ON POSITIVE MULTIPEAK SOLUTIONS OF A NONLINEAR ELLIPTIC PROBLEM

EZZAT S. NOUSSAIR AND SHUSEN YAN

1. *Introduction*

In this paper we continue our investigation in [**5**, **7**, **8**] on multipeak solutions to the problem

$$-\varepsilon^2 \Delta u + u = Q(x)\,|u|^{q-2}u, \quad x \in \mathbb{R}^N,$$
$$u \in H^1(\mathbb{R}^N), \tag{1.1}$$

where $\Delta = \sum_{i=1}^{N} \partial^2/\partial x_i^2$ is the Laplace operator in $\mathbb{R}^N$, $2 < q < \infty$ for $N = 1, 2$, $2 < q < 2N/(N-2)$ for $N \geqslant 3$, and $Q(x)$ is a bounded positive continuous function on $\mathbb{R}^N$ satisfying the following conditions.

(Q_1) Q has a strict local minimum at some point $x_0 \in \mathbb{R}^N$, that is, for some $\delta > 0$

$$Q(x) > Q(x_0)$$

for all $0 < |x - x_0| < \delta$.

(Q_2) There are constants $C, \theta > 0$ such that

$$|Q(x) - Q(y)| \leqslant C|x-y|^{\theta}$$

for all $|x - x_0| \leqslant \delta$, $|y - y_0| \leqslant \delta$.

Our aim here is to show that corresponding to each strict local minimum point x_0 of $Q(x)$ in $\mathbb{R}^N$, and for each positive integer k, (1.1) has a positive solution with k-peaks concentrating near x_0, provided ε is sufficiently small, that is, a solution with k-maximum points converging to x_0, while vanishing as $\varepsilon \to 0$ everywhere else in $\mathbb{R}^N$.

Problem (1.1) arises in various applications, such as chemotaxis, population genetics, chemical reactor theory, and the study of standing wave solutions of certain nonlinear Schrödinger equations.

The study of single and multi-peak solutions to (1.1) and related problems has attracted considerable attention in recent years, and there are several results in the literature on the existence of such solutions.

However, to the best of the authors' knowledge, all previous results on this problem are restricted to solutions with at most one positive peak near x_0, where x_0 is assumed to be a nondegenerate critical point of Q, a strict local maximum point, or some other classes of 'topologically nontrivial critical points' (see [**11**]). We mention the recent results by the authors and D. Cao [**5**, **8**] for the case of a nondegenerate critical point, and [**7**] for the degenerate case. For the related Schrödinger equation

$$-\varepsilon^2 \Delta u + V(x)\,u = |u|^{q-2}u, \quad x \in \mathbb{R}^N,$$
$$u \in H^1(\mathbb{R}^N),$$

Received 19 August 1997; revised 24 June 1999.

2000 *Mathematics Subject Classification* 35J60.

Both authors were supported by the Australian Research Council.

J. London Math. Soc. (2) 62 (2000) 213–227

there are various results by several authors. We mention the works of Floer and Weinstein [**13**], Oh [**22**, **23**], X. Wang [**26**], Z. Q. Wang [**27**], P. Rabinowitz [**24**], M. del Pino and P. Felmer [**11**, **12**], C. Gui [**15**], and Y. Y. Li [**17**], and the references therein.

When $Q(x)$ is a positive constant and $\mathbb{R}^N$ is replaced by an arbitrary domain Ω, problem (1.1) has been considered by several authors. In these studies both the topology of Ω (see, for example, Benci and Cerami [**3**, **4**]) and the geometry of Ω (see [**6**, **9**, **20**, **21**]) play an important role in the existence and multiplicity of positive solutions of (1.1), and on the location of their peaks (see [**6**, **25**]).

To state our results we introduce some notations first. Let w denote the ground state solution of the problem

$$\begin{aligned} -\Delta w + w &= w^{q-1}, \quad x \in \mathbb{R}^N, \\ w &\in H^1(\mathbb{R}^N), \\ w(0) &= \max_{x\in\mathbb{R}^N} w(x). \end{aligned} \tag{1.2}$$

It is well known [**16**] that w is unique and satisfies

$$\begin{cases} w(x) = w(|x|), & \forall\ x \in \mathbb{R}^N, \\ w'(r) < 0, & \forall\ r > 0, \quad w''(0) < 0, \\ \lim_{r\to\infty} r^{(N-1)/2} e^r w(r) = \lambda_0 > 0, \\ \lim_{r\to\infty} \dfrac{w'(r)}{w(r)} = -1. \end{cases}$$

Let

$$v_{\varepsilon,y}(\cdot) = v((\cdot - y)/\varepsilon), \quad y \in \mathbb{R}^N,$$

$$\langle v, v\rangle_\varepsilon = \varepsilon^2 \int \nabla u \cdot \nabla v + \int uv,$$

$$\|u\|_\varepsilon^2 = \langle u, u\rangle_\varepsilon,$$

and we write $\|\ \|_1 = \|\ \|$ for all $u, v \in H^1(\mathbb{R}^N)$, where all our integrals are Lebesgue integrals over $\mathbb{R}^N$, unless otherwise stated. The main results of this paper may be stated as follows.

THEOREM A. *Assume that conditions* (Q_1) *and* (Q_2) *hold. Then, for each* $k = 1, 2, \ldots$, *there exists* $\varepsilon_0 = \varepsilon(k)$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *problem* (1.1) *has a solution* u_ε *of the form*

$$u_\varepsilon = \sum_{j=1}^k \alpha_\varepsilon^j w_{x_\varepsilon^j, \varepsilon} + v_\varepsilon$$

for some positive constants α_ε^j, $j = 1, \ldots, k$, *points* $x_\varepsilon^j \in \mathbb{R}^N$, $j = 1, 2, \ldots, k$, *and* $v_\varepsilon \in H^1(\mathbb{R}^N)$, *satisfying*

$$x_\varepsilon^j \to x_0,$$

$$\frac{|x_\varepsilon^i - x_\varepsilon^j|}{\varepsilon} \to +\infty, \quad i \neq j,$$

$$\alpha_\varepsilon^j \to Q(x_0)^{-1/(q-2)},$$

$$\|v_\varepsilon\|_\varepsilon = o(\varepsilon^{N/2}),$$

as $\varepsilon \to 0$.

Our procedure is based on local reduction methods like those by O. Rey [**24**], A. Bahri [**1**], and Y. Li [**17**]. We should mention that the reduction procedure has been modified here to allow for the degeneracy of the critical point of Q.

We mention that there is a similarity between problem (1.1) and the singularly perturbed elliptic problem with Neumann boundary condition (see [**10**]), where the mean curvature function on $\partial\Omega$ plays a similar role to that of $Q(x)$ here.

We also mention the following example which shows that when x_0 is a local maximum point of Q, one cannot expect, in general, to have a positive solution with more than one peak concentrating at x_0.

EXAMPLE 1.1. Let u be any positive solution of

$$-\varepsilon^2 \Delta u + u = Q(|x|)\, u^{q-1}, \quad \text{in } \mathbb{R}^N,$$
$$u \in H^1(\mathbb{R}^N),$$

where Q is a non-increasing continuous positive function, which is radically symmetric. It is well known [**14**] that u is radially symmetric and non-increasing, and hence has a single peak in $\mathbb{R}^N$.

This paper is organized as follows. In Section 2, we introduce some notations and establish two basic results; one is a decomposition lemma for functions in $H^1(\mathbb{R}^N)$, and the other is a spectrum result. In Section 3 we prove Theorem A.

REMARK 1.2. Recently we learnt from the referee that similar results were obtained by G. Lu and J. Wei [**18**] for similar equations, although not including the one considered here.

