IMA Journal of Mathematical Control and Information (2000) **17**, 363–373

Robust Kalman filtering for continuous-time systems with norm-bounded nonlinear uncertainties

PENG SHI[†] AND C. YALCIN KAYA[‡]

Centre for Industrial and Applicable Mathematics, School of Mathematics, The University of South Australia, The Levels SA 5095, Australia

[Received 24 September 1998 and in revised form 8 February 1999]

In this paper we study the problem of robust Kalman filtering for a class of uncertain linear continuous-time systems. The system under consideration is subjected to time-varying, norm-bounded, nonlinear parameter uncertainties in state and measurement equations. Stability of the above system is analyzed. A state estimator is designed such that the covariance of the estimation error is guaranteed to be within a certain bound for all admissible uncertainties, which is in terms of solutions of two algebraic Riccati equations.

Keywords: Kalman filtering; nonlinear uncertainty; Riccati equations.

1. Introduction

1.1 *A motivation example, BOT*

The basic problem in target motion analysis (TMA) is in estimating the trajectory of an object (or target), i.e. the objects position and velocity at some instant of time, from noise corrupted sensor data. Bearings-only TMA, or bearings-only tracking (BOT) deals with the specific case when the measurement is the angle that the line passing through the moving observer (or receiver) platform and the object makes with some fixed reference axis (Nardone, 1984).

A general setup for BOT is depicted in Fig. 1. The coordinates of the observer and source positions are given as $(x_o(t), y_o(t))$ and $(x_s(t), y_s(t))$, respectively. The respective velocities are denoted by $(\dot{x}_o, \dot{y}_o)$ and $(\dot{x}_s, \dot{y}_s)$. Note that the time variable t has been dropped in Fig. 1 for simplicity.

The source is assumed to be moving at a constant velocity along a fixed rectilinear path. So it has the dynamical equations

$$\ddot{x}_s(t) = 0$$
$$\ddot{y}_s(t) = 0.$$

The dynamics of the observer is given by

$$\ddot{x}_o(t) = u_x(t)$$
$$\ddot{y}_o(t) = u_y(t)$$

[†]Email: peng.shi@dsto.defence.gov.au

[‡]Email: yalcin@phoenix.levels.unisa.edu.au

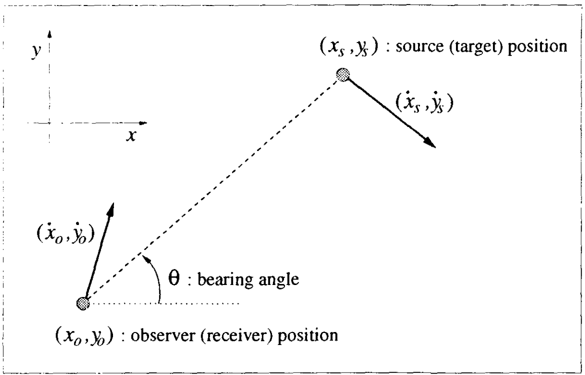

FIG. 1. The bearings-only tracking.

where $u_x(t)$ and $u_y(t)$ are the coordinates of the control input.

The bearing angle $\theta(t)$ can be expressed as

$$\theta(t) = \tan^{-1}\left(\frac{y_s(t) - y_o(t)}{x_s(t) - x_o(t)}\right).$$

Define the state vector $x(t) \in \mathbf{R}^4$ as $x(t) = (x_s(t) - x_o(t), y_s(t) - y_o(t), \dot{x}_s(t) - \dot{x}_o(t), \dot{y}_s(t) - \dot{y}_o(t))$, the control input $u(t) \in \mathbf{R}^2$ as $u(t) = (u_x(t), u_y(t))$, and the output $y(t) \in \mathbf{R}$ as $y(t) = \theta(t)$. Let the dynamical (or process) noise on the system be denoted by $w(t)$, and the measurement noise by $v(t)$. Then the state-space representation of the system is given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + w(t) \\ y(t) &= h(x(t)) + v(t) \end{aligned} \tag{1.1}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad h(x(t)) = \tan^{-1}\left(\frac{x_3(t)}{x_1(t)}\right).$$

In many cases the process noise $w(t)$ is ignored. If, however, $w(t)$ is not ignored then the usual assumption about both of the noise terms $w(t)$ and $v(t)$ is that they are zero-mean, normally distributed and independent. Note that the System (1.1) is nonlinear due to of the output function, or the measurement. In real applications, the measurements are taken at discrete time values, therefore System (1.1) is discretized. Furthermore, a so-called pseudo-linearization of the equations is carried out, which basically models the nonlinearity of the output as noise, additional to that of the measurement (Nardone, 1984). However, for analytical reasons, the continuous-time model is also considered in the literature. For example, Lévine & Marino (1992) investigates the observability properties of the continuous-time model of the system. Grossman (1991) designs an extended Kalman filter using a hybrid coordinate system. While the observer is assumed to be stationary in these two examples (i.e. $u(t) \equiv \mathbf{0}$) Helferty & Mudgett (1993) deal with the problem of choosing an appropriate $u(t)$ so as to optimize the trajectory of the observer and thus minimize the error in the estimation of the position of the source. Le Cadre & Laurent-Michel (1997) carry out a similar analysis using the discretized system equations.

