i,l}),$$

$$Q_2 = \mathrm{mat}\,(e_j,\, e_k,\, \bar{V}^{i,l},\, \bar{V}^{k,l}),$$
$$Q_3 = \mathrm{mat}\,(e_j,\, \bar{V}^{i,l},\, \bar{V}^{k,l},\, \bar{V}^{j,l}).$$

It is indicated in [7] that when applying the algorithm, one can always do without case 5. Indeed, the overall number of negative off-diagonal entries in a (symmetric) matrix is even. Therefore, in a 5×5 matrix, there must be at least one row with an even number of negative off-diagonal entries.

In conclusion, we discuss some applications of the concept of copositivity. First of all, we mention the quadratic Bernstein–Bezier patches. These are functions of the form

$$f = (Au, u), \quad A \in S_n, \tag{4.52}$$

considered over the $(n-1)$-dimensional simplex

$$U_n = \{u \in \mathbf{R}^n \mid u = (u_1, \ldots, u_n)^T,\ \sum_1^n u_i = 1,\ u_i \geq 0\ \ \forall i\}. \tag{4.53}$$

Surfaces of this kind (and more general ones that correspond to homogeneous polynomials of arbitrary degree k in variables $u_1, \ldots, u_n$) are widely used in computer-aided geometric design [10, 33]. The requirement that function (4.52) be nonnegative at all points of simplex (4.53) is exactly equivalent to the copositivity of the matrix A.

A remarkable application of copositivity is given in [18]. Consider a quadratic differential equation, i.e., an autonomous system of ordinary differential equations

$$\dot{y} = f(y), \quad f\,:\,\mathbf{R}^n \to \mathbf{R}^n, \tag{4.54}$$

whose right-hand sides are quadratic polynomials with nonnegative coefficients in the variables $y_1, \ldots, y_n$;

$$f_i(y) = \sum_{j,k=1}^n b_{ijk} y_j y_k, \quad i = 1, \ldots, n. \tag{4.55}$$

Systems of this kind occur, for example, in population genetics.

Setting $e = (1, \ldots, 1)^T$, we introduce the new variables,

$$x = \frac{y}{(y, e)}.$$

When we rescale the time variable, Eq. (4.54) assumes the form

$$\dot{x} = f(x) - (f(x), e)x. \tag{4.56}$$

Since $b_{ijk} \geq 0\ \forall\, i, j, k$ in (4.55), the nonnegative orthant $\mathbf{R}^n_+$ is positively invariant with respect to system (4.54), and hence, positively invariant with respect to system (4.56). Taking the inner product of (4.56) and the vector e yields

$$(\dot{x}, e) = (f(x), e)[1 - (x, e)],$$

which implies that the simplex

$$T = \{x \in \mathbf{R}^n_+ \mid (x, e) = 1\} \tag{4.57}$$

is also positively invariant with respect to (4.56).

The Jacobian of system (4.56) is

$$J(x) = f'(x) - xe^T f'(x) - (f(x), e)I_n.$$

Consider the right-hand side of (4.56) to be a vector field on the simplex T. Then the divergence of this field is

$$\begin{aligned} D_0 = {} & \mathrm{tr}\, J(x) + (f(x), e) \\ = {} & \mathrm{tr}\, f'(x) - (f'(x)x, e) - (n-1)(f(x), e). \end{aligned}$$

On the set T, the divergence $D_0(x)$ coincides with the quadratic function

$$D(x) = \operatorname{tr} f'(x)(x, e) - (f'(x)x, e) - (n-1)(f(x), e).$$

In coordinate notation,

$$D(x) = \sum_{j,k=1}^{n} \left(\sum_{i=1}^{n} b_{iji} + \sum_{i=1}^{n} b_{iik} - (n+1) \sum_{i=1}^{n} b_{ijk} \right) x_j x_k. \tag{4.58}$$

For $n = 3$, simplex (4.57) is just a triangle in the plane

$$x_1 + x_2 + x_3 = 1.$$

By the criterion of Dulac, an autonomous system of ODE in the plane does not have periodic solutions (except for constants) in a given simply connected domain if the divergence does not change sign in this domain. Since

$$D_0(x) = D(x), \quad x \in T,$$

the inequality $D(x) > 0$ on T (or $D(x) < 0$ on T) excludes the existence of periodic orbits for system (4.56). Such an inequality amounts to the requirement that the matrix of the quadratic form (4.58) be strictly copositive (or strictly copositive up to a negative scalar factor). There are vector fields to which this criterion applies, for example, $b_{ijk} = 1\ \forall\, i, j, k$, because then $D(x) = -6\ \forall x \in T$.

5. $\mathcal{K}$-Copositive Matrices

Let $\mathcal{K}$ be a nonempty polyhedral cone in $\mathbf{R}^n$:

$$\mathcal{K} = \{x \mid Bx \geq 0\}, \tag{5.1}$$

B being an $m \times n$ matrix.

Definition. A matrix $A \in S_n$ is called *$\mathcal{K}$-copositive* if

$$(Ax, x) \geq 0 \qquad \forall x \in \mathcal{K}, \tag{5.2}$$

and *strictly $\mathcal{K}$-copositive* if

$$(Ax, x) > 0 \quad \forall x \in \mathcal{K}, \quad x \neq 0. \tag{5.3}$$

In particular, when $\mathcal{K} = \mathbf{R}^n_+$ (i.e., when B in (5.1) is the identity matrix I_n) (5.2) and (5.3) make us return to the definitions of copositive and strictly copositive matrices. Moreover, system (5.1), defining the cone $\mathcal{K}$, may contain hyperplanes in an explicit or implicit way. By an explicit way we mean the situation where the system $Bx \geq 0$ contains a pair of inequalities which, up to positive scalar factors, have the form

$$b_1x_1 + b_2x_2 + \cdots + b_nx_n \geq 0$$

and

$$-b_1x_1 - b_2x_2 - \cdots - b_nx_n \geq 0.$$

Theoretically, the system $Bx \geq 0$ can be equivalent to a system of linear equations. Hence, the case of matrices that are definite or semidefinite with respect to a linear subspace is also covered by definitions (5.2), (5.3).

Thus, polyhedral cones may be highly different. Accordingly, the characterizations of the corresponding copositive matrices differ substantially in complexity. We already had a chance to see this in the previous two sections of this survey. In the subsequent discussion, we shall also distinguish between different kinds of cones. In this respect, the definitions below will be helpful.