2. *Notations and preliminary results*

Without loss of generality, we assume that $x_0 = 0$ and $Q(0) = 1$. Set

$$I_\varepsilon(u) = \tfrac{1}{2} \|u\|_\varepsilon^2 - \frac{1}{q} \int Q(x)\, |u|^q \, dx, \quad u \in H^1(\mathbb{R}^N). \tag{2.1}$$

Let $B_r = \{x \in \mathbb{R}^N, |x| < r\}$, and $\bar{B}_r$ be its closure.

For $\delta, R > 0$ and any positive integer $k = 1, 2, \ldots,$

$$D^k_{\varepsilon, R, \delta} = \{(x^1, \ldots, x^k) \in \mathbb{R}^{kN},\ x^j \in \bar{B}_\delta,\ j = 1, \ldots, k,\ |x^i - x^j|/\varepsilon \geqslant R \text{ for } i \neq j\}$$

$$E_{\varepsilon, k} = \left\{ v \in H^1(\mathbb{R}^N) : \langle v, w_{x^j, \varepsilon} \rangle_\varepsilon = \left\langle v, \frac{\partial w_{x^j, \varepsilon}}{\partial x^j_i} \right\rangle_\varepsilon = 0,\ j = 1, \ldots, k,\ i = 1, \ldots, N \right\}$$

$$\begin{aligned} M_{\varepsilon, R, \delta} = \{ & (\alpha^1, \ldots, \alpha^k, x^1, \ldots, x^k, v) : (x^1, \ldots, x^k) \in D^k_{\varepsilon, R, \delta}, |\alpha^i - 1| \leqslant \delta, \\ & i = 1, \ldots, k,\ v \in E_{\varepsilon, k}, \|v\|_\varepsilon^2 \leqslant \delta \varepsilon^N \}. \end{aligned} \tag{2.2}$$

Set

$$\begin{aligned} J_\varepsilon(\alpha^1, \ldots, \alpha^k; x^1, \ldots, x^k, v) &= I_\varepsilon\left(\sum_{j=1}^k \alpha^j w_{x^j, \varepsilon} + v \right) \\ (\alpha_1, \ldots, \alpha_k, x^1, \ldots, x^k, v) &\in M_{\varepsilon, R, \delta}. \end{aligned} \tag{2.3}$$

LEMMA 2.1. *There exist* $R_0 > 0$, $\delta > 0$ *and* $\varepsilon_0 > 0$ *such that for* $\varepsilon \in (0, \varepsilon_0]$, $R \geqslant R_0$, $\delta \in (0, \delta_0]$, *we have the equivalence* $(\alpha^1, \dots, \alpha^k, x^1, \dots, x^k, v)$ *is a critical point of the functional*

$$J_\varepsilon : M_{\varepsilon, R, \delta} \longrightarrow R$$

$$(\alpha^1, \dots, \alpha^k, x^1, \dots, x^k, v) \longrightarrow I_\varepsilon\left(\sum_{j=1}^k \alpha^j w_{x^j, \varepsilon} + v\right)$$

if and only if

$$u = \sum_{j=1}^k \alpha^j w_{x^j, \varepsilon} + v \tag{2.4}$$

is a critical point of I_ε *in* $H^1(\mathbb{R}^N)$.

LEMMA 2.2. *There exist* $\rho > 0$, $R_0 > 0$, $\delta_0 > 0$ *and* $\varepsilon_0 > 0$ *such that for* $\varepsilon \in (0, \varepsilon_0]$, $\delta \in (0, \delta_0]$ *and* $R \geqslant R_0$, *we have*

$$\|v\|_\varepsilon^2 - (q-1) \int \left(\sum_{j=1}^k w_{x^j, \varepsilon}\right)^{q-2} v^2 \geqslant \rho \|v\|_\varepsilon^2 \tag{2.5}$$

for all $(x^1, \dots, x^k) \in D^k_{\varepsilon, R, \delta}$ *and* $v \in E_{\varepsilon, k}$.

The proofs of Lemmas 2.1 and 2.2 are given in Appendix A.

We notice that $(\alpha^1, \dots, \alpha^k, x^1, \dots, x^k, v) \in M_{\varepsilon, R, \delta}$ is a critical point of J_ε if and only if there are scalars A_ℓ, $B_{\ell, h}$, $\ell = 1, \dots, k$, $h = 1, \dots, N$, such that

$$\frac{\partial J_\varepsilon}{\partial x_i^j} = \sum_{h=1}^N B_{j,h} \left\langle \frac{\partial^2 w_{x^j, \varepsilon}}{\partial x_i^j \, \partial x_h^j}, v \right\rangle_\varepsilon \tag{2.6}$$

$$\frac{\partial J_\varepsilon}{\partial \alpha^i} = 0 \tag{2.7}$$

$$\left(\frac{\partial J_\varepsilon}{\partial v}, \varphi\right) = \sum_{\ell=1}^k A_\ell (w_{x^\ell, \varepsilon}, \varphi) + \sum_{\ell=1}^k \sum_{h=1}^N B_{\ell, h} \left(\frac{\partial w_{x^\ell, \varepsilon}}{\partial x_h^\ell}, \varphi\right) \tag{2.8}$$

for all $\varphi \in H^1(\mathbb{R}^N)$.

In order to prove Theorem A, we show first that for $(x^1, x^2, \dots, x^k)$ given, ε small enough, there exist α^i, $i = 1, \dots, k$, $v_\varepsilon \in E_{\varepsilon, k}$, and scalars A_ℓ, $B_{\ell, h}$, $\ell = 1, \dots, k$, $h = 1, \dots, N$, such that (2.7) and (2.8) are satisfied, and the mappings $(x^1, \dots, x^k) \longrightarrow \alpha^i(x^1, \dots, x^k)$, $(x^1, \dots, x^k) \longrightarrow v(x^1, \dots, x^k) \in E_{\varepsilon, k}$ are $C^1(D^k_{\varepsilon, R, \delta})$. We then show that for sufficiently small ε, there exists a point $(x^1, \dots, x^k) \in D^k_{\varepsilon, R, \delta}$ such that $(\alpha^1, \dots, \alpha^k, x^1, \dots, x^k, v) \in M_{\varepsilon, R, \delta}$ and satisfies (2.6).

3. *Existence of a multi-peak solution*

In this section we fix $k = 1, 2, \dots,$ and write

$$\alpha = (\alpha^1, \dots, \alpha^k) \in \mathbb{R}^k$$
$$x = (x^1, \dots, x^k) \in \mathbb{R}^{kN}.$$

We define inner products in $\mathbb{R}^k$ and $\mathbb{R}^k \times E_{\varepsilon, k}$, respectively, by

$$(\alpha, \gamma)_\varepsilon = \varepsilon^2 \sum_{j=1}^k \alpha^j \gamma^j, \quad \alpha, \gamma \in \mathbb{R}^k, \tag{3.1}$$

$$((\alpha, v), (\gamma, u))_\varepsilon = (\alpha, \gamma)_\varepsilon + (v, u)_\varepsilon, \tag{3.2}$$

for $(\alpha, v), (\gamma, u) \in \mathbb{R}^k \times E_{\varepsilon, k}$. The corresponding norms are denoted by

$$\|\alpha\|_\varepsilon, \quad \|(\alpha, v)\|_\varepsilon.$$

Let

$$H_{\varepsilon,x} = \sum_{j=1}^{k} w_{x^j,\varepsilon}. \tag{3.3}$$

PROPOSITION 3.1. *Assume that* $x_0 = 0$ *and conditions* (Q_1) *and* (Q_2) *hold. Then for any given integer* $k = 1, 2, \ldots$, *there exist constants* $\varepsilon_0, R_0, \delta_0 > 0$ *such that for* $\varepsilon \in (0, \varepsilon_0]$, $R > R_0$, $\delta \in (0, \delta_0]$, *there is a* C^1*-mapping*

$$(\alpha_\varepsilon, v_\varepsilon) : D^k_{\varepsilon, R, \delta} \longrightarrow R^k \times E_{\varepsilon, k}$$

such that (2.7) *and* (2.8) *hold. Moreover,*

$$\varepsilon^{N/2} \sum_{j=1}^{k} |\alpha^i_\varepsilon - 1| + \|v_\varepsilon\|_\varepsilon = \varepsilon^{N/2} O(\varepsilon^\theta + \sum_{i \neq j} e^{-\min((q-1)/2,\, 1)\, |x^i - x^j|/\varepsilon} + \sum_{j=1}^{k} |Q(x^j) - 1|). \tag{3.4}$$

Proof. We follow an argument due to A. Bahri [**1**]. We expand J_ε in a neighbourhood of $(\alpha, v) = (1, 0)$. Let $\beta^i = \alpha^i - 1$, $\beta = (\beta^1, \ldots, \beta^k)$ and $u = (\beta, v) \in \mathbb{R}^k \times E_{\varepsilon,k}$.

