AIP | Journal of Mathematical Physics

The Potts model and the Tutte polynomial

D. J. A. Welsh and C. Merino

Citation: J. Math. Phys. **41**, 1127 (2000); doi: 10.1063/1.533181
View online: http://dx.doi.org/10.1063/1.533181
View Table of Contents: http://jmp.aip.org/resource/1/JMAPAQ/v41/i3
Published by the American Institute of Physics.

The Potts model and the Tutte polynomial

D. J. A. Welsh[a] and C. Merino[b]
Mathematical Institute, Oxford University, Oxford OX1 2HP, United Kingdom

(Received 18 November 1999; accepted for publication 30 November 1999)

This is an invited survey on the relation between the partition function of the Potts model and the Tutte polynomial. On the assumption that the Potts model is more familiar we have concentrated on the latter and its interpretations. In particular we highlight the connections with Abelian sandpiles, counting problems on random graphs, error correcting codes, and the Ehrhart polynomial of a zonotope. Where possible we use the mean field and square lattice as illustrations. We also discuss in some detail the complexity issues involved. © *2000 American Institute of Physics.* [S0022-2488(00)00203-6]

I. INTRODUCTION

The classical Potts model was introduced by Potts in 1952 and in its most basic form can be described as follows.

Consider a finite lattice L_n of N sites or general graph G of N vertices and suppose that each site (=vertex) can have associated with it a spin, which can have one of Q values. The energy between two interacting spins is taken to be zero if the spins are the same and equal to a constant if they are different.

In the simplest description of the Potts model with Q states $\{1,2,...,Q\}$, the Hamiltonian H is given by

$$H=J\sum_{i\sim j}(1-\delta(\sigma_i,\sigma_j)), \tag{1}$$

where the sum is over all nearest-neighbor pairs of sites i, j and σ_i is the spin at site i. Here J is the (constant) interaction. The model is *ferromagnetic* when $J>0$ and *antiferromagnetic* if $J<0$.

The probability of finding the system in state σ is then given by

$$\Pr[\sigma]=e^{-\beta H(\sigma)}/Z, \tag{2}$$

where Z, the normalizing constant, is the *partition function* and $\beta=1/kT$, where k is Boltzmann's constant and T is the temperature.

Thus the partition function is

$$Z(G;Q,K)=\sum_{\sigma}\exp\left(-K\sum_{i\sim j}(1-\delta(\sigma_i,\sigma_j))\right), \tag{3}$$

where $K=J/kT$, the summation in the exponential is over all near-neighbor pairs (i,j), and the first summation is over all possible spin configurations.

The Ising model with zero external field is just the special case when $Q=2$ and then the spins are usually taken to be ± 1.

[a]Electronic mail: dwelsh@maths.ox.ac.uk
[b]Electronic mail: merino@maths.ox.ac.uk

The Tutte polynomial is less familiar and will be precisely defined in Sec. II. However in its basic form it is just a two-variable polynomial $T(G;x,y)$ associated with any finite graph G. Its relation with the classical Potts model on G as described previously is that $Z(G;Q,K)$ as given in (3) is, up to an easy multiplicative constant, just an evaluation of T along the hyperbola

$$H_Q \equiv (x-1)(y-1) = Q.$$

To see and remember this is not difficult. The reparametrization $(Q,K) \leftrightarrow (x,y)$ is just given by

$$x = 1 + \frac{Qe^{-K}}{1-e^{-K}} = \frac{e^K + Q - 1}{e^K - 1},$$

$$y = e^K.$$

Thus the Tutte polynomial T can be regarded as a natural continuation of Z from the countable set of hyperbolae $\{H_Q\}, Q = 1, 2, ...,$ to the whole plane.

The interpretation is quite easy and allows an easy specification of various places of interest. For example the following correspondences are easy to check:

Q-state Potts	Tutte polynomial
Ferromagnetism	Positive branch H_Q^+ of H_Q
Antiferromagnetism	Negative branch H_Q^- of H_Q restricted to $y>0$
High temperature both ferromagnetic and antiferromagnetic	Portion of H_Q asymptotic to $y=1$
Low temperature ferromagnetic	H_Q^+ asymptotic to $x=1$
Absolute zero antiferromagnetic	$x=1-Q,\ y=0$

A partial extension of the Potts model is the *random cluster* model introduced by Fortuin and Kasteleyn in 1972. This extends the ferromagnetic Potts model to the whole of the region $Q>0$ but again this is only a part of the Tutte plane. More precisely the random cluster partition function $Z_{RC}(Q,p)$ which we define in Sec. IV corresponds to the quadrant $x>1$, $y>1$ in the Tutte plane.

In what follows we highlight some of the many other specializations of the Tutte polynomial, concentrating on those in the region of the Potts or random cluster models or those on the boundary of this region, notably the intriguing degenerate hyperbola H_0 corresponding to $Q=0$. We also treat in some detail a curious interpretation in terms of the weight enumerator of codes, whenever Q is a prime power.

We close this introduction by pointing out another way of thinking of the Potts model which is useful in what follows. This is in terms of coloring. The possible colors are the integers 1, 2, ..., Q and the sum of the right-hand side of (3) is just a sum over all possible Q colorings of the vertex set of G. Given a particular coloring σ we see that its contribution to the sum is the term

$$\exp(-K|E \setminus B(\sigma)|),$$

where we use $B(\sigma)$ to denote the set of edges which are *bad*, that is, have end points with the same color, under σ.

Hence, if we write $b_j(G;\lambda)$ to denote the number of λ colorings of G in which exactly j edges are bad, then

$$Z(G;Q,K) = e^{-K|E|} \sum_{j=0}^{\infty} b_j(G;Q)(e^K)^j. \tag{4}$$

In other words, if as in Ref. 1 we define the *bad coloring polynomial* to be the generating function

$$B(G;\lambda,s)=\sum_j b_j(G;\lambda)s^j, \tag{5}$$

then

$$Z(G;Q,K)=e^{-K|E|}B(G;Q,e^K). \tag{6}$$

An excellent, accessible review of the Potts model can be found in Wu.[2]

II. THE TUTTE POLYNOMIAL

The Tutte polynomial is a polynomial in two variables x,y which can be defined for a graph, matrix, or even more generally a matroid. Most of the interesting applications arise when the underlying structure is a graph or a matrix, but matroids are an extremely useful vehicle for unifying the concepts and definitions. For example each of the following is a special case of the general problem of evaluating the Tutte polynomial of a graph (or matrix) along particular curves of the (x, y) plane:

(i) the chromatic and flow polynomials of a graph;
(ii) the all terminal reliability probability of a network;
(iii) the partition function of a Q-state Potts model;
(iv) the Jones polynomial of an alternating knot;
(v) the weight enumerator of a linear code over $GF(q)$.

In this section we will briefly review the standard theory of the Tutte polynomial and in Sec. V we list its well-known evaluations. The graph terminology used is standard. The matroid terminology follows Oxley.[3] Further details of many of the concepts treated here can be found in Welsh.[1]

First consider the following recursive definition of the function $T(G;x,y)$ of a graph G and two independent variables x, y.

If G has no edges, then $T(G;x,y)=1$, otherwise for any $e\in E(G)$.

2.1: $T(G;x,y)=T(G'_e;x,y)+T(G''_e;x,y)$, where G'_e denotes the deletion of the edge e from G and G''_e denotes the contraction of e in G, and the edge e is not a loop or an isthmus,

2.2: $T(G;x,y)=xT(G''_e;x,y)$, whenever e is an isthmus, that is an edge whose removal increases the number of connected components,

2.3: $T(G;x,y)=yT(G'_e;x,y)$, whenever e is a loop.

From this, it is easy to show by induction that T is in fact a two-variable polynomial in x,y, which we call the *Tutte polynomial* of G.

In other words, T may be calculated recursively by choosing the edges in *any* order and repeatedly using 2.1–2.3 to evaluate T. The remarkable fact is that T is well defined in the sense that the resulting polynomial is independent of the order in which the edges are chosen.

Example: In Fig. 1 we show an example of computing the Tutte polynomial of the graph G, that is K_4 minus one edge. By adding the monomials at the bottom of Fig. 1, we get that $T(G;x,y)=x^3+2x^2+x+2xy+y+y^2$.

Alternatively, and this is often the easiest way to prove properties of T, we can show that T has expansion shown in Fig. 1.

First recall that if $A\subseteq E(G)$, the *rank* of $A, r(A)$ is defined by

$$r(A)=|V(G)|-k(A), \tag{7}$$

where $k(A)$ is the number of connected components of the graph $G{:}A$ having vertex set $V=V(G)$ and edge set A.

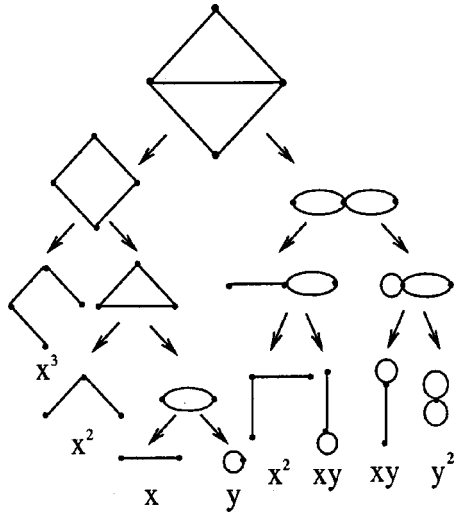


FIG. 1. An example of computing the Tutte polynomial recursively.

It is now straightforward to prove the following.
The Tutte polynomial $T(G;x,y)$ can be expressed in the form

$$T(G;x,y)=\sum_{A\subseteq E}(x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}. \tag{8}$$

One feature of the Tutte polynomial which is rather surprising in view of the states model expansion (8) is that for any graph T has an expansion of the form

$$T(G;x,y)=\sum_{i,j} t_{i,j}x^i y^j,$$

where the $t_{i,j}$ are non-negative integers. Typically the $t_{i,j}$ are represented in matrix form. For example, the following table provides the matrix form for the graph K_6:

$j\backslash i$	0	1	2	3	4	5
0	0	24	50	35	10	1
1	24	106	90	20	0	0
2	80	145	45	0	0	0
3	120	105	15	0	0	0
4	120	60	0	0	0	0
5	96	24	0	0	0	0
6	64	6	0	0	0	0
7	35	0	0	0	0	0
8	15	0	0	0	0	0
9	5	0	0	0	0	0
10	1	0	0	0	0	0

It is easy and useful to extend these ideas to matroids and hence matrices.