Definition. The *dimension* of a cone $\mathcal{K}$ is the dimension of its affine hull, i.e., of the linear space of smallest dimension containing $\mathcal{K}$. A cone $\mathcal{K} \subset \mathbf{R}^n$ is termed *solid* if $\dim \mathcal{K} = n$.

Definition. A cone $\mathcal{K}$ is *pointed* if

$$\mathcal{K} \cap (-\mathcal{K}) = \{0\}. \tag{5.4}$$

Lemma 5.1. *Cone* (5.1) *is pointed if and only if*

$$\operatorname{rank} B = n. \tag{5.5}$$

Proof. Definition (5.4) is equivalent to the requirement that no line

$$y = t\,x_0, \qquad -\infty < t < \infty,$$

belong to the cone $\mathcal{K}$. In other words, the system of linear homogeneous equations $Bx = 0$ should not admit a nontrivial solution x_0. This yields (5.5).

Another important special case of the general definition is where the rows of the matrix B in (5.1) are linearly independent:

$$\operatorname{rank} B = m. \tag{5.6}$$

It is easy to see that the cone $\mathcal{K}$ satisfying (5.6) must be solid.

In [22], it was pointed out that a simple *sufficient* condition for the matrix A to be $\mathcal{K}$-copositive is that A be decomposable in the form

$$A = B^T C B + S, \tag{5.7}$$

where

$$\textit{the } m \times m \textit{ matrix } C \textit{ is copositive} \tag{5.8}$$

and

$$\textit{the } n \times n \textit{ matrix } S \textit{ is positive semidefinite.} \tag{5.9}$$

Clearly, this assertion holds for any polyhedral cone $\mathcal{K}$. A matrix A with decomposition (5.7) is strictly $\mathcal{K}$-copositive if (5.8) is valid and the matrix S in (5.7) is positive definite.

In [27], the question is addressed whether decomposition (5.7)-(5.9) is *necessary* for the matrix $A \in S_n$ to be $\mathcal{K}$-copositive. A similar question related to strict $\mathcal{K}$-copositivity is also treated. Many results of this paper are quite constructive, and therefore we discuss its contents in greater detail.

The case of a cone satisfying (5.6) is the simplest one. Let B be partitioned as

$$B = (B_1\, B_2). \tag{5.10}$$

Without loss of generality, one can assume that the square $m \times m$ submatrix B_1 is nonsingular. We make the following change of variables in (5.1)–(5.3):

$$x = Qy, \tag{5.11}$$

where

$$Q = \begin{pmatrix} B_1^{-1} & -B_1^{-1}B_2 \\ 0 & I_{n-m} \end{pmatrix}. \tag{5.12}$$

Since $BQ = (I_m\, 0)$, the cone $\mathcal{K}$ is now described by the inequality

$$u \geq 0, \tag{5.13}$$

where $u \in \mathbf{R}^m$ is a subvector in the partition of y:

$$y = \begin{pmatrix} u \\ v \end{pmatrix}. \tag{5.14}$$

In the new variables, the quadratic form $\psi(x) = (Ax, x)$ has the matrix

$$\tilde{A} = Q^T A Q = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{12}^T & \tilde{A}_{22} \end{pmatrix}, \tag{5.15}$$

where the block $\tilde{A}_{11}$ is of order m. Vectors y of the form

$$y = \begin{pmatrix} 0 \\ v \end{pmatrix} \qquad \forall\, v \in \mathbf{R}^{n-m}$$

obviously belong to $\mathcal{K}$ (see (5.13)). Using the $\mathcal{K}$-copositivity of $\tilde{A}$ for these vectors, we infer that

$$\textit{the submatrix } \tilde{A}_{22} \textit{ in } (5.15) \textit{ must be positive semidefinite.} \tag{5.16}$$

Let us now consider vectors (5.14), where a subvector u is nonzero (and nonnegative). For these vectors y,

$$\psi = (\tilde{A}y, y) = (\tilde{A}_{11}u, u) + 2(u, \tilde{A}_{12}v) + (\tilde{A}_{22}v, v). \tag{5.17}$$

Suppose that $v \in \ker \tilde{A}_{22}$. Then

$$\psi = \psi(u) = (\tilde{A}_{11}u, u) + 2(u, \tilde{A}_{12}v) \geq 0$$

for all $u \geq 0$, and, hence $\tilde{A}_{12}v = 0$. Thus, the inclusion

$$\ker \tilde{A}_{22} \subset \ker \tilde{A}_{12} \tag{5.18}$$

holds. This implies, in particular, that the rank of the matrix

$$\tilde{S} = \begin{pmatrix} \tilde{A}_{12}\tilde{A}_{22}^{+}\tilde{A}_{12}^T & \tilde{A}_{12} \\ \tilde{A}_{12}^T & \tilde{A}_{22} \end{pmatrix} \tag{5.19}$$

coincides with the rank of the submatrix $\tilde{A}_{22}$. Therefore (see (5.16)),

$$\textit{the matrix } \tilde{S} \textit{ is positive semidefinite.} \tag{5.20}$$

Next, for the fixed vector $u \in \mathbf{R}^m$, the minimum in v of function (5.17) is attained on the vector

$$v = -\tilde{A}_{22}^{+}\tilde{A}_{12}^T u$$

and is equal to

$$(Cu, u), \tag{5.21}$$

where

$$C = \tilde{A}_{11} - \tilde{A}_{12}\tilde{A}_{22}^{+}\tilde{A}_{12}^T. \tag{5.22}$$

For any $u \geq 0$, the quadratic form (5.21) must be nonnegative, and, hence

$$\textit{matrix } (5.22) \textit{ is copositive.} \tag{5.23}$$

The equality

$$\tilde{A} = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} + \tilde{S}$$

is obvious from (5.19) and (5.22). Returning to the original variable x, one obtains a decomposition of A:

$$A = Q^{-T}\tilde{A}Q^{-1} = Q^{-T}\begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}Q^{-1} + S. \tag{5.24}$$

Here (see (5.20))

$$\text{the matrix } S = Q^{-T}\tilde{S}Q^{-1} \text{ is positive semidefinite.} \tag{5.25}$$

Since

$$Q^{-1} = \begin{pmatrix} B_1 & B_2 \\ 0 & I_{n-m} \end{pmatrix},$$

one has

$$Q^{-T}\begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}Q^{-1} = Q^{-T}\begin{pmatrix} I_m \\ 0 \end{pmatrix}C(I_m 0)Q^{-1}$$

$$= \begin{pmatrix} B_1^T \\ B_2^T \end{pmatrix}C(B_1 B_2) = B^T CB.$$

Bearing in mind (5.23) and (5.25), we infer that (5.24) is the required decomposition (5.7)-(5.9) of the $\mathcal{K}$-copositive matrix A. Thus, this decomposition is a necessary condition for the $\mathcal{K}$-copositivity at least for cones $\mathcal{K}$ satisfying (5.6). At the same time, for cones of this type, a quite constructive way was found to check the $\mathcal{K}$-copositivity. We summarize it as the algorithm below.