Define

$$J^*_\varepsilon(x, u) = J_\varepsilon(\alpha, x, v).$$

Then

$$J^*_\varepsilon(x, u) = J^*(x, 0) + f_{\varepsilon,x}(u) + Q_{\varepsilon,x}(u) + R_{\varepsilon,x}(u)$$

where $u = (\beta, v)$,

$$f_{\varepsilon,x}(u) = -\int Q(y)\, H^{q-1}_{\varepsilon,x}(y)\, v(y)\, dy + \sum_{j=1}^{k} \left[(H_{\varepsilon,x}, w_{x^j,\varepsilon})_\varepsilon - \int Q(y)\, H^{q-1}_{\varepsilon,x}\, w_{x^j,\varepsilon} \right] \beta_j \tag{3.5}$$

$$Q_{\varepsilon,x}(u) = Q^{(1)}_{\varepsilon,x}(\beta) + Q^{(2)}_{\varepsilon,x}(v) + Q^{(3)}_{\varepsilon,x}(\beta, v) \tag{3.6}$$

$$Q^{(1)}_{\varepsilon,x}(\beta) = \left[\sum_{i,j=1}^{k} (w_{x^i,\varepsilon}, w_{x^j,\varepsilon})_\varepsilon - (q-1) \sum \int Q(y)\, (H_{\varepsilon,x})^{q-2}\, w_{x^i,\varepsilon}\, w_{x^j,\varepsilon} \right] \beta^i \beta^j \tag{3.7}$$

$$Q^{(2)}_{\varepsilon,x}(v) = \|v\|^2_\varepsilon - (q-1) \int Q(y)\, (H_{\varepsilon,x})^{q-2}\, v^2 \tag{3.8}$$

$$Q^{(3)}_{\varepsilon,x}(\beta, v) = -\sum_{j=1}^{k} \int Q(y)\, (H_{\varepsilon,x})^{q-2}\, w_{x^j,\varepsilon}\, v \beta^j \tag{3.9}$$

and $R_{\varepsilon,x}$ denotes all the higher order terms, and it satisfies

$$\begin{aligned} R_{\varepsilon,x}(u) &= O(\|u\|^{\mathrm{Min}(3,q)}_\varepsilon) \\ R'_{\varepsilon,x}(u) &= O(\|u\|^{\mathrm{Min}(2,q-1)}_\varepsilon) \\ R''_{\varepsilon,x}(u) &= O(\|u\|^{\mathrm{Min}(1,q-2)}_\varepsilon). \end{aligned} \tag{3.10}$$

Now $f_{\varepsilon,x}$ is a continuous linear form over $\mathbb{R}^k \times E_{\varepsilon,k}$ equipped with the scalar product defined in (3.2). Therefore there exists a unique $f_{\varepsilon,x} \in \mathbb{R}^k \times E_{\varepsilon,k}$ such that $f_{\varepsilon,x}(u) = (f_{\varepsilon,x}, u)_\varepsilon$.

In the same way $Q_{\varepsilon,x}$ is a continuous quadratic form over $\mathbb{R}^k \times E_{\varepsilon,k}$, and therefore there exists a continuous linear operator $A_{\varepsilon,x}$ from $\mathbb{R}^k \times E_{\varepsilon,x}$ onto itself such that $Q_{\varepsilon,x}(u) = (A_{\varepsilon,x} u, u)_\varepsilon$,

$$(A_{\varepsilon,x} u, u)_\varepsilon = Q^{(1)}_{\varepsilon,x}(\beta) + Q^{(2)}_{\varepsilon,x}(v) + Q^{(3)}_{\varepsilon,x}(\beta, v). \tag{3.11}$$

We show next that for R sufficiently large, and δ and ε sufficiently small, the operator $A_{\varepsilon,x}$ has a bounded inverse.

From Lemma 2.2 there exists $\rho, R_0, \delta_0, \varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$, $\delta \in (0, \delta_0]$ and $R \geqslant R_0$, we have

$$Q^{(2)}_{\varepsilon,x}(v) \geqslant \rho \|v\|^2_\varepsilon \tag{3.12}$$

for all $x \in D^k_{\varepsilon,R,\delta}$.

On the other hand, if $R_0 > 0$ is sufficiently large and $\delta_0 > 0$ is sufficiently small, we have

$$\begin{aligned} Q^{(1)}_{\varepsilon,x}(\beta) &= \sum_{i=1}^k \left(\|w_{x^i,\varepsilon}\|^2_\varepsilon - (q-1)\int Q(y)\, w^q_{x^i,\varepsilon}\right)|\beta^i|^2 + O(\varepsilon^N e^{-R})\,|\beta|^2 \\ &= \varepsilon^N\{(2-q)\,\|w\|^2 + o_\delta(1) + O(e^{-R})\}\,|\beta|^2 \\ &\leqslant -C_N\,\varepsilon^N|\beta|^2 = -C_N\|\beta\|^2_\varepsilon \end{aligned} \tag{3.13}$$

for some constant $C_N > 0$, which depends only on N, where $o_\delta(1) \to 0$ as $\delta \to 0$, and

$$\begin{aligned} |Q^{(3)}_{\varepsilon,x}(\beta, v)| &\leqslant \sum_{i=1}^k \left|\int Q(y)\, H^{q-2}_{\varepsilon,x}\, w_{x^i,\varepsilon}\, v\right| |\beta^i| \\ &= \sum_{i=1}^k \left(\int Q(y)\, w^{q-1}_{\varepsilon,x^i}\, v + O(\varepsilon^{N/2} e^{-R})\,\|v\|_\varepsilon\right)|\beta|^i \\ &= \sum_{i=1}^k \left(\int w^{q-1}_{\varepsilon,x^i}\, v + o_\delta(1)\,\varepsilon^{N/2}\|v\|_\varepsilon + O(\varepsilon^{N/2}e^{-R})\,\|v\|_\varepsilon\right)|\beta^i| \\ &= (O(e^{-R}) + o(\delta))\,\|\beta\|_\varepsilon\,\|v\|_\varepsilon. \end{aligned} \tag{3.14}$$

From the above estimates, there is, for $\varepsilon \in (0, \varepsilon_0)$, $R > R_0$, $\delta \in (0, \delta_0)$, a unique linear operator from $\mathbb{R}^k \times E_{\varepsilon,k}$ to itself such that

$$(B_{\varepsilon,x}\, u, u)_\varepsilon = Q^{(1)}_{\varepsilon,x}(\beta) + Q^{(2)}_{\varepsilon,x}(v).$$

Furthermore, from (3.12) and (3.13), $B_{\varepsilon,x}$ is invertible and

$$\|B^{-1}_{\varepsilon,x}\| \leqslant C$$

for some constant $C > 0$, independently of ε and x.