A *matroid M* is just a generalization of a matrix and can be simply defined as a pair (E,r) where E is a finite set and r is a submodular *rank function* mapping $2^E\rightarrow\mathbb{Z}$ and satisfying the conditions

$$0 \leqslant r(A) \leqslant |A|, \quad A \subseteq E,$$

$$A \subseteq B \Rightarrow r(A) \leqslant r(B), \text{ and}$$

$$r(A \cup B) + r(A \cap B) \leqslant r(A) + r(B), \quad A, B \subseteq E.$$

The edge set of any graph G with its associated rank function, as defined by (7), is a matroid, but this is just a very small subclass of matroids, known as graphic matroids.

A much larger class is obtained by taking any matrix B with entries in a field $\mathbb{F}$, letting E be its set of columns and for $X \subseteq E$ defining the rank $r(X)$ to be the maximum size of a linearly independent set in X. Any abstract matroid which can be represented in this way is called *representable* over $\mathbb{F}$.

A matroid M is representable over every field if and only if it has a representation over the reals by a matrix B which is *totally unimodular*. Such a matroid is called *regular*. Every graphic matroid is regular.

Given $M = (E, r)$, its *dual matroid* is $M^* = (E, r^*)$, where r^* is defined by

$$r^*(E \setminus A) = |E| - r(E) - |A| + r(A). \tag{9}$$

Duality is of fundamental importance as it allows duality concepts to be extended to nonplanar graphs. When M is the matroid of a planar graph G, M^* is the matroid of any planar dual graph of G. However when G is not planar then M^* is not graphic but is still representable as a matrix.

A set X is *independent* if $r(X) = |X|$, it is a *base* if it is a maximal independent subset of E.

We now just extend the definition of the Tutte polynomial from graphs to matroids by

$$T(M; x, y) = \sum_{A \subseteq E(M)} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)}. \tag{10}$$

Much of the theory developed for graphs goes through in this more general setting and there are many applications as we shall see. For example, routine checking shows that

$$T(M; x, y) = T(M^*; y, x). \tag{11}$$

In particular, when G is a planar graph and G^* is any plane dual of G, (11) becomes

$$T(G; x, y) = T(G^*; y, x).$$

III. INTERPRETATIONS IN TERMS OF THE ISING AND POTTS MODELS

We start this section with what it is called the ''recipe theorem'' from Oxley and Welsh.[4] Its crude interpretation is that whenever a function f on some class of matroids can be shown to satisfy an equation of the form $f(M) = af(M'_e) + bf(M''_e)$, for any $e \in E(M)$, then f is essentially an evaluation of the Tutte polynomial.

Here M'_e is the *restriction* of $M = (E, r)$ to the set $E \setminus \{e\}$ with r unchanged. The *contraction* M''_e can be defined by $M''_e = (M^*)'_e$ or more usefully by its rank function $r''(A) = r(A \cup e) - r(e)$ for $A \subseteq E \setminus \{e\}$ and is the exact analog of contraction in graphs. For matrices it corresponds to *projection* along the column vector e. A *minor* of M is any matroid N obtainable from M by a sequence of contractions and deletions. There is also a natural definition of the *direct sum* of two matroids M and N, where $E(M)$ and $E(N)$ are disjoint sets. The rank function of $M \oplus N$ is given by $r_{M \oplus N}(A \dot{\cup} B) = r_M(A) + r_N(B)$ for $A \subseteq E(M)$ and $B \subseteq E(N)$. Finally, we define a loop (coloop) as a single element matroid $\{e\}$ with rank function $r_1(e) = 0 (r_c(e) = 1)$.

The recipe theorem can now be stated as follows:

Theorem 1: *Let $\mathcal{C}$ be a class of matroids which is closed under direct sums and the taking of minors and suppose that f is well defined on $\mathcal{C}$ and satisfies*

$$f(M)=af(M'_e)+bf(M''_e), \quad e\in E(M), \tag{12}$$

$$f(M_1\oplus M_2)=f(M_1)f(M_2), \tag{13}$$

then f is given by

$$f(M)=a^{|E|-r(E)}b^{r(E)}T\left(M;\frac{x_0}{b},\frac{y_0}{a}\right),$$

where x_0 and y_0 are the values f takes on coloops and loops, respectively.

Any invariant *f* which satisfies (12) and (13) is called a *Tutte–Grothendieck (TG)-invariant.*

Thus, what we are saying is that any TG-invariant has to have an interpretation as an evaluation of the Tutte polynomial. As examples we consider the Ising and Potts models.

Consider the bad coloring polynomial defined in (5),

$$B(G;\lambda,s)=\sum_{i=0}^{|E|} s^i b_i(G;\lambda).$$

Clearly $b_0(G;\lambda)$ is the chromatic polynomial of G and it is easy to check that the following relationships hold.

3.1. If G is connected, then provided e is not a loop or coloop,

$$B(G;\lambda,s)=B(G'_e;\lambda,s)+(s-1)B(G''_e;\lambda,s).$$

3.2. $B(G;\lambda,s)=sB(G'_e;\lambda,s)$, if e is a loop.

3.3. $B(G;\lambda,s)=(s+\lambda-1)B(G''_e;\lambda,s)$, if e is a coloop.

Combining these, we get the following by using the recipe theorem for the class of connected graphs.

3.4. $B(G;\lambda,s)=\lambda(s-1)^{|V|-1}T\left(G;\frac{s+\lambda-1}{s-1},s\right)$.

Consider now the relation with the Potts model. From (6) we get

$$Z_{\text{Potts}}(G;Q,K)=Q(e^K-1)^{|V|-1}e^{-K|E|}T\left(G;\frac{e^K+Q-1}{e^K-1},e^K\right). \tag{14}$$

It is not difficult (with hindsight) to verify that $T(G;x,y)$ can be recovered from the polynomial B and therefore from the Potts partition function by using the following formula:

$$T(G;x,y)=\frac{1}{(y-1)^{|V|}(x-1)}B(G;(x-1)(y-1),y).$$

For connected graphs, the classical Ising model is just the case $Q=2$ in (14). When G has k connected components then there is an extra factor of Q^{k-1} on the right-hand side of (14).

IV. THE RANDOM CLUSTER MODEL AND FERROMAGNETIC POTTS MODEL

The general random cluster model on a finite graph G was introduced by Fortuin and Kasteleyn.[5] It is a correlated bond percolation model on the edge set E of G defined by the probability distribution,

$$\mu(A)=Z_{\text{RC}}^{-1}\left(\prod_{e\in A}(p_e)\right)\left(\prod_{e\notin A}(1-p_e)\right)Q^{k(A)} \quad (A\subseteq E), \tag{15}$$

where $k(A)$ is the number of connected components (including isolated vertices) of the subgraph $G:A=(V,A)$, $p_e(0\leqslant p_e\leqslant 1)$ are parameters associated with each edge of G, $Q\geqslant 0$ is a parameter of the model, and Z_{RC} is the normalizing constant introduced so that

$$\sum_{A\subseteq E}\mu(A)=1.$$

Then $\mu(A)$ is interpreted as the probability that the set of edges of G *open* in the random cluster model is exactly the set A. The complement $E\setminus A$ is *closed*.

We will sometimes use $\omega(G)$ to denote the random configuration produced by μ, and P_μ to denote the associated probability distribution.

Thus, in particular, $\mu(A)=P_\mu\{\omega(G)=A\}$. When $Q=1$, μ is what Fortuin and Kasteleyn call a *percolation model* and when each of the p_e are made equal, say to p, then $\mu(A)$ is clearly seen to be the probability that the set of open edges is A in classical ordinary bond percolation.

For an account of the many different interpretations of the random cluster model we refer to the original paper of Fortuin and Kasteleyn[5] or to Grimmett.[6,7]

Here we shall be concentrating on the percolation problem when each of the p_e are equal, to say p, and henceforth this will be assumed.

Thus we will be concerned with a two parameter family of probability measures $\mu=\mu(p,Q)$ where $0\leqslant p\leqslant 1$ and $Q>0$, which are defined on the edge set of the finite graph $G=(V,E)$ by

$$\mu(A)=p^{|A|}q^{|E\setminus A|}Q^{k(A)}/Z_{\mathrm{RC}},$$

where Z_{RC} is the appropriate normalizing constant, and $q=1-p$.

The reason for studying percolation in the random cluster model is its relation with phase transitions via the two-point correlation function. This was pointed out first by Fortuin and Kasteleyn and given further prominence by Edwards and Sokal[8] in connection with the Swendsen–Wang algorithm[9] for simulating the Potts model. We describe briefly the connection.

The key result is the following:

Theorem 2: *For any pair of sites (vertices) i, j, and positive integer Q, the probability that* σ_i *equals* σ_j *in the Q-state Potts model is given by*

$$\frac{1}{Q}+\frac{(Q-1)}{Q}P_\mu\{i\rightsquigarrow j\},$$

where P_μ *is the random cluster measure on G given by taking* $p=1-\exp(-K)$, *and* $\{i\rightsquigarrow j\}$ *is the event that under* μ *there is an open path from i to j.*

The attractive interpretation of this is that the expression on the right-hand side can be regarded as being made up of two components.

The first term, $1/Q$, is just the probability that under a purely random Q-coloring of the vertices of G, i and j are the same color. The second term measures the probability of long range interaction. Thus we interpret the above as expressing an equivalence between long range spin correlations and long range percolatory behavior.

Phase transition (in an infinite system) occurs at the onset of an infinite cluster in the random cluster model and corresponds to the spins on the vertices of the Potts model having a long range two-point correlation.

Thus the random cluster model can be regarded as the analytic continuation of the Potts model to noninteger $Q>0$.

It is not hard to check that the relation of the random cluster model with T is that

$$Z_{\mathrm{RC}}(G;Q,p)=p^{r(E)}q^{r^*(E)}Q^{k(G)}T\left(G;1+\frac{Qq}{p},\frac{1}{q}\right),\tag{16}$$

where r^* is the dual rank, $k(G)$ is the number of connected components of G, and $q=1-p$.

It follows that for any given $Q>0$, determining the partition function Z_{RC} reduces to determining T along the hyperbola H_Q given by $(x-1)(y-1)=Q$. However, since in its physical interpretations, p is a probability, the reparametrization means that Z_{RC} is evaluated only along the positive branch of this hyperbola. In other words, Z_{RC} is the specialization of T to the quadrant $x>1$, $y>1$.