Algorithm for checking $\mathcal{K}$-copositivity of the matrix A

1. Reorder the variables to ensure the nonsingularity of the block B_1 in matrix (5.10), which defines the cone $\mathcal{K}$. Reorder (symmetrically) the rows and columns of A in the corresponding way.
2. Calculate the matrix $\tilde{A}$ (see (5.15)).
3. Check whether the submatrix $\tilde{A}_{22}$ is positive semidefinite. If not, then A is not a $\mathcal{K}$-copositive matrix. In this case, the execution of the algorithm comes to an end.
4. Calculate the matrix C (see (5.22)).
5. Using the algorithms of the preceding section, check whether C is copositive. If it is, and only in this case, then the matrix A is $\mathcal{K}$-copositive.

Note that the pseudoinversion of the (real) matrix A_{22}, which is required by formula (5.22), can be accomplished over $\mathbf{R}$ by a finite rational procedure [3].

For a strictly $\mathcal{K}$-copositive matrix A, one can repeat the argument above almost word for word. However, some conclusions will be sharper. First, the submatrix $\widetilde{A}_{22}$ in (5.15) must now be positive *definite.* This fact makes the proof of inclusion (5.18) unnecessary. The equality $\operatorname{rank}\widetilde{S} = \operatorname{rank}\widetilde{A}_{22}$ is obvious, as is the positive semidefiniteness of the matrix $\widetilde{S}$. Second, the quadratic form (5.21) is positive for all nonnegative and nonzero vectors u; hence, matrix (5.22) is *strictly* copositive. Thus, we arrive at decomposition (5.7)-(5.9) of a strictly $\mathcal{K}$-copositive matrix A, in which the matrix C is, in fact, strictly copositive. The corresponding alterations should be made in the formulation of the algorithm.

Observe that the algorithm becomes considerably simpler in the important special case where the cone $\mathcal{K}$ not only satisfies (5.6) but is also *pointed*, i.e., where

$$\operatorname{rank} B = m = n. \tag{5.26}$$

Indeed, in this case the (strict) $\mathcal{K}$-copositivity of the matrix A is equivalent to the (strict) copositivity of the matrix

$$\widetilde{A} = B^{-T}AB^{-1}. \tag{5.27}$$

In other words, if the matrix A is (strictly) copositive with respect to a cone of type (5.26), then, the positive semidefinite (definite) term in its decomposition (5.7) is redundant:

$$A = B^T\widetilde{A}B, \qquad \widetilde{A} \text{ is (strictly) copositive.} \tag{5.28}$$

This is consistent with the analysis of the case $\mathcal{K} = \mathbf{R}_+^n$ in the preceding section.

It turns out that a decomposition of type (5.28) is valid for matrices that are *strictly* copositive with respect to *any pointed* cone $\mathcal{K}$. To put it differently, one can waive the first requirement $\operatorname{rank} B = m$ in (5.26). This is implied by the following result from [27].

Theorem 5.1. *Let $\mathcal{K} \subset \mathbf{R}^n$ be a pointed cone. A matrix $A \in S_n$ is strictly $\mathcal{K}$-copositive if and only if the matrix*

$$C_\nu = \nu C_1 + C_2, \tag{5.29}$$

where

$$C_1 = I_m - B(B^T B)^{-1} B^T, \tag{5.30}$$

$$C_2 = B(B^T B)^{-1} A (B^T B)^{-1} B^T, \tag{5.31}$$

is strictly copositive for a positive value $\nu = \nu_0$. If such a ν_0 exists, then C_ν is strictly copositive for all sufficiently large positive ν. Moreover, for all ν, the matrix C_ν satisfies the equality

$$A = B^T C_\nu B. \tag{5.32}$$

Remark. Under the conditions of Theorem 5.1, $\operatorname{rank} B = n$. Hence, the matrix

$$B^+ = (B^T B)^{-1} B^T$$

is the Moore–Penrose pseudoinverse of B and C_1 is the orthoprojector on the orthogonal complement of the subspace $\operatorname{im} B$, the image of B. The matrix $C_2 = (B^+)^T A B^+$ is the closest possible analog of matrix (5.27), and equality (5.32) is an extension of (5.28).

We will not prove Theorem 5.1, but will only note that its proof in [27] is based on the following noteworthy generalization of the Finsler theorem (see Theorem 3.3).

Theorem 5.2. *Let Γ be a closed cone in $\mathbf{R}^n$, and A_1, $A_2 \in S_n$, where A_1 is copositive with respect to Γ. Let*

$$\mathcal{M} = \{x \mid x \in \Gamma,\ (A_1 x, x) = 0,\ x \neq 0\}.$$

The relation

$$(A_2 x, x) > 0 \qquad \forall x \in \mathcal{M}$$

holds if and only if the matrix

$$A_\nu = \nu A_1 + A_2$$

is strictly Γ-copositive for all sufficiently large positive ν.

If $\Gamma = \mathbf{R}^n$, then A_1 is a positive semidefinite matrix. Suppose that the system of linear equations $Bx = 0$ defines the null space $\ker A$. Then one can replace A_1 in Theorem 5.2 by the matrix $B^T B$ without altering the conclusion of the theorem. In this case, Theorem 5.2 turns into the original Finsler theorem.

This generalized Finsler theorem makes it possible for Martin and Jacobson [27] to get a complete answer to the question whether decomposition (5.7)-(5.9) is necessary for the case where $\mathcal{K}$ is an *arbitrary* polyhedral cone and A is *strictly* $\mathcal{K}$-copositive.