From (3.14) we have

$$\|A_{\varepsilon,x} - B_{\varepsilon,x}\| = \|Q^{(3)}_{\varepsilon,x}\| \leqslant O(e^{-R}) + o_\delta(1).$$

Hence we can choose $R_0, \delta_0, \varepsilon_0$ such that for $\varepsilon \in (0, \varepsilon_0]$, $R \geqslant R_0$, $\delta \in (0, \delta_0)$, $A_{\varepsilon,x}$ is invertible and

$$\|A^{-1}_{\varepsilon,x}\| \leqslant C$$

for some constant $C > 0$.

We now follow the argument in Rey [**24**].

Since

$$\left.\frac{\partial J^*_\varepsilon}{\partial u}\right|_{\mathbb{R}^k \times E_{\varepsilon,k}}(x, u) = f_{\varepsilon,x} + 2A_{\varepsilon,x}\, u + R'_{\varepsilon,x}(u)$$

there is an equivalence between the existence of $u = (\alpha, v)$ such that (2.7) and (2.8) are satisfied and

$$f_{\varepsilon,x} + 2A_{\varepsilon,x}\, u + R'_{\varepsilon,x}(u) = 0. \tag{3.15}$$

As in [**24**] we employ the implicit function theorem to conclude that for some $\varepsilon_0, R_0, \delta_0$, we have a C^1-mapping

$$u_\varepsilon = (\alpha_\varepsilon, v_\varepsilon) : D^k_{\varepsilon, R, \delta} \longrightarrow \mathbb{R}^k \times E_{\varepsilon, k}$$

for $\varepsilon \in (0, \varepsilon_0]$, $R \geqslant R_0$, $\delta \in (0, \delta_0]$, satisfying (3.15), and

$$\|u_\varepsilon\|_\varepsilon \leqslant C \|f_{\varepsilon, x}\|. \tag{3.16}$$

We estimate next $\|f_{\varepsilon, x}\|$.

We claim that

$$\int Q(y)\, H^{q-1}_{\varepsilon, x}\, v = \sum_{i=1}^{k} \int Q(y)\, w^{q-1}_{x^i, \varepsilon} v + O(\varepsilon^{N/2} \sum_{i \neq j} e^{-\operatorname{Min}((q-1)/2,\, 1) |x^i - x^j|/\varepsilon})\, \|v\|_\varepsilon. \tag{3.17}$$

In fact, if $2 < q < 3$, from the inequality

$$\begin{aligned} ||a+b|^{q-1} - |a|^{q-1} - |b|^{q-1}| &\leqslant \begin{cases} C|a|\,|b|^{q-2} & \text{if } |a| \leqslant |b| \\ C|b|\,|a|^{q-2} & \text{if } |b| \leqslant |a| \end{cases} \\ &\leqslant C|a|^{(q-1)/2}\, |b|^{(q-1)/2}, \end{aligned}$$

for some constant $C > 0$, we obtain

$$\begin{aligned} \left| \int Q(y) \left(H^{q-1}_{\varepsilon, x} - \sum_{i=1}^{k} w^{q-1}_{x^i, \varepsilon} \right) v \right| &\leqslant C \int \sum_{i \neq j} w^{(q-1)/2}_{x^i, \varepsilon}\, w^{(q-1)/2}_{x^j, \varepsilon} |v| \\ &\leqslant C \varepsilon^{N/2} \|v\|_\varepsilon \left(\int \left(\sum_{i \neq j} w^{(q-1)/2} w^{(q-1)/2}_{1, |x^i - x^j|/\varepsilon} \right)^{q/(q-1)} \right)^{1 - 1/q} \\ &= O(\varepsilon^{N/2} \sum_{i \neq j} e^{-(q-1)|x^i - x^j|/2\varepsilon})\, \|v\|_\varepsilon; \end{aligned} \tag{3.18}$$

if $q > 3$, then

$$\left| \int Q(y) \left(H^{q-1}_{\varepsilon, x} - \sum_{i=1}^{k} w^{q-1}_{x^i, \varepsilon} \right) v \right| \leqslant C \sum_{i \neq j} \int w^{q-2}_{x^i, \varepsilon}\, w_{x^j, \varepsilon} |v| = O(\varepsilon^{N/2} \sum_{i \neq j} e^{-|x^i - x^j|/\varepsilon})\, \|v\|_\varepsilon. \tag{3.19}$$

Combining (3.18) and (3.19), we obtain (3.17).
However,

$$\begin{aligned} \int Q(y)\, w^{q-1}_{x^i, \varepsilon} v &= \int_{B_\delta(x^i)} Q(y)\, w^{q-1}_{x^i, \varepsilon} v + O\left(\int_{|y - x^i| \geqslant \delta} w^{q-1}_{x^i, \varepsilon} |v| \right) \\ &= Q(x^i) \int_{B_\delta(x^i)} w^{q-1}_{x^i, \varepsilon} v + O\left(\int_{B_\delta(x^i)} |y - x^i|^\theta\, w^{q-1}_{x^i, \varepsilon} |v| \right) \\ &\quad + O\left(\int_{|y - x^i| \geqslant \delta} w^{q-1}_{x^i, \varepsilon} |v| \right) \\ &= O\left(\int_{B_\delta(x^i)} |y - x^i|^\theta\, w^{q-1}_{x^i, \varepsilon} |v| \right) + O\left(\int_{|y - x^i| \geqslant \delta} w^{q-1}_{x^i, \varepsilon} |v| \right) \\ &= O(\varepsilon^{N+\theta} \int |y|^\theta\, w^{q-1} |v(\varepsilon y + x^i)|) + O\left(\varepsilon^N \int_{|y| \geqslant \delta/\varepsilon} w^{q-1} |v(\varepsilon y + x^i)| \right) \\ &= O(\varepsilon^{N/2+\theta} + \varepsilon^{N/2} e^{-\delta/\varepsilon})\, \|v\|_\varepsilon \\ &= O(\varepsilon^{N/2+\theta})\, \|v\|_\varepsilon. \end{aligned} \tag{3.20}$$

Using (3.17) and (3.20), we obtain

$$\int Q(y)\, H_{\varepsilon,x}^{q-1} = O(\varepsilon^{N/2+\theta} + \varepsilon^{N/2} \sum_{i \neq j} e^{-\mathrm{Min}((q-1)/2,\,1)\,|x^i - x^j|/\varepsilon})\, \|v\|_\varepsilon. \tag{3.21}$$

We also have

$$\begin{aligned}
\int Q(y)\, H_{\varepsilon,x}^{q-1}\, w_{x^i,\varepsilon} & \int_{B_\delta(x^i)} Q(y)\, H_{\varepsilon,x}^{q-1}\, w_{x^i,\varepsilon} + O\left(\varepsilon^N \left(\int_{|y| \geqslant \delta/\varepsilon} w^q\right)^{1/q}\right) \\
&= Q(x^i) \int_{B_\delta(x^i)} H_{\varepsilon,x}^{q-1}\, w_{x^i,\varepsilon} + O\left(\int_{B_\delta(x^i)} |y - x^i|^\theta\, H_{\varepsilon,x}^{q-1}\, w_{x^i,\varepsilon}\right) + O(\varepsilon^N e^{-\delta/\varepsilon}) \\
&= Q(x^i) \int_{B_\delta(x^i)} w_{x^i,\varepsilon}^q + O\,(\varepsilon^{N+\theta} + \varepsilon^N \sum_{i \neq j} e^{-|x^i - x^j|/\varepsilon}) \\
&= Q(x^i)\, \varepsilon^N \|w\|^2 + O\,(\varepsilon^{N+\theta} + \varepsilon^N \sum_{i \neq j} e^{-|x^i - x^j|/\varepsilon}).
\end{aligned} \tag{3.22}$$