The antiferromagnetic Ising and Potts models are contained in T along the negative branches of the hyperbolae H_Q, but do not have representations in the random cluster model. For more on this model and its relation to T see Ref. 1, Chap. 4.

V. SOME WELL-KNOWN INVARIANTS

Having shown in detail how the Potts, Ising, and random cluster models are related to the Tutte polynomial we now collect together some of the naturally occurring interpretations of the Tutte polynomial. Throughout G is a graph, M is a matroid, and E will denote $E(G)$, $E(M)$, respectively.

In each of the following cases, the interesting quantity (on the left-hand side) is given (up to an easily determined term) by an evaluation of the Tutte polynomial. We shall use the phrase ''*specializes to*'' to indicate this.

When talking about the Tutte polynomial and Potts model, it turns out that the hyperbolas H_α defined by

$$H_\alpha=\{(x,y):(x-1)(y-1)=\alpha\}$$

seem to have a special role in the theory. We note several important specializations in the following.

(1) Along H_1, $T(G;x,y)=x^{|E|}(x-1)^{r(E)-|E|}$.

(2) Along H_2, when G is a graph, T specializes to the partition function of the Ising model.

(3) Along H_Q, for general positive integer Q,T specializes to the partition function of the Q-state Potts model.

(4) Along H_Q for any positive, not necessarily integer, Q,T specializes to the partition function of the random cluster model discussed in Sec. IV.

(5) At (1, 1), T counts the number of bases of M (spanning trees in a connected graph).

(6) At (2, 1), T counts the number of independent sets of M (forests in a graph).

(7) At (1, 2), T counts the number of spanning connected subgraphs of the graph G.

(8) At (2, 0), T counts the number of acyclic orientations of G.[10]

(9) Another interpretation at (2,0), and this for any real matrix was discovered by Zaslavsky.[11] If $\{H_1,\ldots,H_r\}$ is a set of hyperplanes in d-dimensional Euclidean space with nonempty intersection, then T counts the number of unbounded regions of this hyperplane arrangement.

(10) At (0, 2), T counts the number of totally cyclic orientations, that is, those in which every edge of the graph G is contained in some directed cycle.

(11) At (1, 0), T counts the number of acyclic orientations with exactly one source.

(12) At (0, 1), if G is a directed graph having a fixed ordering on its edges, T counts the number of totally cyclic reorientations τ of G such that in each cycle of τ the lowest edge is not reoriented. If G is planar, T counts the number of totally cyclic orientations in which there is no clockwise cycle.

(13) When λ is a positive integer $T(G;1-\lambda,0)$ gives the number of λ colorings because, the chromatic polynomial $\chi(G;\lambda)$ is given by

$$\chi(G;\lambda)=(-1)^{r(E)}\lambda^{k(G)}T(G;1-\lambda,0),$$

where $k(G)$ is the number of connected components.

(14) Similarly $T(G;0,1-\lambda)$ counts the number of nowhere zero flows over any Abelian group of order λ. Then the flow polynomial $F(G;\lambda)$, is given by

$$F(G;\lambda)=(-1)^{|E|-r(E)}T(G;0,1-\lambda).$$

(15) The (all terminal) reliability $R(G;p)$ defined as the probability that when each edge of the connected graph G is independently deleted with probability $1-p$ the remaining graph stays connected is given by

$$R(G;p)=q^{|E|-r(E)}p^{r(E)}T(G;1,1/q),$$

where $q=1-p$.

(16) At $(0,-2)$, if G is a four-regular graph, T counts the number of ice configurations of G. An *ice configuration* of G is an orientation of the edges so that at each vertex exactly two edges are directed in and two out. It is easy to see that this counts exactly the number of nowhere zero three-flows on G.

(17) $T(G;-1,-1)=(-1)^{|E|}(-2)^{d(B)}$ where B is the bicycle space of G, see Read and Rosenstiehl.[12] When G is planar it also has interpretations in terms of the Arf invariant of the associated knot.

(18) The number of forests of size i of G, $f_i(G)$, is related to T by the following:

$$\sum_{i=0}^{|V|-1} f_i(G)s^i=s^{|V|-1}T\left(G;\frac{1}{s}+1,1\right).$$

(19) Also, the generating function of connected subgraphs of size k of G, $c_k(G)$, is related to T by

$$\sum_{k=0}^{|E|-|V|+1} c_k(G)s^k=s^{|E|-|V|+1}T\left(G;1;\frac{1}{s}+1\right).$$

(20) Along H_q, when q is a prime power, for a matrix M of column vectors over $GF(q)$, T specializes to the weight enumerator of the linear code over $GF(q)$, with generator matrix M. Equation (11) relating $T(M)$ to $T(M^*)$ gives the MacWilliams identity of coding theory, we return to this in Sec. IX.

(21) Along the hyperbola $xy=1$ when G is planar, T specializes to the Jones polynomial of the alternating link or knot associated with G. This connection was first discovered by Thistlethwaite[13] and is explained in Ref. 1.

Other more specialized interpretations can be found in the survey of Brylawski and Oxley[14] and the book of Welsh.[1]

VI. MEAN FIELD RESULTS

The mean field Potts model refers to the case where the underlying graph is the complete graph. This has been considered by Wu[2] and Kesten and Schonmann[15] for the classical Potts model, and more generally for the random cluster model by Bollobás, Grimmett, and Janson.[16] First, however, as a useful example to illustrate what is known, we consider the behavior of $T_n(x,y)=T(K_n;,x,y)$ for some of the points and curves described in Sec. V.

First some easy and not so easy known evaluations.

(1) Along H_1, $(x-1)(y-1)=1$,

$$T_n(x,y)=\left(\frac{x}{x-1}\right)^{\binom{n}{2}}(x-1)^{n-1}.$$

(2) $T_n(1,1)$ is the number of trees on n vertices and hence

$$T_n(1,1)=n^{n-2}.$$

(3) The number of acyclic orientations is

$$T_n(2,0)=n!.$$

(4) The number of acyclic orientations with exactly one source is

$$T_n(1,0)=(n-1)!.$$

(5) The number of k-colorings, for any fixed k, is given by the Stirling polynomials so

$$T_n(1-k,0)=(-1)^{n-1}k(k-1)\cdots(k-n+1).$$

(6) The number of forests is

$$T_n(2,1)\sim\sqrt{e}n^{n-2} \quad \text{as} \quad n\to\infty.$$

See for example Renyi[17] or Dénes.[18]

Hence this gives a good picture of the sort of asymptotics one might expect.

It would be nice if there was a compact useful formula for the Tutte polynomial of the complete graph as there is for the chromatic polynomial. Unfortunately this does not seem to be the case and all that seems possible is to obtain a generating function expansion which is not that useful. It was originally obtained by Tutte.[19] Two different forms of this are in Refs. 20 and 21.

Welsh[20] gives

$$1+(x-1)\sum_{n=1}^{\infty}\frac{(y-1)^n s^n T_n(x,y)}{n!}=\left(\sum_{j=0}^{\infty}\frac{s^j y^{\binom{n}{2}}}{j!}\right)^{(x-1)(y-1)}. \tag{17}$$

Substituting $(x-1)(y-1)^n T_n(x,y)=B_n(Q,y)$ we get the following generating function for the bad coloring polynomial or equivalently $Z(K_n)$:

$$1+\sum_{n=1}^{\infty}\frac{s^n B_n(Q,y)}{n!}=\left(\sum_{j=0}^{\infty}\frac{s^j y^{\binom{n}{2}}}{j!}\right)^{Q}.$$

One way of obtaining this directly is the following.

Let

$$\sum_{n=0}^{\infty}\frac{B_n(Q;s_1,\ldots,s_Q)}{n!}u^n$$

be the generating function in which the coefficient of

$$s_1^{m_1}s_2^{m_2}\cdots s_Q^{m_Q}u^n$$

is the number of ways of coloring K_n with Q colors $\{1, 2, ..., Q\}$ where m_i is the number of edges with both end points colored i, for $1\leqslant i\leqslant Q$.

Then clearly as

$$B(K_n;s_1,\ldots,s_Q)=\sum_{k_1+\cdots+k_Q=n}\frac{n!}{k_1!\cdots k_Q!}s_1^{\binom{k_1}{2}}\cdots s_Q^{\binom{k_Q}{2}}$$

we conclude that

$$\sum_{n=0}^{\infty} \frac{B(K_n;s_1,\ldots,s_Q)}{n!} u^n = \prod_{i=1}^{Q} \left(\sum_{r=0}^{\infty} \frac{s_i^{\binom{r}{2}}}{r!} u^r \right).$$

Now, this gives more information than we are asking for and putting

$$s_1 = s_2 = \cdots = s_Q = s$$

we get

$$\sum_{n=0}^{\infty} \frac{B(K_n;Q,s)}{n!} u^n = \left(\sum_{r=0}^{\infty} \frac{s^{\binom{n}{2}}}{r!} u^r \right)^Q.$$

One of the annoying features of the above-mentioned expansions is that they seem to be of little help in attacking problems we wish to solve. As an example of this consider the evaluation at $x=2$, $y=1$ which gives $F(n)$ the number of forests in the complete graph. Direct substitution in (17) does not seem to give us anything useful, in particular, we do not see how to get the following result from (17).