Theorem 5.3. *Let $\mathcal{K}$ be cone* (5.1). *Then the following properties of the matrix $A \in S_n$ are equivalent:*

(1) *A is strictly $\mathcal{K}$-copositive.*

(2) *There exist a strictly copositive matrix C and a positive-definite matrix S such that decomposition* (5.7) *holds.*

(3) *For a positive value $\nu = \nu_0$,*

$$\text{the matrix } A + \nu B^T B \text{ is positive definite} \tag{5.33}$$

and

$$\text{the matrix } D_\nu = I_m - \nu B(A + \nu B^T B)^{-1} B^T \text{ is strictly copositive.} \tag{5.34}$$

If such a ν_0 exists, then conditions (5.33) *and* (5.34) *are fulfilled for all sufficiently large positive* ν.

Remark. We can see that Theorem 5.3 ensures decomposition (5.7)-(5.9) with an excess: one can choose C to be even *strictly* copositive.

For small m and n, conditions (5.33) and (5.34) can be verified with the use of symbolic computations (or even on paper). Hence, they can be considered to be a criterion for strict $\mathcal{K}$-copositivity. As an illustration, we reproduce below two examples from [27].

Suppose that a cone $\mathcal{K} \subset \mathbf{R}^3$ is defined by the inequalities

$$3x + y - 8z \geq 0,$$
$$-x - 3y + 8z \geq 0.$$

One must determine whether the quadratic form

$$\psi = x^2 + y^2 - z^2$$

is strictly $\mathcal{K}$-copositive.

For this example,

$$A = \operatorname{diag}(1,\,1,\,-1), \quad B = \begin{pmatrix} 3 & 1 & -8 \\ -1 & -3 & 8 \end{pmatrix}.$$

The matrix in (5.33) is

$$E_\nu \equiv \begin{pmatrix} 10\nu + 1 & 6\nu & -32\nu \\ 6\nu & 10\nu + 1 & -32\nu \\ -32\nu & -32\nu & 128\nu - 1 \end{pmatrix}.$$

The leading principal minors of E_ν are the polynomials

$$10\nu + 1, \quad 64\nu^2 + 20\nu + 1$$

and

$$d(\nu) \equiv \det E_\nu = 448\nu^2 + 108\nu - 1.$$

The leading coefficients of these three polynomials are positive. Thus, the matrix E_ν is positive definite for all sufficiently large positive ν, and condition (5.33) is fulfilled.

If $d(\nu) > 0$, then one can examine the matrix

$$F_\nu \equiv d(\nu)D(\nu) = d(\nu)I_m - \nu B \operatorname{adj}(A + \nu B^T B)B^T$$

instead of D_ν when checking condition (5.34). For our example, F_ν is

$$\begin{pmatrix} 54\nu - 1 & 58\nu \\ 58\nu & 54\nu - 1 \end{pmatrix}.$$

This matrix is positive and, hence, strictly copositive for $\nu > \frac{1}{54}$. Applying Theorem 5.3, we infer that the form ψ is strictly $\mathcal{K}$-copositive.

Now we preserve the form ψ, but change sign in the second inequality defining $\mathcal{K}$. The new cone $\mathcal{K}$ is given by

$$3x + y - 8z \geq 0,$$
$$x + 3y - 8z \geq 0.$$

Neither the matrix $B^T B$ nor the matrix in (5.33) will change. Thus, condition (5.33) is again satisfied for all sufficiently large positive ν. As for F_ν, this matrix assumes the form

$$F_\nu = \begin{pmatrix} 54\nu - 1 & -58\nu \\ -58\nu & 54\nu - 1 \end{pmatrix}.$$

It will not be positive as $\nu \to \infty$; therefore, we check inequality (4.5). It is obvious that

$$\{F_\nu\}_{12}^2 = (58\nu)^2 > \{F_\nu\}_{11}\{F_\nu\}_{22} = (54\nu - 1)^2$$

for all sufficiently large ν, i.e., for these ν the matrix F_ν is not copositive. Hence, the form ψ is not copositive with respect to the new cone $\mathcal{K}$.

It has already been observed that the system of linear inequalities that defines the cone $\mathcal{K}$ may contain, explicitly or implicitly, linear equations. Suppose that one can single out these equations in an explicit way. Then the original definition (5.1) of $\mathcal{K}$ is replaced by

$$\mathcal{K} = \{x \mid B_1 x \geq 0,\ B_2 x = 0\}. \tag{5.35}$$

In this case, it is possible to obtain a decomposition of A that is more economical than that of (5.7)–(5.9). Here we cite one more result from [27].

Theorem 5.4. *Let $\mathcal{K}$ be cone (5.35). Then a matrix $A \in S_n$ is strictly $\mathcal{K}$-copositive if and only if A can be decomposed as*

$$A = B_1^T C_1 B_1 + S_1, \tag{5.36}$$

where C_1 is a strictly copositive matrix and S_1 is positive definite with respect to the linear subspace $B_2 x = 0$.

The economy of decomposition (5.36) consists in the order of the copositive matrix C_1 being equal to the number of "genuine" inequalities in system (5.1), and not to the overall number of inequalities in this system.

As we will shortly see, the question whether decomposition (5.7)–(5.9) is necessary for $\mathcal{K}$-copositive matrices proves to be much more delicate. However, for this case as well, one can get a number of useful implications from the results already stated above. One need only make use of the following simple observation: a matrix A is $\mathcal{K}$-copositive if and only if the matrix $A_\delta = A + \delta I_n$ is *strictly* $\mathcal{K}$-copositive for any $\delta > 0$. Applying Theorem 5.3 to A_δ, we obtain

Theorem 5.5. *Let $\mathcal{K}$ be cone (5.1). Then the following properties of a matrix $A \in S_n$ are equivalent.*

(1) *A is copositive with respect to the cone $\mathcal{K}$.*

(2) *For each $\delta > 0$, there exist a strictly copositive matrix C_δ and a positive-definite matrix S_δ such that*

$$A_\delta = A + \delta I_n = B^T C_\delta B + S_\delta. \tag{5.37}$$

(3) *For each $\delta > 0$, there exists a positive value $\nu = \nu_0$ (which may depend upon δ) such that*

$$\text{the matrix } A_\delta + \nu B^T B \text{ is positive definite} \tag{5.38}$$

and

$$\text{the matrix } D_{\nu,\delta} = I_m - \nu\, B(A_\delta + \nu B^T B)^{-1} B^T \text{ is strictly copositive.} \tag{5.39}$$

As was the case with Theorems 5.1 and 5.3, statements (2) and (3) become considerably simpler when the cone $\mathcal{K}$ is pointed. Indeed, for such a cone, already the matrix $B^T B$ is positive definite, and (5.38) is fulfilled automatically.