Thus

$$\begin{aligned}
(H_{\varepsilon,x}, w_{x^i,\varepsilon})_\varepsilon - \int Q(y)\, H_{\varepsilon,x}^{q-1}\, w_{x^i,\varepsilon} &= \varepsilon^N\,(\|w\|^2 + O\,(\sum_{i \neq j} e^{-|x^i - x^j|/\varepsilon})) \\
&\quad - (Q(x^i)\, \varepsilon^N \|w\|^2 + O\,(\varepsilon^{N+\theta} + \varepsilon^N \sum_{i \neq j} e^{-|x^i - x^j|/\varepsilon})) \\
&= O\left(\varepsilon^\theta + \sum_{i=1}^k |1 - Q(x^i)| + \sum_{i \neq j} e^{-|x^i - x^j|/\varepsilon}\right) \varepsilon^N.
\end{aligned} \tag{3.23}$$

From (3.21), (3.23) and (3.5), we have

$$|(f_{\varepsilon,x}, u)_\varepsilon| = \varepsilon^{N/2} O\left(\varepsilon^\theta + \sum_{i=1}^k |1 - Q(x^i)| + \sum_{i \neq j} e^{-\mathrm{Min}((q-1)/2,\,1)\,|x^i - x^j|/\varepsilon}\right) \|v\|_\varepsilon$$

which implies that

$$\|f_{\varepsilon,x}\|_\varepsilon = \varepsilon^{N/2} O\left(\varepsilon^\theta + \sum_{i \neq j} e^{-\mathrm{Min}((q-1)/2,\,1)\,|x^i - x^j|/\varepsilon} + \sum_{i=1}^k |Q(x^i) - 1|\right). \tag{3.24}$$

Combining the above estimate and (3.16), (3.4) follows. This completes the proof of Proposition 3.1. □

Let $\varepsilon_0, R_0, \delta_0$ be as in Proposition 3.1. For $\varepsilon \in (0, \varepsilon_0]$, $R \geqslant R_0$, $\delta \in (0, \delta_0]$, let $(\alpha_\varepsilon(x), v_\varepsilon(x))$ be the C^1-mapping established in Proposition 3.1. Define

$$\tilde{J}_\varepsilon(x) = J_\varepsilon(\alpha_\varepsilon(x), x, v_\varepsilon(x)),$$
$$x \in D_{\varepsilon,R,\delta}^k.$$

Let $x_\varepsilon = (x_\varepsilon^1, \ldots, x_\varepsilon^k) \in D_{\varepsilon,R,\delta}$ be any point for which

$$\tilde{J}_\varepsilon(x_\varepsilon) = \mathrm{Max}\{\tilde{J}_\varepsilon(x) : x \in D_{\varepsilon,R,\delta}^k\}. \tag{3.25}$$

In the following proposition we show that x_ε for small ε is an interior point of $D_{\varepsilon,R,\delta}$, and hence a critical point of $\tilde{J}_\varepsilon$.

PROPOSITION 3.2. *Let x_ε satisfy* (3.25). *Then as $\varepsilon \to 0$,*

$$x_\varepsilon^i \to 0, \qquad i = 1, 2, \ldots, k,$$
$$|x_\varepsilon^i - x_\varepsilon^j|/\varepsilon \to \infty, \qquad i \neq j.$$

Proof. With the notations of Proposition 3.1 and from Appendix B and the estimates (3.4) and (3.24), we have

$$\begin{aligned} J_\varepsilon(\alpha_\varepsilon(x_\varepsilon), x_\varepsilon, v_\varepsilon(x_\varepsilon)) &= J_\varepsilon^*(x_\varepsilon, u_\varepsilon) \\ &= J_\varepsilon^*(x_\varepsilon, 0) + O(\|f_{\varepsilon, x_\varepsilon}\|_\varepsilon^2 + \|u\|_\varepsilon^2) \\ &= J_\varepsilon^*(x_\varepsilon, 0) + O\left(\varepsilon^N\left(\varepsilon^{2\theta} + \sum_{i \neq j} e^{-\mathrm{Min}(q-1,2)|x^i - x^j|/\varepsilon} + \sum_{j=1}^k |Q(x^j) - 1|^2\right)\right) \\ &= \left(\frac{k}{2} - \frac{1}{q}\sum_{j=1}^k Q(x^j)\right)\|w\|^2\varepsilon^N - \int Q(y)\sum_{j=1}^{k-1} w_{x^i,\varepsilon}\left(\sum_{j=i+1}^k w_{x^j,\varepsilon}\right)^{q-1} \\ &\quad + O\left(\varepsilon^N\left(\varepsilon^{2\theta} + \sum_{j=1}^k |1 - Q(x^j)|^2 + \sum_{i \neq j} e^{-\mathrm{Min}(q-1,2)|x^i - x^j|/\varepsilon}\right)\right). \end{aligned} \tag{3.26}$$

Let

$$z_\varepsilon^i = \varepsilon^\tau e_i, \quad i = 1, 2, \ldots, k,$$

for some $\tau \in (\frac{1}{2}, 1)$ and some vectors $e_1, \ldots, e_k$ with $e_i \neq e_j$ $(i \neq j)$. Then

$$|z_\varepsilon^i - z_\varepsilon^j|/\varepsilon = |e_i - e_j|/\varepsilon^{1-\tau} \to \infty$$

as $\varepsilon \to 0$.

Thus $z_\varepsilon = (z_\varepsilon^1, \ldots, z_\varepsilon^k) \in D_{\varepsilon,R,\delta}^k$ for ε sufficiently small, and by (3.25) and (3.26) we obtain

$$\begin{aligned} J_\varepsilon(\alpha_\varepsilon(x_\varepsilon), x_\varepsilon, v_\varepsilon(x_\varepsilon)) &\geqslant J_\varepsilon(\alpha_\varepsilon(z_{\varepsilon'}), z_\varepsilon, v_\varepsilon(z_\varepsilon)) \\ &\geqslant \left(\frac{k}{2} - \frac{1}{q}\sum_{i=1}^k Q(z_\varepsilon^i)\right)\|w\|^2\varepsilon^N \\ &\quad + O\left(\varepsilon^N\left(\varepsilon^{2\theta} + \sum_{j=1}^k |1 - Q(z_\varepsilon^i)|^2 + \sum_{i \neq j} e^{-(1+\sigma)|e_i - e_j|/\varepsilon^{1-\tau}}\right)\right) \\ &= \left(\frac{1}{2} - \frac{1}{q}\right)k\|w\|^2\varepsilon^N + O(\varepsilon^{N+\theta_1}) \end{aligned} \tag{3.27}$$

where $\theta_1 = \mathrm{Min}(\tau, 2\theta)$ and $1 + \sigma = \mathrm{Min}(q-1, 2)$.

Using (3.26) and (3.27), we obtain

$$\begin{aligned} &\left(\frac{k}{2} - \frac{1}{q}\sum_{i=1}^k Q(x_\varepsilon^i)\right)\|w\|^2\varepsilon^N - \int Q(y)\sum_{i=1}^{k-1} w_{x_\varepsilon^i,\varepsilon}\left(\sum_{j=i+1}^k w_{x_\varepsilon^j,\varepsilon}\right)^{q-1} \\ &\qquad + O\left(\varepsilon^N\left(\varepsilon^{2\theta} + \sum_{i=1}^k |1 - Q(x_\varepsilon^i)|^2 + \sum_{i \neq j} e^{-(1+\sigma)|x_\varepsilon^i - x_\varepsilon^j|/\varepsilon}\right)\right) \\ &\qquad \geqslant \left(\frac{1}{2} - \frac{1}{q}\right)k\|w\|^2\varepsilon^N + O(\varepsilon^{N+\theta_1}). \end{aligned}$$