Takacs[22] gives the following exponential generating function for $F(n)$:

$$\sum_{n=1}^{\infty} \frac{F(n)}{n!} s^n = \exp\left[\sum_{n=1}^{\infty} \frac{n^{n-2}}{n!} s^n \right].$$

He also gives the more useful

$$F(n) = \sum_{r=0}^{n} \binom{n}{r} n^{n-r} H_{r+1}(1),$$

where $H_n(x)$ is the nth Hermite polynomial defined by

$$H_n(x) = n! \sum_{j=0}^{[n/2]} \frac{(-1)^j x^{n-2j}}{2^j j!(n-2j)!}.$$

An extension of this by Stanley[23] gives the number $F(i,n)$ of forests with i edges on K_n as having generating function

$$\sum_{n \geqslant 0} \sum_{i} F(i,n) \frac{s^i t^n}{n!} = \exp\left(n^{n-2} \frac{s^{n-1} t^n}{n!} \right).$$

We now turn to the recent work of Bollobás, Grimmett, and Janson[16] on the asymptotics of the random cluster model. Recall that if $Z(n,p,Q)$ denotes the partition function of $Z_{\mathrm{RC}}(K_n;p,Q)$ then this gives the Potts model on K_n by the substitution

$$p = 1 - e^{-k}$$

One of the main results (Theorem 2.6[16]) is the following:

Theorem 3: *If* $Q \geqslant 1$ *and* $\lambda > 0$, *then*

$$\frac{1}{n} \log Z\left(n, \frac{\lambda}{n}, Q \right) \to \phi(\lambda, Q)$$

as $n \to \infty$, *where the free energy* $\phi(\lambda,Q)$ *is given by*

$$\phi(\lambda,Q)=\frac{g(\theta(\lambda))}{2Q}-\frac{(Q-1)\lambda}{2Q}+\log Q$$

and where $g(\theta)$ *is defined by*

$$g(\theta)=-(Q-1)(2-\theta)\log(1-\theta)-\{2+(Q-1)\theta\}\log\{1+(Q-1)\theta\}.$$

The function $\theta(\lambda)(=\theta(\lambda,Q))$ *is defined as*

$$\theta(\lambda,Q)=\begin{cases}0 & if\ \ \lambda<\lambda_c(Q)\\ \theta_{\max} & \text{if}\ \ \lambda\geqslant\lambda_c(Q),\end{cases}$$

where

$$\lambda_c(Q)=\begin{cases}Q & \text{if}\ \ 0<Q\leqslant 2\\ 2\left(\dfrac{Q-1}{Q-2}\right)\log(Q-1) & \text{if}\ \ Q>2\end{cases}$$

and $\theta_{\max}$ *is the largest root of the equation*

$$e^{-\lambda\theta}=\frac{1-\theta}{1+(Q-1)\theta}.$$

This explains the asymptotics in the region $x\geqslant 1$, $y\geqslant 1$ of the Tutte plane but note that it says nothing about the antiferromagnetic part.

VII. THE COMPLEXITY OF THE TUTTE PLANE

We have seen that along different curves of the x,y plane, the Tutte polynomial evaluates many diverse quantities. Since it is also the case that for particular curves and at particular points the computational complexity of the evaluation can vary from being polynomial time computable to being $\#P$-hard a more detailed analysis of the complexity of evaluation is needed in order to give a better understanding of what is and is not computationally feasible for these sort of problems. The main result of Jaeger, Vertigan, and Welsh[24] is the following:

Theorem 4: *The problem of evaluating the Tutte polynomial of a graph at a point (a,b) is* $\#P$*-hard except when (a,b) is on the special hyperbola*

$$H_1\equiv(x-1)(y-1)=1$$

or when (a,b) is one of the special points (1, 1), (−1, −1), (0, −1), (−1, 0), $(i,-i)$, $(-i,i)$, (j,j^2) *and* (j^2,j), *where* $j=e^{2\pi i/3}$. *In each of these exceptional cases the evaluation can be done in polynomial time.*

As far as the easy real points are concerned, with one exception, the explanation is straightforward. The hyperbola H_1 is trivial, (1, 1) gives the number of spanning trees, (−1, 0) and (0, −1) give the number of two-colorings and two flows respectively, and are easy evaluations of the antiferromagnetic Ising. The point (−1, −1) is less well known but has been explained, see Sec. V. It lies on the four-state Potts curve but as far as we are aware has no natural explanation there.

Finally the complex points $(i,-i)(-i,i)$ lie on the Ising curve and the points $(j,j^2)(j^2,j)$ lie on the three-state Potts. Again there seems to be no natural interpretation to explain why their evaluation is easy. The only reason why they appear in Theorem 4 is that they ''turn up in the calculations.''

For planar graphs there is a significant difference. The technique developed using the Pfaffian to solve the Ising problem for the plane square lattice by Kasteleyn[25] can be extended to give a polynomial time algorithm for the evaluation of the Tutte polynomial of any planar graph along the special hyperbola

$$H_2 \equiv (x-1)(y-1) = 2.$$

However H_3 cannot be easy for planar graphs since it contains the point $(-2, 0)$ which counts the number of three-colorings and since deciding whether a planar graph is three-colorable is NP-hard, this must be at least NP-hard. However it does not seem easy to show that H_4 is hard for planar graphs. The decision-problem is after all trivial by the four-color theorem. The fact that it is $\#P$-hard is just part of the following extension of Theorem 4 due to Vertigan and Welsh.[26]

Theorem 5: *The evaluation of the Tutte polynomial of bipartite planar graphs at a point (a,b) is #P-hard except when*

$$(a,b) \in H_1 \cup H_2 \cup \{(1,1),(-1,-1),(j,j^2),(j^2,j)\}$$

when it is computable in polynomial time.

It follows immediately from the fact that any graph can be represented as a totally unimodular matrix that if a problem is hard (in any formal sense) for graphs then it will be at least as hard for matrices.

VIII. APPROXIMATIONS

Since exact evaluation is provably hard, we turn to the possibility of obtaining good approximations or Monte Carlo estimates.

For positive numbers a and $r \geqslant 1$, we say that a third quantity $\hat{a}$ *approximates a within ratio r or is an r-approximation to a, if*

$$r^{-1}a \leqslant \hat{a} \leqslant ra. \tag{18}$$

In other words the ratio $\hat{a}/a$ lies in $[r^{-1},r]$.

First consider what it would mean to be able to find a polynomial time algorithm which gave an approximation within r to the number of three-colorings of a graph. We would clearly have a polynomial time algorithm which would decide whether or not a graph is three-colorable. But this is NP-hard. Thus no such algorithm can exist unless NP=P.

The same argument can be applied to any function which counts objects whose existence is NP-hard to decide. Hence

Proposition 6: Unless NP=P *there can be no polynomial time approximation to* $T(G;1-k,0)$ *for integer* $k \geqslant 3$.

However this argument only applies to a few points of the Tutte plane and it seems a difficult problem to decide on the existence of good approximations elsewhere.

We now consider a randomized approach to counting problems and make the following definition.

An *ϵ-δ-approximation scheme* for a counting problem f is a Monte Carlo algorithm which on every input $\langle x, \epsilon, \delta \rangle$, $\epsilon > 0$, $\delta > 0$, outputs a number $\tilde{Y}$ such that

$$\Pr\{(1-\epsilon)f(x) \leqslant \tilde{Y} \leqslant (1+\epsilon)f(x)\} \geqslant 1-\delta.$$

Now let f be a function from input strings to the natural numbers. A *randomized approximation scheme for f is a probabilistic* algorithm that takes as an input a string x and a rational number ϵ, $0 < \epsilon < 1$, and produces as output a random variable Y, such that Y approximates $f(x)$ within ratio $1+\epsilon$ with probability greater or equal 3/4.

In other words,

$$\Pr\left\{\frac{1}{1+\epsilon}\leqslant\frac{Y}{f(x)}\leqslant 1+\epsilon\right\}\geqslant\frac{3}{4}. \tag{19}$$

A fully polynomial randomized approximation scheme (fpras) for a function $f:\Sigma^*\to\mathbf{N}$ is a randomized approximation scheme which runs in time which is a polynomial function of *n and* ϵ^{-1}.

Suppose now we have such an approximation scheme and suppose further that it works in polynomial time. Then we can boost the success probability up to $1-\delta$ for any desired $\delta>0$, by using the following trick of Jerrum, Valiant, and Vazirani.[27] This consists of running the algorithm $O(\log\delta^{-1})$ times and taking the median of the results.

We make this precise as follows:

Proposition 7: If there exists a fpras for computing f then there exists an ϵ–δ *approximation scheme for f which on input* $\langle x, \epsilon, \delta\rangle$ *runs in time which is bounded by* $O(\log\delta^{-1})\mathrm{poly}(x,\epsilon^{-1})$.

The existence of a fpras for a counting problem is a very strong result, it is the analog of a randomized polynomial time (RP) algorithm for a decision problem and corresponds to the notion of tractability. However we should also note

Proposition 8: If $f:\Sigma^*\to\mathbb{N}$ *is such that deciding if f is nonzero is NP-hard then there cannot exist a fpras for f unless NP is equal to random polynomial time RP*.

Hence we have immediately from the NP-hardness of *k*-coloring, for $k\geqslant 3$, that:

Unless NP=RP there cannot exist a fpras for evaluating $T(G;-k,0)$ for any integer $k\geqslant 2$.

Recall now that along the hyperbola, H_Q, for positive integer Q,T evaluates the partition function of the *Q*-state Potts model.

In an important paper, Jerrum and Sinclair[28] have shown that there exists a fpras for the ferromagnetic Ising problem. Their result can be restated in our terminology as follows.

(1) There exists a fpras for estimating T along the positive branch of the hyperbola H_2.

However it seems to be difficult to extend the argument to prove a similar result for the *Q*-state Potts model with $Q>2$ and this remains one of the outstanding open problems in this area.

A second result of Jerrum and Sinclair is the following:

(2) There is no fpras for estimating the antiferromagnetic Ising partition function unless NP=RP.

In the context of its Tutte plane representation this can be restated as follows.

(3) Unless NP=RP, there is no fpras for estimating T along the curve

$$\{(x,y):(x-1)(y-1)=2,\ 0<y<1\}.$$

The following extension of this result is proved in Welsh.[29]

Theorem 9: *On the assumption that* $NP\neq RP$, *the following statements are true*.

(a) *Even in the planar case, there is no fully polynomial randomized approximation scheme for T along the negative branch of the hyperbola* H_3, *that is for the antiferromagnetic three-state Potts model*.

(b) *For* $Q=2,4,5,...$, *there is no fully polynomial randomized approximation scheme for T along the curves*

$$H_Q^-\cap\{x<0\}.$$

It is worth emphasizing that the above-mentioned statements do not rule out the possibility of there being a fpras at *specific points* along the negative hyperbolas. For example;

(1) T can be evaluated exactly at $(-1, 0)$ and $(0, -1)$, which both lie on H_2^-.
(2) There is no inherent obstacle to there being a fpras for estimating the number of *k*-colorings of a planar graph for any $k\geqslant 4$.

Positive results: Mihail and Winkler[30] have shown that there exists a fpras for counting the number of ice configurations in a four-regular graph. This is equivalent to the statement:

There is a fpras for computing T at $(0, -2)$ for four-regular graphs.