Theorem 5.6. *Let $\mathcal{K}$ in (5.1) be a pointed cone. Then the following properties of a matrix $A \in S_n$ are equivalent.*

(1) *A is copositive with respect to the cone $\mathcal{K}$.*

(2) *For each $\delta > 0$, there exists a strictly copositive matrix C_δ such that*

$$A_\delta = A + \delta I_n = B^T C_\delta B. \tag{5.40}$$

(3) *For each $\delta > 0$, the matrix*

$$C_{\nu,\delta} = \nu C_1 + C_{2\,\delta}, \tag{5.41}$$

where C_1 is given by formula (5.30) *and*

$$C_{2,\delta} = B(B^TB)^{-1}A_\delta(B^TB)^{-1}B^T, \tag{5.42}$$

is strictly copositive for all sufficiently large ν.

(4) *For each $\delta > 0$, the matrix* (*see* (5.30) *and* (5.31))

$$C_\nu + \delta I = \nu C_1 + C_2 + \delta I \tag{5.43}$$

is strictly copositive for all sufficiently large ν.

Theorems 5.5 and 5.6 indicate one more way of checking $\mathcal{K}$-copositivity. This method, involving two parameters ν and δ, is even less convenient than the criterion of strict $\mathcal{K}$-copositivity of Theorems 5.1 and 5.3. However, for small m and n, the new criterion can also be implemented by means of computer algebra. Again we illustrate this by an example from [27].

The matrix

$$A = \operatorname{diag}(-1, -1, 1) \tag{5.44}$$

is obviously copositive with respect to the circular (or ice cream) cone

$$\sqrt{x^2 + y^2} \le z. \tag{5.45}$$

We inscribe in (5.45) the polyhedral cone $\mathcal{K}$ for which the vectors

$$\begin{pmatrix} 5 \\ 0 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} -3 \\ -4 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix} \tag{5.46}$$

indicate the direction of the edges. Then A is also copositive with respect to $\mathcal{K}$ but *not strictly copositive* since the quadratic form $\psi(u) = (Au, u)$ vanishes on each vector (5.46). Thus, all assertions of Theorems 5.5 and 5.6 must hold for A.

The cone $\mathcal{K}$ can be defined by a system of linear inequalities of type (5.1), where

$$B = \begin{pmatrix} -2 & -1 & 2 \\ 0 & -5 & 4 \\ 5 & 0 & 3 \\ 0 & 5 & 4 \\ -2 & 1 & 2 \end{pmatrix}.$$

The rank of this matrix is 3. We verify that, for each $\delta > 0$, matrix (5.43) corresponding to the pair (A, B) is strictly copositive for all sufficiently large ν. To this end, the Motzkin criterion (see Theorem 4.15) is applied in [27]. For matrix (5.43), we have

$$C_1 = \frac{1}{2548} \begin{pmatrix} 1875 & -765 & 520 & 275 & -575 \\ -765 & 465 & -416 & 367 & -275 \\ 520 & -416 & 416 & -416 & -520 \\ 275 & 367 & -416 & 465 & -765 \\ -575 & -275 & 520 & -765 & 1875 \end{pmatrix}.$$

The diagonal entries of this matrix are positive, which ensures that the diagonal entries of $C_\nu + \delta I$ will be positive as $\nu \to +\infty$. Now one has to examine in consecutive order all 26 principal submatrices of order ≥ 2. If a submatrix with a nonpositive determinant is detected, then one has to verify that, among the cofactors of its last column, there are nonpositive cofactors. This verification is based on the signs of the leading coefficients of the corresponding polynomials. In our example, we come to the conclusion that matrix (5.43) is strictly copositive for each $\delta > 0$ as $\nu \to +\infty$. For an illustration, consider the leading principal submatrix of order 3. Its determinant is

$$522645766\delta\nu^2 - (25500 + 94689606\delta - 4494264326\delta^2)\nu$$

$$+(225 - 336000\delta - 100089600\delta^2 + 4155074560\delta^3)$$

and is positive for all sufficiently large ν if $\delta > 0$ is fixed. No further analysis of this submatrix is required, and we can begin examining the next one.

Now we return to the discussion of decomposition (5.7)–(5.9) for the $\mathcal{K}$-copositive matrix A. Recall that for the cone $\mathcal{K}$ satisfying (5.6) the necessity of this decomposition was proved at the beginning of this section. Also recall that for strictly $\mathcal{K}$-copositive matrices we managed to remove eventually all limitations on the type of cone (see Theorem 5.3). The following simple example from [27] shows that the situation is different for the $\mathcal{K}$-copositive case.

In $\mathbf{R}^2$ we take

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5.47}$$

and let the cone $\mathcal{K}$ be defined by system (5.1) with the matrix

$$B = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}. \tag{5.48}$$

This system defines, in reality, the line $x = 0$. The quadratic form

$$\psi = (Au, u) = 2xy$$

vanishes identically along this line. Thus, matrix (5.47) is $\mathcal{K}$-copositive. We shall prove that this matrix does not admit decomposition (5.7)–(5.9). Indeed, for any symmetric matrix

$$C = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix},$$

the matrix

$$S = A - B^T C B = \begin{pmatrix} -\alpha + 2\beta - \gamma & 1 \\ 1 & 0 \end{pmatrix}$$

cannot be positive semidefinite because $\det S = -1 < 0$.

Consider the set of matrices C_δ in decompositions (5.37) of matrix (5.47). The example above shows that this set does not have a limit point as $\delta \to 0$. Indeed, if such a limit point C_0 existed, then it would necessarily be a copositive matrix. The matrix

$$S_0 = \lim_{\delta\to 0} S_\delta = \lim_{\delta\to 0}(A_\delta - B^T C_\delta B) = A - B^T C_0 B$$

is obviously positive semidefinite, which yields a decomposition of type (5.7)–(5.9) of A:

$$A = B^T C_0 B + S_0.$$

However, it has been proved that this decomposition is impossible.

The cone defined by matrix (5.48) is not solid, and one could attribute the counterexample above to this fact. However, a more detailed analysis of matrix (5.44) and the *solid* cone $\mathcal{K}$ given by the system of edges (5.46) shows that this $\mathcal{K}$-copositive matrix does not admit decomposition (5.7)-(5.9) either.