Thus

$$\frac{1}{q}\sum_{i=1}^{k}(Q(x_\varepsilon^i)-1)\,\|w\|^2\varepsilon^N+\int Q(y)\sum_{i=1}^{k-1}w_{x_\varepsilon^i,\varepsilon}\left(\sum_{j=i+1}^{k}w_{x_\varepsilon^j,\varepsilon}\right)^{q-1}$$

$$\leqslant O\left(\varepsilon^N\left(\varepsilon^{2\theta}+\sum_{i=1}^{k}|1-Q(x_\varepsilon^i)|^2+\sum_{i\neq j}e^{-(1+\sigma)|x_\varepsilon^i-x_\varepsilon^j|/\varepsilon}\right)\right) \tag{3.28}$$

but

$$\int Q(y)\sum_{i=1}^{k-1}w_{x_\varepsilon^i,\varepsilon}\left(\sum_{j=i+1}^{k}w_{x_\varepsilon^i,\varepsilon}\right)^{q-1}\geqslant\int_{\mathbb{R}^N}\sum_{i=1}^{k-1}w_{x_\varepsilon^i,\varepsilon}\left(\sum_{j=i+1}^{k}w_{x_\varepsilon^j,\varepsilon}\right)^{q-1}$$

$$+\int_{|y|\leqslant\delta}(Q(y)-1)\sum_{i=1}^{k-1}w_{x_\varepsilon^i,\varepsilon}\left(\sum_{j=i+1}^{k}w_{x_\varepsilon^j,\varepsilon}\right)^{q-1}$$

$$+\int_{|y|\geqslant\delta}(Q(y)-1)\sum_{i=1}^{k-1}w_{x_\varepsilon^i,\varepsilon}\left(\sum_{j=i+1}^{k}w_{x_\varepsilon^j,\varepsilon}\right)^{q-1} \tag{3.29}$$

and

$$\left|\int_{|y|\geqslant\delta}(Q(y)-1)\sum_{i=1}^{k-1}w_{x_\varepsilon^i,\varepsilon}\left(\sum_{j=i+1}^{k}w_{x_\varepsilon^j,\varepsilon}\right)^{q-1}dy\right|$$

$$\leqslant C\varepsilon^N\sum_{j\neq i}\int_{|y|\geqslant\delta/\varepsilon}w(y)\,w^{q-1}(y-(x_\varepsilon^j-x_\varepsilon^i)/\varepsilon)\,dy$$

$$\leqslant C\varepsilon^N\sum_{i\neq j}\int_{|y|\geqslant\delta/\varepsilon}e^{-|y|}e^{-(q-1)|y-(x_\varepsilon^i-x_\varepsilon^j)|/\varepsilon}\,dy$$

$$=O(\varepsilon^N e^{-\delta/\varepsilon}). \tag{3.30}$$

From (3.29), (3.30), and assumptions (Q_1), (Q_2), we have

$$\int Q(y)\sum_{i=1}^{k-1}w_{x_\varepsilon^i,\varepsilon}\left(\sum_{j=i+1}^{k}w_{x_\varepsilon^i,\varepsilon}\right)^{q-1}\geqslant C\sum_{i\neq j}e^{-(1+\sigma/2)|x_\varepsilon^i-x_\varepsilon^j|/\varepsilon}\varepsilon^N+O(\varepsilon^N e^{-\varepsilon/\delta}) \tag{3.31}$$

where C is a positive constant.

From (3.28) and (3.31) we have

$$\varepsilon^N\sum_{i=1}^{k}(Q(x_\varepsilon^i)-1)+C\sum_{i\neq j}e^{-(1+\sigma/2)|x_\varepsilon^i-x_\varepsilon^j|/\varepsilon}\varepsilon^N\leqslant O(\varepsilon^{N+2\theta}),$$

which implies that

$$Q(x_\varepsilon^i)\to 1=Q(0),$$
$$x_\varepsilon^i\to 0,$$
$$|x_\varepsilon^i-x_\varepsilon^j|/\varepsilon\to\infty,\quad i\neq j,$$

as $\varepsilon\to 0$, $i,j=1,2,\ldots,k$.

This completes the proof of Proposition 3.2. □

Proof of Theorem A. From Proposition 3.1, for each k, there exist ε_0, R_0, δ_0, and a C^1-mapping $(\alpha_\varepsilon(x_\varepsilon), v_\varepsilon(x_\varepsilon)): D_{\varepsilon,R,\delta}\longrightarrow\mathbb{R}^k\times E_{\varepsilon,k}$ for each $\varepsilon\in(0,\varepsilon_0]$, $R>R_0$, $\delta\in(0,\delta_0]$, such that (2.7) and (2.8) hold. By Proposition 3.2 we can choose x_ε such that $x_\varepsilon^i \to 0$, $|x_\varepsilon^i - x_\varepsilon^j|/\varepsilon \to \infty$, as $\varepsilon \to 0$, $i, j = 1, 2, \ldots, k$, $i \neq j$, and $\partial \tilde{J}_\varepsilon(x_\varepsilon)/\partial x_i^j = 0$. That is,

$$\begin{aligned} 0 &= \sum_{\ell=1}^{k} \frac{\partial J_\varepsilon}{\partial \alpha^\ell} \frac{\partial \alpha^\ell}{\partial x_i^j} + \frac{\partial J_\varepsilon}{\partial x_i^j} + \left\langle \frac{\partial J_\varepsilon}{\partial v}, \frac{\partial v_\varepsilon}{\partial x_i^j} \right\rangle_\varepsilon \\ &= \frac{\partial J_\varepsilon}{\partial x_i^j} + \sum_{\ell=1}^{k} A_\ell \left\langle w_{x_\varepsilon^\ell, \varepsilon}, \frac{\partial v_\varepsilon}{\partial x_i^j} \right\rangle_\varepsilon \\ &\quad + \sum_{\ell=1}^{k} \sum_{h=1}^{N} B_{\ell, h} \left\langle \frac{\partial w_{x_\varepsilon^\ell, \varepsilon}}{\partial x_h^\ell}, \frac{\partial v_\varepsilon}{\partial x_i^j} \right\rangle_\varepsilon \end{aligned} \tag{3.32}$$

by (2.7) and (2.8). However $\langle w_{x_\varepsilon^\ell, \varepsilon}, v_\varepsilon \rangle_\varepsilon = 0$, $\langle \partial w_{x_\varepsilon^\ell, \varepsilon}/\partial x_h^\ell, v_\varepsilon \rangle_\varepsilon = 0$, since $v_\varepsilon \in E_{\varepsilon, k}$. Therefore, $\langle w_{x_\varepsilon^\ell, \varepsilon}, \partial v_\varepsilon/\partial x_i^j \rangle_\varepsilon = 0$, and from (3.32) we obtain

$$\frac{\partial J_\varepsilon}{\partial x_i^j} = \sum_{h=1}^{N} B_{j, h} \left\langle \frac{\partial^2 w_{x_\varepsilon^\ell, \varepsilon}}{\partial x_i^j \partial x_h^j}, v_\varepsilon \right\rangle_\varepsilon,$$

which is (2.6). Theorem A follows easily. □

REMARK 3.3. If $Q(x)$ has several strict local minimum points, say $a_1, \ldots, a_m$, the above arguments may be used to show that (1.1) has, for any given integers k_ℓ, $\ell = 1, \ldots, m$, a solution u_ε of the form

$$u_\varepsilon = \sum_{\ell=1}^{m} \sum_{i=1}^{k_\ell} \alpha_{\varepsilon, \ell, i} w_{x_{\varepsilon, \ell}^i, \varepsilon} + v_\varepsilon$$

for sufficiently small ε, and as $\varepsilon \to 0$,

$$\begin{aligned} x_{\varepsilon, \ell}^i &\to a_\ell, & i &= 1, 2, \ldots, k_\ell, \\ |x_{\varepsilon, \ell}^i - x_{\varepsilon, \ell}^j|/\varepsilon &\to \infty, & i &\neq j, \\ \alpha_{\varepsilon, \ell, i} &\to 1/(Q(a_\ell))^{1/(q-2)}, & i &= 1, 2, \ldots, k_\ell, \\ \|v_\varepsilon\|^2 &= o(\varepsilon^N). \end{aligned}$$