The reader will note that all the ''negative results'' are about evaluations of T in the region outside the quadrant $x \geqslant 1$, $y \geqslant 1$. In Welsh[1] it is conjectured that the following is true:

Conjecture 10: There exists a fpras for evaluating T at all points of the quadrant $x \geqslant 1$, $y \geqslant 1$.

Some evidence in support of this is the following.

If we let $\mathcal{G}_\alpha$ be the collection of graphs $G=(V,E)$ such that each vertex has at least $\alpha|V|$ neighbors, then we call a class $\mathcal{C}$ of graphs *dense* if $\mathcal{C} \subseteq \mathcal{G}_\alpha$ for some fixed $\alpha > 0$.

Annan[31] showed that:

Proposition 11: For any class of dense graphs, there is a fpras for evaluating $T(G;x,1)$ *for positive integer x.*

Extending this, Alon, Frieze, and Welsh[32] show

Theorem 12: (a) *There exists a fully polynomial randomized approximation scheme for evaluating* $T(G;x,y)$ *for all* $x \geqslant 1$, $y \geqslant 1$, *for any dense class of graphs.*

(b) *For any class of strongly dense graphs, meaning* $G \in \mathcal{G}_\alpha$ *for* $\alpha > \frac{1}{2}$, *there is also such a scheme for* $x < 1$, $y \geqslant 1$.

Even more recently Karger[33] has proved the existence of a similar scheme for the class of graphs with no small edge cut set. This can be stated as follows.

For $c > 0$ define the class $\mathcal{G}^c$ by $G \in \mathcal{G}^c$ if and only if its edge connectivity is at least $c \log|V(G)|$. A class of graphs is *well connected* if it is contained in $\mathcal{G}^c$ for some fixed c.

Theorem 13: *For any fixed (x,y),* $y > 1$, *there exists c, depending on (x,y), such that for any class* $\mathcal{C} \subseteq \mathcal{G}^c$, *there is a fpras for evaluating* $T(G;x,y)$.

Notice that though the properties of being well connected and dense are very similar neither property implies the other.

Notice also that part (a) of Theorem 12 can be loosely reinterpreted as (a) There is a good Monte Carlo scheme for estimating the partition function of the random cluster model on any class of dense graphs.

Unfortunately there are several important classes of graphs, in particular lattices, which are not dense.

IX. THE POTTS MODEL AND ERROR CORRECTING CODES

We now turn to a curious correspondence between the partition functions of the Q-states Potts model whenever Q is a power of a prime and the weight enumerator polynomial of linear codes over the finite field with Q elements. This correspondence is reasonably well known for the case $Q=2$, the Ising model. It has been pointed out for example by Hoede[34] and Rosengren and Lindström.[35] In Ref. 25 this correspondence was used to derive terms of the low temperature series expansion of the partition function Z of the three-dimensional cubic lattice.

For the purpose of this section take q to be any prime power and let $\mathcal{C}$ be a *linear code* over the field $\mathrm{GF}(q)$.

A compact description of $\mathcal{C}$ is by a $k \times n$ *generator matrix* G. The code words of $\mathcal{C}$ are all linear combinations of rows of G. Now let $E=\{e_1,\ldots,e_n\}$ be the column vectors of G and take M to be the matroid (E,r), where each subset A of columns has rank $r(A)$ equal to the maximum number of linearly independent columns in A as in Sec. II.

The *weight* of a code word is the number of nonzero entries. Given a code $\mathcal{C}$, let A_i denote the number of code words which have weight i. The *weight enumerator* of $\mathcal{C}$ is

$$\mathcal{A}(\mathcal{C};t) = \sum A_i t^i. \tag{20}$$

Then we have the following theorem of Greene.[36]

Theorem 14: *Let* $\mathcal{C}$ *be a linear code of dimension k and length n over the field* $\mathrm{GF}(q)$. *Let G be a* $k \times n$ *generator matrix of* $\mathcal{C}$ *and let M be the matroid on the set of columns. Then the weight enumerator* $\mathcal{A}(\mathcal{C};t)$ *is given by*

$$\mathcal{A}(\mathcal{C};t)=(1-t)^k t^{n-k} T\left(M;\frac{1+(q-1)t}{1-t},\frac{1}{t}\right). \tag{21}$$

The proof of this is not that difficult and can be found for example in Ref. 37.

If we compare this with the expression (14) for the partition function $Z_{\text{Potts}}(G)$ we can rewrite it as

$$Z_{\text{Potts}}(G;Q,K)=Q(1-e^{-K})^{|V|-1}e^{-K(|E|-|V|+1)}T\left(G;\frac{1+(Q-1)e^{-K}}{1-e^{-K}},\frac{1}{e^{-K}}\right)$$

so that there is a direct translation via

$$Q\to q=p^{\alpha},$$

$$e^{-K}\to\text{parameter } t.$$

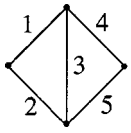
Under this correspondence we get

$$\mathcal{A}(\mathcal{C};t)=Q^{-1}Z_{\text{Potts}}(e^{-K}=t).$$

Now let us consider what this means in the context of a graphic matroid. Given any finite graph it is easy to find a representation of it as a generator matrix of a linear code over any finite field.

The edges of the graph correspond to the columns of the matrix and a set A of columns is linearly independent if and only if the corresponding edges form a forest.

Example: Working with the field GF(2), K_4 minus one edge has a representation



$\Leftrightarrow$

1	2	3	4	5
1	0	1	0	1
0	1	1	0	1
0	0	0	1	1

The resulting code generated by this matrix has eight code words of length 5 and weight enumerator $1+2z^2+4z^3+z^4$.

Now lets consider how this can be interpreted in general. Writing the weight enumerator as in (20), we see that we have another expansion for the Potts partition function namely,

$$Z_{\text{Potts}}(G;Q,K)=Q\sum_{i=0}^{|E|}A_i e^{-Ki}.$$

Note however that this only works when Q is a prime power.

X. THE POTTS MODEL AND COUNTING IN RANDOM GRAPHS

Although the theory of random graphs is highly developed, less attention seems to have been paid to counting problems. Here we give some results obtained in Welsh[20] which give new interpretations of the Tutte polynomial as the expected value of classical counting functions.

Given an arbitrary graph G and $p\in[0,1]$ we denote by G_p the *random subgraph* of G obtained by deleting each edge of G independently with probability $1-p$.

This is a generalization of the standard random graph model $G_{n,p}$ which corresponds to $(K_n)_p$.

First an easy result to illustrate the notation. If $f(G_p)$ denotes the number of forests in G_p then, for G and p fixed this is a random variable and has an expectation which we denote by $\langle f(G_p)\rangle$.

Routine calculation gives that for any connected graph G,

$$\langle f(G_p)\rangle = p^{|V|-1}T\left(G;1+\frac{1}{p},1\right).$$

Turning now to colorings, we have:

Theorem 15: *For any connected graph G and* $0<p\leqslant 1$, *the random subgraph* G_p *has chromatic polynomial whose expectation is given by*

$$\langle \chi(G_p;\lambda)\rangle = (-p)^{|V|-1}\lambda T(G;1-\lambda p^{-1},1-p).$$

For the flow polynomial there is a similar, but more complicated evaluation, namely

Theorem 16: *For any graph G the flow polynomial* $F(G_p;\lambda)$ *has expectation given by*

(a) *if* $p\in(0,\frac{1}{2})\cup(\frac{1}{2},1)$, *then*

$$\langle F(G_p;\lambda)\rangle = p^{r(G)}(q-p)^{r^*(G)}T\left(G;qp^{-1},1+\frac{\lambda p}{q-p}\right),$$

where $q=1-p$;

(b) *if* $p=\frac{1}{2}$, *then*

$$\langle F(G_{1/2};\lambda)\rangle = \lambda^{|E|-|V|+k(G)}2^{-|E|}.$$

Notice that parametrized in terms of the Potts model these give interpretations in the antiferromagnetic region.

XI. THE LIMIT AS $Q\to 0$

Several authors (See Wu[2]) have considered the formal limiting behavior of the Q-state Potts model as $Q\to 0$. This makes more sense in the context of the random cluster model which we recall is defined for all $Q>0$. Let us now consider this convergence in more detail.

Suppose in the random cluster model, p and Q both tend to zero with p/Q kept constant at 1. Then easy calculations show that in this case

$$\lim \frac{Z_{\mathrm{RC}}(G;p,Q)}{p^{r(E)}} = T(G;2,1).$$

In other words, from Sec. V, the limit is the number of forests of G.

There are various other cases to consider.

(a) If $Q\to 0$ with p fixed then

$$\lim_{Q\to 0} Z_{\mathrm{RC}}(G;p,Q) = cT\left(G;1,\frac{1}{1-p}\right),$$

where c is a constant. In other words we are getting

(1) the reliability probability, which we have already mentioned and is a much studied topic,

(2) the chip-firing game (Abelian sandpile model),

as two different realizations of this limiting behavior. We consider the latter in some detail in Sec. XI B.

(b) For the other part of the hyperbola H_0, consisting of $y=1, x\geqslant 1$, it is clear that if we let $Q\to 0$ in such a way that Q/p is fixed at $\alpha>0$, then in the random cluster model

$$x=1+\frac{Qq}{p}\to 1+\alpha,$$

$$y=\frac{1}{1-p}\to 1$$

and so

$$p^{-r}(1-p)^{-|E|+r(E)}Z_{\mathrm{RC}}(G;Q,p)\to T(G;1+\alpha,1).$$

We have already mentioned the case $\alpha=1$ where the limit is the number of forests. More generally we have the interesting specialization

$$\lambda^r T(G;1+1/\lambda,1)=i(P(G),\lambda),$$

where i is the Ehrhart polynomial of a particular family of zonotopes $P(G)$ determined by a graph G.

We now discuss in more detail these two separate problems areas.

A. The Ehrhart polynomial

Let $\mathbb{Z}^n$ denote the n-dimensional integer lattice in $\mathbb{R}^n$ and let P be an n-dimensional lattice polytope in $\mathbb{R}^n$, that is a convex polytope whose vertices have integer coordinates. Consider the function $i(P;t)$ which when t is a positive integer counts the number of lattice points which lie inside the dilated polytope tP. Ehrhart[38] initiated the systematic study of this function by proving that it was always a polynomial in t, and that in fact

$$i(P,t)=\chi(P)+c_1 t+\cdots+c_{n-1}t^{n-1}+\mathrm{vol}(P)t^n.$$

Here $c_0=\chi(P)$ is the Euler characteristic of P and $\mathrm{vol}(P)$ is the volume of P.