In [28], a complete description is given of polyhedral cones $\mathcal{K}$ such that any $\mathcal{K}$-copositive matrix can be represented in the form (5.7)–(5.9). Let us consider a cone in the matrix space S_n, generated by cone (5.1) with the use of the formula

$$C_B = \{B^T C B \,|\, C \text{ copositive}\}. \tag{5.49}$$

Definition. Cone (5.1) is said to have the *closure property* if the matrix cone (5.49) is closed.

Theorem 5.7. *Let $\mathcal{K}$ be cone* (5.1)*. Any $\mathcal{K}$-copositive matrix A admits decomposition* (5.7)–(5.9) *if and only if the cone $\mathcal{K}$ has the closure property and is either solid or pointed.*

Unfortunately, it is not clear how the presence or absence of the closure property for a particular cone $\mathcal{K}$ can be constructively verified. Thus, equality (5.6) is still the most general sufficient condition ensuring that the $\mathcal{K}$-copositive matrix A can be decomposed as in (5.7)–(5.9). However, two more partial results of this kind can be found in [27].

Theorem 5.8. *Let $\mathcal{K}$ in* (5.1) *be a solid cone. Suppose that the matrix B in* (5.1) *has no more than four rows. Then any $\mathcal{K}$-copositive matrix A can be decomposed as in* (5.7)-(5.9)*. Moreover, the copositive matrix C can be chosen to be nonnegative.*

This assertion differs from condition (5.6) in that the rows of the matrix B need not be linearly independent.

Theorem 5.9. *Suppose that $n = 2$ and cone* (5.1) *is solid. Then any $\mathcal{K}$-copositive matrix A either is positive semidefinite or can be represented as*

$$A = B^T N B,$$

where N is a nonnegative matrix.

Now we shall describe another type of criteria for $\mathcal{K}$-copositivity, which can be called enumerative type. One can think of criteria based on a sequential analysis of principal submatrices as their prototype in the case where $\mathcal{K} = \mathbf{R}^n_+$. For cone (5.1), the enumeration is governed by the set of submatrices that are formed from the full rows of B. Such a submatrix will be called a *row submatrix* of B (this includes the vacuous row submatrix of size $0 \times n$).

Suppose that $\alpha \subset \{1, 2, \dots, m\}$ is an index set defining the row submatrix B_α. Then $\overline{\alpha}$ will denote the complementary index set and $B_{\overline{\alpha}}$ will denote the complementary row submatrix. In the procedures under consideration, each step consists of certain tests for a matrix of the form

$$D_\alpha = \begin{pmatrix} A & B_{\overline{\alpha}}^T \\ B_{\overline{\alpha}} & 0 \end{pmatrix}, \tag{5.50}$$

α being an index set chosen for the current step.

The description of tests will be given after some preliminary observations. If a matrix A is strictly copositive with respect to cone (5.1), then it must also be $\mathcal{L}$-definite, $\mathcal{L}$ being the null space of the matrix B. In other words,

$$(Ax, x) > 0 \text{ if } Bx = 0 \text{ and } x \neq 0. \tag{5.51}$$

Thus, property (5.51) is a necessary condition for the matrix A to be strictly $\mathcal{K}$-copositive. As such, it will be called the *strong kernel condition.*

By the *weak kernel condition* we shall mean the property

$$(Ax, x) > 0 \text{ if } Bx = 0 \text{ and } Ax \neq 0. \tag{5.52}$$

It is shown in [26] that for a solid cone $\mathcal{K}$, property (5.52) holds for any $\mathcal{K}$-copositive matrix A.

As in the preceding sections, the symbol e stands for the vector $(1, 1, \dots, 1)^T$, its dimension being defined by the context. For a chosen nonvacuous row submatrix B_α, consider the system

$$D_\alpha \begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} -B_\alpha^T e \\ 0 \end{pmatrix} \tag{5.53}$$

which consists of $n + m - |\alpha|$ linear equations. We say that B_α *fails* Test 1 if system (5.53) defines x in a unique way and this unique vector x satisfies the inequality

$$B_\alpha x > 0. \tag{5.54}$$

We say that B_α *fails* Test 2 if the homogeneous system

$$D_\alpha \begin{pmatrix} x \\ u \end{pmatrix} = 0 \tag{5.55}$$

has a one-dimensional solution space for x and one ray of it satisfies condition (5.54). Otherwise, B_α is said to *pass* the corresponding test.

Suppose, for example, that $\ker B_{\overline{\alpha}} = \{0\}$. Of course, this is only possible when $m > n$. Both systems (5.53) and (5.54) define the unique x; namely, $x = 0$. Thus, both tests are trivially passed by B_α in this case.

The procedures verifying whether a given matrix A is (strictly) copositive with respect to a given cone of type (5.1) are based on the following two assertions, which are the main results of [26].

Theorem 5.10. *Let $\mathcal{K}$ in* (5.1) *be a solid cone. Then a matrix $A \in S_n$ is $\mathcal{K}$-copositive if and only if the weak kernel condition holds and every nonvacuous row submatrix B_α of the matrix B passes Test* 1.

Theorem 5.11. *Let $\mathcal{K}$ be cone* (5.1). *Then a matrix $A \in S_n$ is strictly $\mathcal{K}$-copositive if and only if the strong kernel condition holds and every nonvacuous row submatrix B_α of the matrix B passes Test* 1 *and Test* 2.

Both procedures are simpler in the so-called *regular case*, i.e., when the matrix A is nonsingular and every set of n or fewer rows of B is linearly independent. The tests that are carried out in the regular case can be conveniently stated in terms of the Schur complements in the matrix

$$M = BA^{-1}B^T. \tag{5.56}$$

For an index set $\alpha \subset \{1, 2, \ldots, m\}$, we associate with the row submatrices B_α and $B_{\overline{\alpha}}$ the principal submatrices of matrix (5.56) that are defined by the formulas

$$R = B_{\overline{\alpha}} A^{-1} B_{\overline{\alpha}}^T,$$

$$\tilde{R} = B_\alpha A^{-1} B_\alpha^T.$$

Also, let

$$S_R = B_\alpha A^{-1} B_{\overline{\alpha}}^T.$$

If one reorders (symmetrically) the rows and columns in M so that R becomes the *leading* principal submatrix, then R, $\tilde{R}$, and S_R will be the blocks of the reordered matrix M_α:

$$M_\alpha = \begin{pmatrix} R & S_R^T \\ S_R & \tilde{R} \end{pmatrix}.$$

We say that the principal submatrix R of M *fails* Test 3 if R is nonsingular and the inequality

$$(M/R)\, e < 0 \tag{5.57}$$

holds for the corresponding Schur complement M/R. We say that R *fails* Test 4 if R is singular, its null space is of dimension one, and one ray l of it satisfies

$$S_R u > 0 \qquad \forall u \in l. \tag{5.58}$$

Theorem 5.12 [26]. *Let (A, B) be a regular pair. Then a row submatrix B_α of the matrix B fails Test* 1 (*Test* 2) *if and only if the corresponding principal submatrix R of M fails Test* 3 (*Test* 4).