REMARK 3.4. The requirement that $Q(x)$ has a strict local minimum may be replaced by the weaker condition that there is a set $\Lambda \subset \mathbb{R}^N$ such that

$$\mathrm{Min}_{\bar{\Lambda}} Q(x) < \mathrm{Min}_{\partial \Lambda} Q(x).$$

We may use the same arguments to construct a solution u_ε of the form

$$u_\varepsilon = \sum_{i=1}^{k} \alpha_{\varepsilon, i} w_{x_\varepsilon^i, \varepsilon} + v_\varepsilon$$

for any given positive integer k, provided $\varepsilon = \varepsilon(k)$ is sufficient small, and as $\varepsilon \to 0$,

$$\begin{aligned} x_\varepsilon^i &\to x_0^i \\ Q(x_0^i) &= \mathrm{Min}_{\bar{\Lambda}} Q(x) \\ \alpha_{\varepsilon, i} &\to 1/(Q(x_0^i))^{1/(q-2)} \\ \|v_\varepsilon\|_\varepsilon^2 &= o(\varepsilon^N). \end{aligned}$$

Appendix A

We give here a sketch of the proofs of Lemmas 2.1 and 2.2. The results are essentially known, and have been used in one form or another by several authors.

To prove Lemma 2.1, we need the following decomposition lemma.

Let $k = 1, 2, \dots$ be fixed. Let $x = (x^1, \dots, x^k) \in \mathbb{R}^{kN}$, $\alpha = (\alpha^1, \dots, \alpha^k) \in \mathbb{R}^k$,

$$\sum_{\delta} = \{(\alpha^1, \alpha^2, \dots, \alpha^k, x^1, \dots, x^k) : |x^i|, |\alpha^i - 1| \leqslant \delta, i = 1, 2, \dots, k\}$$

$$W(\delta, R, \varepsilon) = \left\{ u : u \in H^1, \left\| u - \sum_{i=1}^{k} \alpha^i w_{x^i_x, \varepsilon} \right\| \leqslant \delta \varepsilon^N \text{ for some } x \in D^k_{\varepsilon, R, \delta} \right\}.$$

LEMMA A.1. *There are $R_0, \varepsilon_0, \delta_0 > 0$ such that for $R \geqslant R_0$, $\varepsilon \in (0, \varepsilon_0]$, $\delta \in (0, \delta_0]$ and $u \in W(\delta, R, \varepsilon)$, the minimization problem*

$$\inf \left\{ \left\| u - \sum_{i=1}^{k} \alpha^i w_{x^i_\varepsilon, \varepsilon} \right\|^2_\varepsilon : (\alpha, x) \in \sum_{4\delta} \right\} \tag{A.1}$$

is achieved in $\sum_{2\delta}$ and not in $\sum_{4\delta} \backslash \overline{\sum_{2\delta'}}$. Furthermore, the above minimization problem admits a unique solution.

For the proof of the above lemma we refer the reader to [**7**, **8**].

REMARK A.2. Let (α, x) be the minimizer of (A.1), as given by Lemma A.1. Set

$$v = u - \sum_{i=1}^{k} \alpha^i w_{x^i, \varepsilon}.$$

Then v satisfies

$$\langle v, w_{x^j, \varepsilon} \rangle_\varepsilon = \left\langle v, \frac{\partial w_{x^j, \varepsilon}}{\partial x^j_i} \right\rangle_\varepsilon = 0,$$

$j = 1, \dots, k$, $i = 1, \dots, N$. Therefore, $(\alpha, x, v) \in M_{\varepsilon, R, \delta}$ for $\varepsilon \in (0, \varepsilon_0]$, $R \geqslant R_0$, $\delta \in [0, \delta_0]$.

The proof of Lemma 2.1 then follows as in [**24**, Proposition 3]. See also [**8**].

LEMMA A.3. *There exist $R_0, \varepsilon_0, \delta_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$, $\delta \in (0, \delta_0]$, $R \geqslant R_0$,*

$$\|v\|^2_\varepsilon - (q-1) \int_{\mathbb{R}^N} \left(\sum_{j=1}^{k} w_{x^j_\varepsilon, \varepsilon} \right)^{q-2} v^2 \geqslant \rho \|v\|^2_\varepsilon$$

for all $x \in D_{\delta, R, \varepsilon}$, $v \in E_{\varepsilon, k}$, where $\rho > 0$ is a positive constant.

This lemma can be proved in exactly the same way as [**8**, Lemma B.2].

Appendix B

We establish here an estimate for $J^*_\varepsilon(x, 0)$.

LEMMA B.1. *For any $x \in D_{\delta, R, \varepsilon}$, $x = (x^1, \dots, x^k)$ we have*

$$\begin{aligned} J^*(x, 0) = &\left(\frac{k}{2} - \frac{1}{q} \sum_{i=1}^{k} Q(x^i) \right) \|w\|^2 \varepsilon^N \\ &- \int Q(y) \sum_{i=1}^{k-1} w_{x^i, \varepsilon} \left(\sum_{j=i+1}^{k} w_{x^j, \varepsilon} \right)^{q-1} \\ &+ O\left(\varepsilon^N \left(\varepsilon^\theta + \sum_{i=1}^{k} |1 - Q(x^i)|^2 + \sum_{i \neq j} e^{-(1+\sigma)|x^i - x^j|/\varepsilon} \right) \right). \end{aligned}$$

Proof. Let $H_{\varepsilon,x} = \sum_{j=1}^{k} w_{x^j,\varepsilon}$. Then

$$\|H_{\varepsilon,x}\|_\varepsilon^2 = k\varepsilon^N \|w\|^2 + \sum_{i \neq j} (w_{x^i,\varepsilon}, w_{x^j,\varepsilon}) = k\varepsilon^N \|w\|^2 + 2\sum_{i<j} \int w_{x^i,\varepsilon}^{q-1} w_{x^j,\varepsilon}. \tag{B.1}$$

We also have

$$\begin{aligned}\int Q(y)\, H_{\varepsilon,x}^q - \int Q(y) \sum_{j=1}^{k} w_{x^j,\varepsilon}^q &= \int Q(y) \Big(\sum_{j=2}^{q} w_{x^j,\varepsilon} \Big)^q - \int Q(y) \sum_{j=2}^{k} w_{x^j,\varepsilon}^q \\ &\quad + q \int Q(y)\, w_{x^1}^{q-1} \Big(\sum_{j=2}^{k} w_{x^j,\varepsilon} \Big) \\ &\quad + q \int Q(y)\, w_{x^1} \Big(\sum_{j=2}^{k} w_{x^j,\varepsilon} \Big)^{q-1} \\ &\quad + O\big(\varepsilon^N \sum_{i \neq j} e^{-(1+\sigma)|x^i - x^j|/\varepsilon}\big)\end{aligned} \tag{B.2}$$

for some $\sigma > 0$, as $\varepsilon \to 0$, where we have used the following inequalities:

(i) For $2 < q \leqslant 3$

$$\begin{aligned}||a+b|^q - a^q - b^q - qa^{q-1}b - qab^{q-1}| &\leqslant \begin{cases} C|b|^{q-1}|a|, & |b| \leqslant |a|, \\ C|a|^{q-1}|b|, & |b| > |a|, \end{cases} \\ &\leqslant C|a|^{q/2}|b|^{q/2}.\end{aligned}$$

(ii) For $p > 3$

$$||a+b|^q - a^q - b^q - qa^{q-1}b - qab^{q-1}| \leqslant C(a^{q-2}b^2 + a^2 b^{q-2}).$$