Until recently the other coefficients of $i(P,t)$ remained a mystery, even for simplices, see for example Diaz and Robins.[39]

However, in the special case that P is a unimodular zonotope there is a nice interpretation of these coefficients. First recall that if A is an $r\times n$ matrix, written in the form $A=[a_1,\ldots,a_n]$, then it defines a *zonotope* $Z(A)$ which consists of those points p of $\mathbb{R}^r$ which can be expressed in the form

$$p=\sum_{i=1}^{n}\lambda_i a_i,\quad 0\leqslant\lambda_i\leqslant 1.$$

In other words, $Z(A)$ is the *Minkowski sum* of the line segments $[0,a_i],1\leqslant i\leqslant n$.

$Z(A)$ is a convex polytope which, when A is a totally unimodular matrix, has all integer vertices and in this case it is described as a *unimodular zonotope*. For these polytopes a result from Stanley[40] shows that

$$i(Z(A);t)=\sum_{k=0}^{r} i_k t^k,$$

where i_k is the number of subsets of columns of the matrix A which are linearly independent and have cardinality k.

In other words, the Ehrhart polynomial $i(Z(A);t)$ is the generating function of the number of independent sets in the matroid $M(A)$. But from (2.3) we know that for any matroid M, the evaluation of $T(M;x,y)$ along the line $y=1$ also gives this generating function. Hence, combining these observations we have the result

Theorem 17: *If M is a regular matroid and A is any totally unimodular representation of M then the Ehrhart polynomial of the zonotope $Z(A)$ is given by*

$$i(Z(A);\lambda)=\lambda^r T\left(M;1+\frac{1}{\lambda},1\right),$$

where r is the rank of M.

Another new interpretation of T follows from what is sometimes known as the Ehrhart–Macdonald reciprocity law. This states that for any convex polytope P with integer vertices in $\mathbb{R}^n$ and for any positive integer t, the function $k(P;t)$ counting the number of lattice points lying strictly inside tP is given by

$$k(P;t)=(-1)^n i(P;-t).$$

This gives

Corollary 18: If A is an $r\times n$ totally unimodular matrix of rank r then for any positive integer λ the number of lattice points of $\mathbb{R}^r$ lying strictly inside the zonotope $\lambda Z(A)$ is given by

$$k(Z(A);\lambda)=(-\lambda)^r T\left(M(A);1-\frac{1}{\lambda},1\right).$$

In particular we have the following new interpretations:

The number of lattice points strictly inside $Z(A)$ is $(-1)^{r(M)}T(M;0,1)$.

B. Sandpiles

Self-organized criticality is a concept widely considered in various domains since Bak, Tang, and Wiesenfeld[41] introduced it ten years ago. One of the paradigms in this framework is the Abelian sandpile model, introduced by Dhar.[42]

We start by recalling the definition of the general Abelian sandpile model on a set of N sites labeled 1, 2, ..., N, that we referred to as the system. At each site the height of the sandpile is given by an integer h_i. The set $\vec{h}=\{h_i\}$ is called the *configuration* of the system. For every site i, a threshold H_i is defined; configurations with $h_i<H_i$ are called *stable*. For every stable configuration, the height h_i increases in time at a constant rate, this is called the *loading* of the system. This loading continues until at some site i, its height h_i exceeds the threshold H_i, then the site i topples and all the values h_j, $1\leqslant j\leqslant N$, are updated according to the rule:

$$h_j=h_j-\Delta_{ij} \quad \text{for all } j, \tag{22}$$

where Δ_{ij} is an integer matrix satisfying

$$\Delta_{ii}>0, \quad \Delta_{ij}\leqslant 0, \quad s_i=\sum_j \Delta_{ij}\geqslant 0.$$

If after this redistribution some height exceeds its threshold we apply the toppling rule (22) and so on, until we arrive at a stable configuration and the loading resumes. The sequence of topplings is called an *avalanche*. We assume that an avalanche is ''instantaneous,'' and thus, no loading occurs during an avalanche.

The value s_i is called the *dissipation* at site i. It may happen that an avalanche continues without end. We can avoid this possibility by requiring that from every *nondissipative* site i, i.e., $s_i=0$, there exists a path to a *dissipative* site j, i.e., $s_j>0$. In other words, there is a sequence $i_0,\ldots,i_n$, with $i_0=i$, $i_n=j$, and $\Delta_{i_{k-1},i_k}<0$, for $k=1,\ldots,n$. In this case we said that the system is *weakly dissipative*.[43] From now on, we assume that the system is always weakly dissipative.

When the matrix Δ_{ij} is symmetric and the loading of the system at site i equals the dissipation at i, the Abelian sandpile model coincides with the chip-firing game on a graph.[44] We now explain this.

Every site of the Abelian sandpile model corresponds to a vertex in a graph G containing $N+1$ vertices, that is the number of vertices is one more than the number of sites in the system. We label the vertices 0, 1, ..., N. The graph has multiple edges, and the number of edges between sites i and j, i and j both nonzero, equals $|\Delta_{ij}|$. For all $i \neq 0$, we connect site i to site 0 using $|\Sigma_{j=1}^{N} \Delta_{ij}|$ edges.

Every vertex i, $1 \leqslant i \leqslant N$, has a number of chips θ_i that represents its height h_i (when seen as a site of the system) at every moment of time and vertex 0 has a negative number of chips given by $\Sigma_{i=1}^{N}(-h_i)$. A toppling at site i corresponds to *firing* vertex i, that is, to redistribute some of the chips at vertex i according to the following rule: At vertex j, the new number of chips is $\theta_j - \Delta_{ij}$, for all j, that is, each neighbor k of i in G receives $|\Delta_{ik}|$ chips and vertex i loses Δ_{ii} chips. The loading of the system is represented by the firing of the vertex 0, in this case the height of site i (its number of chips in G) is increased by the number of edges from 0 to i. The vertex 0 may (must) fire only when no ordinary vertex can fire.

This process of firing vertices in the graph G is called a chip-firing game. The process is infinite, although the number of firings corresponding to an avalanche in the system, that is, a sequence of firings of the vertices $1, ..., N$ without firing the vertex 0, is finite. The number of stable configurations is also finite, hence certain configurations are *recurrent*, that is, a configuration is recurrent if there exists an avalanche which starts and ends with it. A configuration is *critical* if it is recurrent and stable.

Using the chip-firing game it can be proved that this sandpile model has an important Abelian property, namely the stable configuration of the system after an avalanche, and the number of breaks at any site during an avalanche, do not depend on the order of breaks during the avalanche.[44] Even more, there is a close relation between the critical configurations of the system and the Tutte polynomial of G. We now explain this more precisely.

The *level* of a configuration $\vec{h}$ is defined by

$$\text{level}(\vec{h}) = \sum_{i \neq 0} h_i + \deg(0) - |E(G)|.$$

The theorem conjectured by Biggs[45] and proved by Merino[46] is the following

Theorem 19: *If c_i denotes the number of critical configurations of level i in a graph G with special site 0, then*

$$P_q(G;y) = \sum_{i=0}^{\infty} c_i y^i = T(G;1,y).$$

A first, nontrivial consequence of this is that it shows $P_q(G;y)$ is independent of choice of the vertex 0 in G.

Critical configurations possess some interesting mathematical properties: they form a finite Abelian group whose order equals the number of spanning trees of the graph G. For the structure of this group for planar graphs, n-wheels, and complete graphs, and in this case its relation with parking functions see Ref. 47.

XII. RESISTOR NETWORKS

The problem of finding the effective resistance in a network of resistors was solved by Kirchhoff (1847) but Fortuin and Kasteleyn[5] showed that it also appears naturally as a limit of the Potts partition function as $Q \to 0$.

Suppose we let J_{ij}, the interaction energy between neighbor vertices i, j, be given by

$$J_{ij} \sim -kTr_{ij}^{-1},$$

where r_{ij} is the resistance of the resistor connecting sites i and j.

Now we wish to find the effective resistance R_{kl} between two fixed sites k and l, where k and l are joined in G. Then the result in Ref. 5 can be stated as in Ref. 2, namely

$$R_{kl} = \lim_{Q \to 0} \frac{\partial}{\partial x_{kl}} \ln Z_{\mathrm{RC}}(G;Q,Q^{\alpha} x_{ij}),$$

where $x_{ij} = r_{ij}^{-1}$ for each edge $(i,j) \in E(G)$ and α is arbitrary in the open interval (0, 1).

As explained in Ref. 2 this is essentially obtained from a fairly well known interpretation of the effective resistance in terms of a *spanning tree polynomial* which goes back to Kirchhoff. This spanning tree polynomial which is denoted by $S(G;x_{ij})$ is multivariate and defined by

$$S(G;x_{ij}) = \sum_{A \subseteq E} \left(\sum_{T \subseteq G:A} \left(\prod_{(i,j) \in T} x_{ij} \right) \right),$$

where the variables x_{ij} are indeterminates associated with each edge and the inner sum is over all spanning trees T of the subgraph $G:A = (V(G),A)$.

Then the claim [(Ref. 2), 4.26] is that for any α, $0 < \alpha < 1$,

$$\lim_{Q \to 0} Q^{\alpha(1-N)-1} Z_{\mathrm{RC}}(G;Q,Q^{\alpha} x_{ij}) = S(G;x_{ij}).$$

Taking $x_{ij} = r_{ij}^{-1}$ gives the result of Kirchhoff.

Now let us reappraise this in terms of the Tutte polynomial. First of all, we should emphasize that because it is a general result with variables r_{ij} and hence variable interaction strengths J_{ij}, its description cannot be exactly covered by the Tutte polynomial which is just two variable. However the basic ingredients are there. It is well known and easy to prove that in the case where all resistances are constant, say equal to 1, then the effective resistance R_e between two vertices of G which are joined by an edge e is given by

$$R_e = \frac{T(G''_e;1,1)}{T(G;1,1)}.$$

Putting $x_{ij} = r^{-1}$ in $S(G;x_{ij})$ gives

$$S\left(G;\frac{1}{r}\right) = r^{-n+1} \sum_{A \subseteq E} T(G|A;1,1) = \frac{2^{|E|-|V|+1}}{r^{|V|-1}} T(G;1,1).$$

Hence

$$\frac{T(G''_e;1,1)}{T(G;1,1)} = r \frac{S(G''_e;r^{-1})}{S(G;r^{-1})}.$$

XIII. THE SQUARE LATTICE

For obvious reasons the two-dimensional lattice is a graph of fundamental importance in the Potts model.