Thus, for the regular case, Test 1 and Test 2 can be replaced by Test 3 and Test 4, respectively.

One can verify the kernel conditions by means of computations much similar to those that were used at the beginning of this section for analyzing case (5.6). Let Q be a nonsingular $n \times n$ matrix such that

$$BQ = (\tilde{B}\ 0), \qquad \ker \tilde{B} = \{0\}. \tag{5.59}$$

We change the variables:

$$x = Qy = Q\begin{pmatrix} u \\ v \end{pmatrix},$$

the size of u being equal to the number t of columns of the block $\tilde{B}$ in (5.59). In the new variables, the subspace $Bx = 0$ is defined by the condition $u = 0$ and the quadratic form $\psi(x) = (Ax, x)$ is associated with the matrix

$$\tilde{A} = Q^T A Q = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{12}^T & \tilde{A}_{22} \end{pmatrix},$$

where $\tilde{A}_{11}$ is a block of order t. Now the strong kernel condition amounts to the requirement that the submatrix $\tilde{A}_{22}$ be positive definite, and the weak kernel condition amounts to the combination of two requirements, namely, that $\tilde{A}_{22}$ be positive semidefinite and the inclusion $\ker \tilde{A}_{22} \subset \ker \tilde{A}_{12}$ hold. It is obvious that both conditions can be verified via finite rational procedures.

Let us see how Tests 1 and 2 look in the case where $\mathcal{K} = \mathbf{R}_+^n$, i.e., when the matrix B in (5.1) is the identity matrix I_n. Suppose that an index set $\alpha = \{1, 2, \ldots, s\}$, $s < n$, has been chosen (we can reduce the general case to this special case by reordering variables). Then

$$B_\alpha = (I_s \; 0), \quad B_{\overline{\alpha}} = (0 \; I_{n-s}) \tag{5.60}$$

for the zero submatrices of appropriate sizes. We partition x in accordance with (5.60):

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad x_1 \in \mathbf{R}^s. \tag{5.61}$$

The substitution of (5.60) and (5.61) into (5.53) yields

$$\begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{12}^T & A_{22} & I_{n-s} \\ 0 & I_{n-s} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ u \end{pmatrix} = \begin{pmatrix} -e \\ 0 \\ 0 \end{pmatrix}. \tag{5.62}$$

Here A_{11} is a submatrix of order s. Now the third block row of system (5.62) implies

$$x_2 = 0.$$

Then the first block row gives

$$A_{11} x_1 = -e. \tag{5.63}$$

Thus, the submatrix B_α fails Test 1 if system (5.63) admits a unique solution, i.e., the principal submatrix A_{11} is nonsingular, and this unique solution $x_1 = B_\alpha x$ is positive.

A similar analysis of Test 2 leads to the following conclusion: the submatrix B_α fails Test 2 if the principal submatrix A_{11} is singular, its rank deficiency is one, and, for any vector belonging to the null space of A_{11}, all nonzero entries are of the same sign.

The criteria thus obtained are very close to the statements of Theorems 4.12–4.15 and are precisely the criteria of Theorems 3.1 and 3.2 in [12].

There exist situations that allow one to reduce substantially the amount of enumeration in the procedures indicated by Theorems 5.10 and 5.11. They are described in [38] and may be regarded as an extension of the situations discussed in Theorems 4.16–4.18.

Definition. Let $\mathcal{K}$ be a given cone (5.1) and let k be a fixed positive integer, $0 \le k \le n$. A matrix $A \in S_n$ is said to be (*strictly*) $\mathcal{K}$-*copositive of order* k if A is (strictly) copositive with respect to every cone

$$\mathcal{K}' = \mathcal{K} \cap \ker B_\alpha.$$

Here B_α is any row submatrix of the matrix B such that the null space $\ker B_\alpha$ is of dimension k. If A is a (strictly) $\mathcal{K}$-copositive matrix of order k, but not of order $k+1$, then k is said to be its *exact order* or *index* of (strict) $\mathcal{K}$-copositivity.

Theorem 5.13. *Let $\mathcal{K}$ in (5.1) be a pointed cone. Suppose that a matrix $A \in S_n$ has p positive eigenvalues, $p < n$. Then A is (strictly) $\mathcal{K}$-copositive if and only if it is (strictly) $\mathcal{K}$-copositive of order $p+1$.*

Theorem 5.14. *Let $\mathcal{K}$ in (5.1) be a pointed cone. Suppose that a matrix $A \in S_n$ is singular of rank r. Then A is $\mathcal{K}$-copositive if and only if it is $\mathcal{K}$-copositive of order r.*

Theorem 5.15. *Let $\mathcal{K}$ in (5.1) be a pointed cone. Suppose that a matrix $A \in S_n$ is singular of rank r. Then A is strictly $\mathcal{K}$-copositive if and only if it is strictly $\mathcal{K}$-copositive of order $r+1$.*

One more result of this kind can be found in [38]. For obvious reasons, it has no correspondence in the case $\mathcal{K} = \mathbf{R}^n_+$.

Theorem 5.16. *Suppose that cone (5.1) is of dimension $k < n$. Then a matrix $A \in S_n$ is (strictly) $\mathcal{K}$-copositive if and only if it is (strictly) $\mathcal{K}$-copositive of order k.*

The criteria of Theorems 5.10 and 5.11 can be thought of as extensions of the inner criteria from Sec. 4. The extensions of the outer criteria are discussed in [38]. Since the descriptions of these extended criteria are rather bulky, they are not given here. We only want to point out that the principal pivoting scheme for quadratic programming is used as the main tool in outer criteria, by analogy with the case $\mathcal{K} = \mathbf{R}^n_+$.

We conclude this section by indicating two applications related to the notion of $\mathcal{K}$-copositivity.