By repeated application of the above inequalities to (B.2), we obtain

$$\begin{aligned}\int Q(y)\, H_{\varepsilon,x}^q &= \int Q(y) \sum_{j=1}^{k} w_{x^j,\varepsilon}^q + q \int Q(y) \sum_{i<j} w_{x^i,\varepsilon}^{q-1} w_{x^j,\varepsilon} \\ &\quad + q \int Q(y) \sum_{i=1}^{k-1} w_{x^i,\varepsilon} \Big(\sum_{j=i+1}^{k} w_{x^j,\varepsilon} \Big)^{q-1} \\ &\quad + O\big(\varepsilon^N \sum_{i \neq j} e^{-(1+\sigma)|x^i - x^j|/\varepsilon}\big).\end{aligned} \tag{B.3}$$

Using the estimates (1.3) we also have

$$\begin{aligned}\int Q(y)\, w_{x^j,\varepsilon}^q &= \int_{B_\delta(x^j)} Q(y)\, w_{x^j,\varepsilon}^q + O(\varepsilon^N e^{-q\delta/\varepsilon}) \\ &= Q(x^j) \int_{B_\delta(x^j)} w_{x^j,\varepsilon}^q + O(\varepsilon^{N+\theta}) \\ &= Q(x^j)\, \|w\|^2 \varepsilon^N + O(\varepsilon^{N+\theta})\end{aligned} \tag{B.4}$$

$$\begin{aligned}\int Q(y)\, w_{x^i,\varepsilon}^{q-1} w_{x^j,\varepsilon} &= \int_{B_\delta(x^i)} Q(y)\, w_{x^i,\varepsilon}^{q-1} w_{x^j,\varepsilon} + O(\varepsilon^N e^{-(q-1)\delta/\varepsilon}) \\ &= Q(x^i) \int_{B_\delta(x^j)} w_{x^i,\varepsilon}^{q-1} w_{x^j,\varepsilon} + O(\varepsilon^{N+\theta}) \\ &= Q(x^i) \int w_{x^i,\varepsilon}^{q-1} w_{x^j,\varepsilon} + O(\varepsilon^{N+\theta}).\end{aligned} \tag{B.5}$$

Combining (B.1)–(B.5), we obtain

$$\begin{aligned} J^*(x^1, \ldots, x^k, 0) &= \tfrac{1}{2}\|H_{\varepsilon,x}\|_\varepsilon^2 - \frac{1}{q}\int Q(y)\, H_{\varepsilon,x}^q \\ &= \left(\frac{k}{2} - \frac{1}{q}\sum_{j=1}^k Q(x^j)\right)\|w\|^2\varepsilon^N \\ &\quad + \sum_{i<j}\left(1 - Q(x^i)\int w_{x^i,\varepsilon}^{q-1} w_{x^i,\varepsilon}\right) \\ &\quad - \int Q(y)\sum_{i=1}^{k-1} w_{x^i,\varepsilon}\left(\sum_{j=i+1}^{k} w_{x^j,\varepsilon}\right)^{q-1} \\ &\quad + O\,(\varepsilon^{N+\theta} + \varepsilon^N \sum_{i\neq j} e^{-(1+\sigma)|x^i - x^j|/\varepsilon}). \qquad \square \end{aligned}$$

References

1. A. Bahri, *Critical points at infinity in some variational problems*, Research Notes in Mathematics 182 (Pitman, 1989).
2. A. Bahri and J. M. Coron, 'On a nonlinear elliptic equation involving the Sobolev exponent: the effect of the topology of the domain', *Comm. Pure Appl. Math.* 41 (1988) 253–294.
3. V. Benci and G. Cerami, 'The effect of the domain topology on the number of positive solutions of nonlinear elliptic problems', *Arch. Rational Mech. Anal.* 114 (1991) 79–93.
4. V. Benci and G. Cerami, 'The multiple positive solutions of some elliptic problems via the Morse theory and the domain topology', *Calc. Var.* 2 (1994) 29–48.
5. D. Cao and E. S. Noussair, 'Multiplicity of positive and nodal solutions for nonlinear elliptic problems in $\mathbb{R}^N$', *Ann. Inst. Henri Poincaré* 13 (1996).
6. D. Cao, N. E. Dancer, E. S. Noussair and S. Yan, 'On the existence and profile of multi-peaked solutions to singularly perturbed semilinear Dirichlet problems', *Discrete Contin. Dynam. Systems* 2 (1996) 221–236.
7. D. Cao, E. S. Noussair and S. Yan, 'Existence and uniqueness results on single-peaked solutions of a semilinear problem', *Ann. Inst. H. Poincaré* 15 (1998) 73–111.
8. D. Cao, E. S. Noussair and S. Yan, 'Solutions with multiple "peaks" for nonlinear elliptic equations', *Proc. Roy. Soc. Edinburgh. Sect.* A (1999) 235–264.
9. E. N. Dancer, 'The effect of the domain shape on the number of positive solutions of certain nonlinear equations II', *J. Differential Equations* 87 (1990) 120–156.
10. N. Dancer and S. Yan, 'Multi-peaks solutions for a singularly perturbed Neumann problem'. *Pacific J. Math.* 189 (1999) 241–262.
11. M. del Pino and P. Felmer, 'Multi-peak bound states for nonlinear Schrödinger equations'. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 15 (1998) 127–149.
12. M. del Pino and P. L. Felmer, 'Semi-classical states for nonlinear Schrödinger equations', *J. Funct. Anal.* 149 (1997) 245–265.
13. A. Floer and A. Weinstein, 'Nonspreading wave packets for the cubic Schrödinger equations with a bounded potential', *J. Funct. Anal.* 69 (1986) 397–408.
14. B. Gidas, W. M. Ni and L. Nirenberg, 'Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^N$', *Adv. Math. Supplementary Studies* 7A (1981) 369–402.
15. C. Gui, 'Existence of multi-bump solutions for nonlinear Schrödinger equations via variational method', *Comm. Partial Differential Equations* 21 (1996) 787–820.
16. M. K. Kwong, 'Uniqueness of positive solutions for $\Delta u - u + u^p = 0$ in $\mathbb{R}^N$', *Arch. Rational Mech. Anal.* 105 (1989) 243–266.
17. Y. Y. Li, 'On a singularly perturbed elliptic equation', *Adv. Differential Equations* 2 (1997) 955–980.
18. G. Lu and J. Wei, 'On nonlinear Schrödinger equations with totally degenerate potentials', *C. R. Acad. Sci. Paris Sér.* 1 326 (1998) 1–7.
19. W. M. Ni and J. Wei, 'On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems', *Comm. Pure. Appl. Math.* 48 (1995) 731–768.
20. E. S. Noussair and S. Yan, 'The effect of the domain geometry in singular perturbation problems', *Proc. London Math. Soc.* 76 (1998) 427–452.
21. E. S. Noussair and S. Yan, 'On the existence and multiplicity of nodal solutions in singular perturbation problems', preprint.
22. Y. J. Oh, 'Existence of semi-classical bound states of nonlinear Schrödinger equations with potentia of the class $(V)_\alpha$', *Comm. Partial Differential Equations* 13 (1988) 731–768.

23. Y. J. OH, 'On positive multi-lump bounded states of nonlinear Schrödinger equations under multiple well potential', *Comm. Math. Phys.* 131 (1990) 223–253.

24. P. H. RABINOWITZ, 'On a class of nonlinear Schrödinger equations', *Z. Angew Math. Phys.* 43 (1992) 270–291.

25. O. REY, 'The role of the Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent', *J. Funct. Anal.* 89 (1990) 1–52.

26. X. WANG, 'On the concentration of positive bound states of nonlinear Schrödinger equations', *Comm. Math. Phys.* 153 (1993) 229–244.

27. Z. Q. WANG, 'On the existence of multiple, single-peaked solutions of a semilinear Neumann problem', *Arch. Rational Mech. Anal.* 120 (1992) 375–399.

School of Mathematics
University of New South Wales
Sydney
NSW 2052
Australia

School of Mathematics and Statistics
University of Sydney
Sydney
NSW 2006
Australia