It is also the case that the square lattice is the fundamental separation point between the classes of graphs of bounded tree width and unbounded tree width, in the sense of Robertson and Seymour.[48]

Thus, in a very technical sense, it can be regarded as the separation point between hard and easy problems as all evaluations of the Tutte polynomial are known to be in polynomial time for graphs of bounded tree width (see Andrzejak[49] and Noble[50]). This was the motivation for the Merino–Welsh paper[51] which is the basis for the work of this section.

Here we review what is known about the Tutte polynomial on the square lattice.

The *square lattice* L_n is the set of ordered pairs $\{(i,j)\in\mathbb{N}^2 | 0\leqslant i,j\leqslant n-1\}$. There is an edge between the vertices (i,j) and (i',j') if $|i-i'|+|j-j'|=1$.

It is easy to show that as $n\to\infty$ for any fixed (x,y), $T(L_n;x,y)$ is $O(\theta^{n^2})$ for a suitable $\theta=\theta(x,y)$. We focus on the limit of the sequence

$$\{(T(L_n,i,j))^{1/n^2}\}$$

for certain values of the integers i and j.

We note that we already know from the results of Grimmett[52,53] and Biggs,[54] that except in certain special cases, these limits exist.

We first consider the trivial hyperbola H_1 where

$$T(L_n;x,y)=x^{n^2-1}y^{(n-1)^2}.$$

We next highlight the special hyperbola $(x-1)(y-1)=2$. On the positive branch of this hyperbola, which corresponds to the ferromagnetic version of the Ising model, convergence is to the classical limit of the Onsager solution, see, e.g., Ref. 55.

Consider now $\chi(L_n;k)$, the number of k-colorings of the square lattice L_n. Clearly the number of two-colorings of L_n is 2. Hence

$$\lim_{n\to\infty}(\chi(L_n;2))^{1/n^2}=1.$$

For $k>2$ the problem becomes much harder and exact results are not known. An easy argument gives

$$k-2\leqslant(\chi(L_n;k))^{1/n^2}\leqslant k-1.$$

For the rest of this section we assume $n>2$ to avoid trivialities. Let L_n^T be the graph obtained from the square lattice L_{n+1} by identifying the boundary vertices $(i,0)$ and (i,n), for $0\leqslant i\leqslant n$, and the vertices $(0,j)$ and (n,j), for $0\leqslant j\leqslant n$, and deleting any parallel edge. This is the toroidal square lattice. Let $\chi(L_n^T;k)$ be the number of k-colorings of L_n^T. It is known[56] that for a fixed integer $k\geqslant 3$ the limits of the sequences $\{(\chi(L_n^T;k))^{1/n^2}\}$ and $\{(\chi(L_n;k))^{1/n^2}\}$ are equal and we call this limit $\hat{\chi}(k)$.

In a classical paper, Lieb[57] showed that the number of ice configurations, see Sec. V, of L_n^T is asymptotically $(4/3)^{3/2}$. If we now assume L_n^T is self-dual, which is not strictly true because it is nonplanar, it is generally accepted (Ref. 56, p. 56) that the result of Lieb implies that

$$\hat{\chi}(3)=(4/3)^{3/2}\approx 1.539\,600\,718.$$

Biggs and Meredith in Ref. 58 obtained the estimate

$$\hat{\chi}(k)\sim\tfrac{1}{2}(k-3+\sqrt{k^2-2k+5}).$$

Lower and upper bounds for $\hat{\chi}(k)$ were given by Biggs in Ref. 56. He used the transfer matrix technique to obtain

$$\frac{k^2-3k+3}{k-1}\leqslant\hat{\chi}(k)\leqslant\frac{1}{2}(k-2+\sqrt{k^2-4k+8}).$$

In Ref. 59, Nagel used an induced subgraph expansion for the chromatic polynomial to obtain the first terms of a power series that converges to $\hat{\chi}(k)$.

Kim and Enting[60] gave a more accurate approximation of the same power series by combining an expansion of $\hat{\chi}(k)$ due to de Neef and the transfer matrix technique.

Numerical values obtained by using this approximation give for example $\hat{\chi}(10)\sim 8.111\ldots$.

We use $t(n)$ to denote the number of spanning trees of L_n. Let a_n be the number of one-factors or perfect matchings of L_{2n}. It is shown in Ref. 61 that

$$\lim_{n\to\infty}\frac{\ln a_n}{n^2}=\frac{4}{\pi^2}\int_0^{\pi/2}\int_0^{\pi/2}\ln(4\cos^2 x+4\cos^2 y)\,dx\,dy=c\approx 1.166\,243\,696.$$

Now, let b_n be the number of one-factors in the graph L_n', which is obtained from the $(2n-1)\times(2n-1)$ square lattice by taking out one corner vertex, that is, $L_n'=L_{2n-1}\backslash(0,0)$. In Ref. 62, a bijection has been established between the one factors of L_n' and the spanning trees of L_n. Since $a_n/b_n\sim 1$ as $n\to\infty$ we get

$$\lim_{n\to\infty}(t(n))^{1/n^2}=e^c\approx 3.209\,912\,556.$$

The number of spanning forests of L_n, which we denote by $f(n)$, seems a much more elusive quantity to approximate accurately. Now $f(n)$ corresponds to $T(L_n;2,1)$ and the related points $T(L_n;2,0)$ and $T(L_n;0,2)$ are the number of acyclic orientations of L_n, $\alpha(n)$, and the number of acyclic orientations with (0,0) as the only source, $\alpha_0(n)$, respectively. In Ref. 51 we show

$$\tfrac{7}{3}\leqslant\lim_{n\to\infty}(\alpha_0(n))^{1/n^2}\leqslant\lim_{n\to\infty}(t(n))^{1/n^2}\approx 3.209\,912\,556,$$

$$\frac{22}{7}\leqslant\lim_{n\to\infty}(\alpha(n))^{1/n^2}\leqslant 3.709\,259\,278\ldots,$$

and

$$3.209\,912\,556\leqslant\lim_{n\to\infty}(f(n))^{1/n^2}\leqslant 3.841\,619\,541\ldots.$$

More recently, Calkin *et al.*[63] have improved some of these upper bounds. By using the transfer matrix method, they obtain

$$\lim_{n\to\infty}(\alpha(n))^{1/n^2}\leqslant 3.563\,221\,504\,771\,6\ldots,$$

$$\lim_{n\to\infty}(f(n))^{1/n^2}\leqslant 3.746\,981\,401\,399\,4\ldots.$$

Also, Merino and Noy[64] have improved previous lower bounds by using generating function techniques, their results are

$$\lim_{n\to\infty}(\alpha(n))^{1/n^2}\geqslant\sqrt{\frac{13+\sqrt{61}}{2}}\approx 3.225\,697\,573\,851\,8\ldots,$$

$$\lim_{n\to\infty}(f(n))^{1/n^2}\geqslant 2+\sqrt{2}\approx 3.414\,213\,562\,373\,1\ldots.$$

By using the transfer matrix method together with the Perron–Frobenius Theorem, they have also obtained the following improvements:

$$\lim_{n\to\infty}(\alpha(n))^{1/n^2}\geqslant 3.413\,580\,975\,034\,92...,$$

$$\lim_{n\to\infty}(f(n))^{1/n^2}\geqslant 3.644\,975\,653\,386\,48...\,.$$

To sum up, the above-mentioned results give

$$\tfrac{7}{3}\leqslant\lim_{n\to\infty}(\alpha_0(n))^{1/n^2}\leqslant\lim_{n\to\infty}(t(n))^{1/n^2}\approx 3.209\,912\,556,$$

$$3.413\,580\,975\,034\,92...\leqslant\lim_{n\to\infty}(\alpha(n))^{1/n^2}\leqslant 3.563\,221\,504\,771\,6...,$$

and

$$3.644\,975\,653\,386\,48...\leqslant\lim_{n\to\infty}(f(n))^{1/n^2}\leqslant 3.746\,981\,401\,399\,4...\,.$$

Our objective is to find exact results for other evaluations $T(L_n;x,y)$ but this includes some very difficult problems.

XIV. CONCLUSION

We hope that the above gives a reasonably coherent picture of the intimate relationship between the Tutte polynomial and its physical interpretations associated with the Potts model.

One problem which has particularly engaged us is the question of whether there exists a good Monte Carlo scheme for the ferromagnetic Potts or random cluster model. A recent attack on this problem in Ref. 65 works as follows.

For any graph G, the *win polytope* W_G is the convex polytope defined by

$$\sum_{i\in U}x_i\leqslant e(U),\quad U\subseteq V,\quad x_i\geqslant 0,$$

where $e(U)$ is the number of edges incident with U.

It has the property that its bounding base face is combinatorially equivalent to $Z(A)$ where A is any totally unimodular representation of the graph G as in Sec. XI A. Now carry out simple random walk X_t in a slightly dilated version of W_G, call it W_G'. Associate with each lattice point a box of equal volume, ensuring that the boxes are disjoint but otherwise as large as possible. Now let t be large enough, say $t=T$ so that the stopping point X_T is almost uniform in W_G', and map X_T to the lattice point associated with the box containing it. Accept the output as an almost uniform point of W_G if it lies inside it. Repeat N times, where N is large enough to ensure we have a good estimate of the number of lattice points inside W_G. Ideally this process would work successfully enough to enable us also to get a good estimate of the number of lattice points in the bounding face and hence in $Z(A)$.

Curiously, and somewhat depressingly, in order for the method to work in polynomial time we need exactly the same density condition on the underlying graph as did Annan.[31] This suggests that it might be more profitable to look for a mathematical reason why good approximation schemes should not exist for $Z(G;p,Q)$ for general p and Q.

Accordingly, problems which seem to us particularly interesting are the following:

(a) Settle the Conjecture 10 that in the ferromagnetic region $x\geqslant 1$, $y\geqslant 1$ there is a good Monte Carlo approximation for $T(x,y)$.

(b) Decide the question of whether the number of acyclic orientations has a good approximation.
(c) Clarify, or at least explain more convincingly than the reasons given in Ref. 65 why it is so hard to approximate the number of forests.
(d) Understand better the region of the Tutte plane where the random cluster model is not positively correlated.