Let

$$\dot{x} = F(t)x + G(t)u, \quad x \in \mathbf{R}^n, \quad u \in \mathbf{R}^m,$$

$$x(t_0) = x_0$$

be a linear system with the control function $u(t)$. The problem is to minimize the quadratic cost functional

$$J(x_0, u) = \int_{t_0}^{t_1} [u^T(t)R(t)u(t) + x^T(t)Q(t)x(t)]\, dt + x^T(t_1)Hx(t_1)$$

subject to controller constraints of the form

$$B(t)\,u(t) \geq 0 \ \text{ a.e. on } [t_0, t_1],$$

where $B(t)$ is a given $l \times m$ matrix function.

An analysis of this problem with the use of the analog of the classical condition of Legendre in the calculus of variations leads to the following inference: a necessary condition for J to be bounded below is that for almost all $t \in [t_0, t_1]$ the matrix $R(t)$ be copositive with respect to the cone defined by the matrix $B(t)$ via formula (5.1). Note that this example is taken from [28].

An interesting application of Theorem 5.3 is given in [27]. Consider, along with the matrix $A \in S_n$ and cone (5.1), the hyperellipsoid

$$(Px, x) = 1, \quad P \in S_n, \ P \text{ is positive definite.} \tag{5.64}$$

One needs to evaluate the minimum m of the quadratic form $\psi(x) = (Ax, x)$ on hyperellipsoid (5.64) subject to constraints $Bx \geq 0$.

The following approach to this problem can be proposed. The number m is the supremum of positive δ such that the matrix $A - \delta P$ is copositive with respect to cone (5.1). Applying Theorem 5.3, we infer that the supremum should be found of those $\delta > 0$ for which the matrix $A - \delta P + \nu B^T B$ is positive definite and the matrix $I_m - \nu B(A - \delta P + \nu B^T B)^{-1} B^T$ is strictly copositive for all sufficiently large ν.

We illustrate this approach by an example from [27]. Assume that $n = 3$, $P = I_3$ (thus, hyperellipsoid (5.64) is, in this case, a two-dimensional sphere), $A = \operatorname{diag}(1, 1, -1)$, and

$$B = \begin{pmatrix} 3 & 1 & -8 \\ -1 & -3 & 8 \end{pmatrix}.$$

The leading principal minors of the symmetric matrix $A - \delta I_3 + \nu B^T B$ are

$$10\nu + 1 - \delta,$$

$$64\nu^2 + (20 - 20\,\delta)\nu + (1 - \delta)^2,$$

and

$$d(\nu, \delta) = (448 - 576\,\delta)\nu^2 + (108 - 256\,\delta + 148\,\delta^2)\nu - 1 + \delta + \delta^2 - \delta^3.$$

The first and second minors are clearly positive as $\nu \to +\infty$. The determinant d will only be positive for sufficiently large ν if $\delta < 448/576 = 7/9$.

For the symmetric 2×2 matrix

$$F_{\nu,\delta} = d(\nu, \delta) I_2 - \nu B \operatorname{adj}(A - \delta I_3 + \nu B^T B) B^T$$

the entries are

$$\{F_{\nu,\delta}\}_{1,1} = \{F_{\nu,\delta}\}_{2,2} = (54 - 128\,\delta + 94\,\delta^2)\nu - 1 + \delta + \delta^2 - \delta^3$$

and

$$\{F_{\nu,\delta}\}_{1,2} = (58 - 128\,\delta + 70\,\delta^2)\nu.$$

If $\delta = 7/9$, the leading coefficients of both polynomials are positive. Thus, the matrix $F_{\nu,7/9}$ is positive (hence, strictly copositive) as $\nu \to +\infty$.

This analysis shows that the required supremum is equal to

$$\delta_0 = 7/9.$$

In other words,

$$(Ax, x) \geq \frac{7}{9}(x, x)$$

if $Bx \geq 0$, and the minimum of the form $\psi(x)$ on the intersection of the unit sphere and the cone $Bx \geq 0$ is $7/9$.

6. Concluding Remarks

In [15, 34], finite procedures are described for deciding whether or not a given inhomogeneous quadratic function $\psi(x)$ is bounded below on a polyhedron M defined by a system of linear inhomogeneous inequalities. In the homogeneous case, the boundedness of a quadratic *form* below is equivalent to its nonnegativity on M. Indeed, assuming that $\psi(x_0) < 0$ for some point $x_0 \in M$, one finds that

$$\psi(t\,x_0) = t^2\,\psi(x_0) \to -\infty$$

as $t \to +\infty$ (recall that the polyhedron M is a cone for the homogeneous case). Hence, the procedures in [15, 34] can also be used for checking $\mathcal{K}$-copositivity. However, their descriptions were not included in Sec. 5, the reason being that, since these procedures are meant for a more general situation, they will be less efficient for the homogeneous problem than algorithms specially constructed for this case.

In the main body of this survey, the matrix property of copositivity was discussed with respect to the three types of subsets of the n-dimensional space $\mathbf{R}^n$, namely, linear subspaces, the nonnegative orthant $\mathbf{R}^n_+$, and, finally, arbitrary convex polyhedral cones. However, the copositivity can be defined and may be of interest for other types of subsets as well. By way of example, consider the ice cream cone

$$K_n = \{x \in \mathbf{R}^n \mid (x_1^2 + \cdots + x_{n-1}^2)^{1/2} \le x_n\}. \tag{6.1}$$

Definition. A matrix $A \in S_n$ is called *K_n-copositive* if $(Ax, x) \ge 0\ \forall x \in K_n$.

Using the matrix $J_n = \operatorname{diag}(-1, -1, \ldots, -1, 1)$, one can rewrite (6.1) as

$$K_n = \{x \in \mathbf{R}^n \mid (J_n x, x) \ge 0 \wedge x_n \ge 0\}.$$

Theorem 6.1 [25]. *A matrix $A \in S_n$ is K_n-copositive if and only if the matrix $A - \mu J_n$ is positive semidefinite for some $\mu \ge 0$.*

Unfortunately, the criterion of K_n-copositivity indicated in Theorem 6.1 cannot be regarded as a constructive one because it does not show how to find the required scalar μ (or show its absence). Unlike the criteria of the type of Theorem 3.3, here one cannot reduce the analysis to the investigation of the behavior of the principal minors as $\mu \to \infty$. Indeed, the matrix $A - \mu J_n$ has a negative diagonal entry (n, n) as $\mu \to +\infty$ and, hence, it cannot be positive semidefinite. As $\mu \to -\infty$, all diagonal entries, excluding the last one, are negative.

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