ACKNOWLEDGMENTS

D.J.A. Welsh is supported in part by Esprit Working Group No. 21726, ''RAND2.'' C.M. is supported by a grant from D.G.A.P.A.

[1] D. J. A. Welsh, *Complexity: Knots, Colourings and Counting*, in London Mathematical Society Lecture Note Series, Vol. 186 (Cambridge University Press, Cambridge, 1993).
[2] F. Wu, ''The Potts model,'' Rev. Mod. Phys. **54**, 235–268 (1982).
[3] J. G. Oxley, *Matroid Theory* (Oxford University Press, Oxford, 1992).
[4] J. G. Oxley and D. J. A. Welsh, ''The Tutte polynomial and percolation,'' *Graph Theory and Related Topics*, edited by J. A. Bondy and U.S.R. Murty (Academic, London, 1979), pp. 329–339.
[5] C. M. Fortuin and P. W. Kasteleyn, ''On the random cluster model. I. Introduction and relation to other models,'' Physica (Amsterdam) **57**, 536–564 (1972).
[6] G. R. Grimmett, *Precolation* (Springer, Berlin, 1989).
[7] G. R. Grimmett, *Precolation and Disordered Systems, Ecole dÉteé de probabilités de Saint Flour XXXVI–1996*, edited by P. Bernard [Lect. Notes Math. **1665**, 153–300 (1997)].
[8] R. G. Edwards and A. D. Sokal, ''Generalization of the Fortuin–Kasteleyn–Swendesen–Wang representation and Monte Carlo algorithms,'' Phys. Rev. D **38**, 2009–2012 (1988).
[9] R. H. Swendsen and J.-S. Wang, ''Nonuniversal critical dynamics in Monte Carlo simulations,'' Phys. Rev. Lett. **58**, 86–88 (1987).
[10] R. P. Stanley, ''Acyclic orientations of graphs,'' Discrete Math. **5**, 171–178 (1973).
[11] T. Zaslavsky, ''Facing up to arrangements: Face count formulas for partitions of space by hyperplanes,'' Memoirs of American Mathematical Society **154** (1975).
[12] R. C. Read and P. Rosenstiehl, ''On the principal edge tripartition of a graph,'' Ann. Discrete Math. **3**, 195–226 (1978).
[13] M. B. Thistlethwaite, ''A spanning tree expansion of the Jones polynomial,'' Topology **26**, 297–309 (1987).
[14] T. H. Brylawski and J. G. Oxley, *The Tutte Polynomial and its Applications Matroid Applications*, edited by N. White (Cambridge University Press, Cambridge, 1992), pp. 123–225.
[15] H. Kesten and R. H. Schonmann, ''Behavior in large dimensions of the Potts and Heisenberg models,'' Rev. Math. Phys. **1**, 147–182 (1989).
[16] B. Bollobás, G. Grimmett, and S. Janson, ''The random-cluster model on the complete graph,'' Probab. Theory Related Fields **104**, 283–317 (1996).
[17] A. Rényi, ''Some remarks on the theory of trees,'' Pub. Math. Inst. Hungarian Acad. Sci. **4**, 73–85 (1959).
[18] J. Dénes, ''The representation of a permutation as the product of a minimal number of transpositions, and its connection with the theory of graphs,'' Publ. Math. Inst. Hungarian Acad. Sci. **4**, 63–71 (1959).
[19] W. T. Tutte, ''On dichromatic polynomials,'' J. Comb. Theory, Sec. A **2**, 301–320 (1967).
[20] D. J. A. Welsh, ''Counting, colourings and flows in random graphs,'' Bolyai Society Mathematical Studies **2**, 491–506 (1996).
[21] I. M. Gessel and B. E. Sagan, ''The Tutte polynomial of a graph, depth-first search, and simplicial complex partitions,'' Electron. J. Combin. **3**, 1–36 (1996).
[22] L. Takacs, ''On the number of distinct forests,'' SIAM J. Discrete Math. **3**, 574–581 (1990).
[23] R. P. Stanley, *A Zonotope Associated with Graphical Degree Sequences*, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Vol. 4, 555-570 (1991).
[24] F. Jaeger, D. L. Vertigan, and D. J. A. Welsh, ''On the computational complexity of the Jones and Tutte polynomials,'' Math. Proc. Cambridge Philos. Soc. **108**, 35–53 (1990).
[25] P. W. Kasteleyn, ''The statistics of dimers on a lattice,'' Physica (Amsterdam) **27**, 1209–1225 (1961).
[26] D. L. Vertigan and D. J. A. Welsh, ''The computational complexity of the Tutte plane: The bipartite case,'' Comb. Probab. Comput. **1**, 181–187 (1992).
[27] M. R. Jerrum, L. G. Valiant, and V. V. Vazirani, ''Random generation of combinatorial structures from a uniform distribution,'' Theor. Comput. Sci. **43**, 169–188 (1986).
[28] M. R. Jerrum and A. Sinclair, ''Polynomial-time approximation algorithms for the Ising model,'' SIAM J. Comput. **22**, 1087–1116 (1993).
[29] D. J. A. Welsh, ''Randomised approximation in the Tutte plane,'' Comb. Probab. Comput. **3**, 137–143 (1993).
[30] M. Mihail and P. Winkler, ''On the number of Eulerian orientations of a graph,'' Algorithmica **16**, 402–414 (1996).
[31] J. D. Annan, ''A randomized approximation algorithm for counting the number of forests in dense graphs,'' Comb. Probab. Comput. **3**, 273–283 (1994).
[32] N. Alon, A. M. Frieze, and D. J. A. Welsh, ''Polynomial time randomized approximation schemes for Tutte–Grothendieck invariants: The dense case,'' Random Struct. Algorithms **6**, 459–478 (1995).
[33] D. R. Karger, ''A randomized fully polynomial time approximation scheme for the all terminal network reliability problem,'' in Proceedings of the 36th Annual IEEE Symposium on Foundations of Computer Science, 1995 (unpublished), pp. 328–337.

[34] C. Hoede, ''The Ising model and linear codes,'' Congr. Numer. **33**, 81–108 (1981).
[35] A. Rosengren and B. Lindström, ''A combinatorial series expansion for the Ising model,'' Eur. J. Combin. **8**, 317–323 (1987).
[36] C. Greene, ''Weight enumeration and the geometry of linear codes,'' Stud. Appl. Math. **55**, 119–128 (1976).
[37] D. J. A. Welsh, *Matroid Theory* (Academic, New York, 1976).
[38] E. Ehrhart, ''Sur un problème de géometrie diophantienne linéaire I, II,'' J. Reine Angew. Math. **226**, 1–29 (1967); **227**, 25–49 (1967); correction **231**, 220 (1968).
[39] R. Diaz and S. Robins, ''The Ehrhart, polynomial of a lattice *n*-simplex,'' Electron. Res. Announc. Am. Math. Soc. **2**, 1–6 (1996).
[40] R. P. Stanley, ''Decompositions of rational convex polytopes,'' Ann. Discrete Math. **6**, 333–342 (1980).
[41] P. Bak, C. Tang, and K. Wiesenfeld, ''Self-organized criticality,'' Phys. Rev. A **38**, 364–374 (1988).
[42] D. Dhar, ''Self-organized critical state of sandpile automaton models,'' Phys. Rev. Lett. **64**, 1613–1616 (1990).
[43] A. Gabrielov, ''Abelian avalanches and the Tutte polynomials,'' Physica A **195**, 253–274 (1993).
[44] A. Björner, L. Lovász, and P. Shor, ''Chip-firing games on graphs,'' Eu. J. Combin. **12**, 283–291 (1991).
[45] N. Biggs, ''The Tutte polynomial as a growth function'' (unpublished).
[46] C. Merino, ''Chip-firing and the Tutte polynomial,'' Ann. Comb. **1**, 253–259 (1997).
[47] R. Cori and D. Rossin, ''On the sandpile group of a graph'' (unpublished).
[48] N. Robertson and P. D. Seymour, ''Graph Minors. II. Algorithmic aspects of tree-width,'' J. Algorithm **7**, 309–322 (1986).
[49] A. Andrzejak, ''An algorithm for the Tutte polynomials of graphs of bounded tree width,'' Discrete Math. **190**, 39–54 (1998).
[50] S. Noble, ''Evaluating the Tutte polynomial for graphs of bounded tree-width,'' Comb. Probab. Comput. **7**, 307–322 (1998).
[51] C. Merino and D. J. A. Welsh, ''Forests, colorings and acyclic orientations of the square lattice,'' Ann. Comb. **3**, 417–429 (1999).
[52] G. R. Grimmett, ''Multidimensional lattices and their partition function,'' Q. J. Math. **29**, 141–157 (1978).
[53] G. R. Grimmett, ''The rank polynomials of large random lattices,'' J. London Math. Soc. **18**, 567–575 (1978).
[54] N. L. Bíggs, ''Chromatic and thermodynamic limits,'' J. Phys. A **8**, L110–112 (1975).
[55] C. J. Thompson, *Mathematical Statistical Mechanics* (Princeton University Press, Princeton, NJ, 1979), pp. 131–135.
[56] N. L. Biggs, ''Colouring square lattice graphs,'' Bull. London Math. Soc. **9**, 54–56 (1977).
[57] E. Lieb, ''The residual entropy of square ice,'' Phys. Rev. **162**, 162–172 (1967).
[58] N. L. Biggs and G. H. J. Meredith, ''Approximations for chromatic polynomials,'' J. Comb. Theory, Ser. B **20**, 5–19 (1976).
[59] J. F. Nagle, ''A new sugraph expansion for obtaining coloring polynomials for graphs,'' J. Comb. Theory, Ser. B **10**, 42–59 (1971).
[60] D. Kim and I. G. Enting, ''The limit of chromatic polynomials,'' J. Comb. Theory, Ser. B **26**, 327–336 (1979).
[61] L. Lovász, *Combinatorial Problems and Exercises* (North-Holland, Amsterdam, 1979), pp. 238–243.
[62] H. N. V. Temperley, *Combinatorics*, in London Mathematical Society, Lecture Note Series Vol. 13 (Cambridge University Press, Cambridge 1974), pp. 202–204.
[63] N. J. Calkin, C. Merino, S. Noble, and D. J. A. Welsh (private communication).
[64] C. Merino and M. Noy (private communication).
[65] E. Bartels, J. Mount, and D. J. A. Welsh, ''The win polytope of a graph,'' Ann. Comb. **1**, 1–15 (1997).