System (1.1) motivated the development of a state estimator which is given in the following sections. The class of systems considered in the design of the estimator is allowed to be more general than the form given in (2.1). Application of the estimator to System (1.1) and further relevant analysis are intended for future work.

1.2 *Recent advancements in Kalman filtering*

Kalman filtering is one of the most popular estimation approaches. In the past three decades, considerable effort has been devoted to its theory and applications; see for example, Anderson & Moore (1979). This filtering approach assumes that both the state equation and output measurement are subjected to stationary Gaussian noises. The applications of the Kalman filtering theory may be found in a large spectrum of different fields ranging from various engineering problems to biology, geoscience, economics and management, etc.

Recently, the research of robust estimation is very attractive, and many developments have been made. In Bernstein & Haddad (1989), a Kalman filtering with an H_∞ norm constraint has considered. Xie *et al.* (1991) have studied the design of filters guaranteeing both robust stability and a prescribed H_∞ performance for the filtering error, in the presence of parameter uncertainty. Note, however, that in Xie *et al.* (1991) the adopted performance measure is in terms of the induced norm of the operator from the noise input to the estimation error. The design of digital filters with an H_∞ like performance for a linear system, has been tackled in Sun *et al.* (1991) whereas Shi (1993, 1996); Shi *et al.* (1997); Shi (1998) have considered the H_∞ filtering for sampled-data systems with parameter uncertainties. Very recently, Petersen & McFarlane (1991) considered a robust Kalman filtering problem for systems with bounded parameter uncertainty in the state matrix. A different approach has been proposed by Xie & Soh (1994), to the robust Kalman filtering problem for systems with bounded parameter uncertainty in both the state and measurement matrices. Also, Xie *et al.* (1994) studied the above problem for discrete-time systems. However, to the best of the authors knowledge, to date the problem of robust Kalman filtering for uncertain continuous-time linear systems with nonlinear uncertainties has not yet been investigated.

In this paper we consider the problem of state estimation for linear systems subject to real time-varying nonlinear parametric uncertainty. We address the designing of a stable quadratic state estimator such that the estimation error covariance will have a guaranteed bound for all admissible uncertainties. A Riccati equation approach is proposed to solve the above problem. We demonstrate that the above problem can be solved in terms of two algebraic Riccati equations (ARE).

Notation. The notations in this paper are quite standard. $\mathbf{R}^n$ and $\mathbf{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript 'T' denotes the transpose and the notation $X \geqslant Y$ (respectively, $X > Y$) where

X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrix with compatible dimension. $\mathsf{E}\{\cdot\}$ denotes the expectation operator with respective to some probability measure P. $\mathrm{tr}(M)$ is the trace of a square matrix M.

2. Problem formulation and preliminaries

2.1 *The uncertain system*

Consider the following class of uncertain dynamical systems:

$$\dot{x}(t) = Ax(t) + f_1(t, x(t)) + Bu(t) + w(t), \quad x(0) = x_0 \tag{2.1}$$

$$y(t) = Cx(t) + f_2(t, x(t)) + v(t) \tag{2.2}$$

where $x(t) \in \mathbf{R}^n$ is the system state, $u(t) \in \mathbf{R}^m$ is the system input, $y(t) \in \mathbf{R}^r$ is the measurement, $w(t) \in \mathbf{R}^n$ and $v(t) \in \mathbf{R}^r$ are the process and measurement noises, respectively. A, B and C are known constant matrices of appropriate dimensions that describe the nominal system, and $f_1(t, x(t))$ and $f_2(t, x(t))$ are unknown matrices with compatible dimensions which represent time-varying parametric uncertainties and satisfy

$$\|f_1(t, x(t))\| \leqslant a_1\|x(t)\|, \quad \|f_2(t, x(t))\| \leqslant a_2\|x(t)\|, \quad \text{for all } x \in \mathbf{R}^n \tag{2.3}$$

where $a_1 \geqslant 0$ and $a_2 \geqslant 0$ are known constant numbers.

As is in the standard Kalman filtering case, we shall make the following assumptions on the process noise $w(t)$ and measurement noise $v(t)$.

ASSUMPTION 2.1 For all $t \geqslant 0$, $\tau \geqslant 0$ and $i \in \mathcal{S}$,

(a) $\mathsf{E}w(t) = 0, \ \mathsf{E}w(t)w^T(\tau) = W\delta(t - \tau), \ W > 0$
(b) $\mathsf{E}v(t) = 0, \ \mathsf{E}v(t)v^T(\tau) = V\delta(t - \tau), \ V > 0$
(c) $\mathsf{E}w(t)v^T(\tau) = 0$

where $\delta(\cdot)$ is the Dirac function. Since in this paper, we are dealing with only the steady state filtering problem, it is assumed that $t_0 \to -\infty$.

2.2 *The state estimation problem*

Our objective in this paper is to design a stable estimator such that the error covariance of state $x(t)$ and its estimate $\hat{x}(t)$ is bounded for all admissible uncertainties $f_1(t, x(t))$ and $f_2(t, x(t))$.

DEFINITION 2.1 Given the Systems (2.1)–(2.2), the state equations,

$$\dot{\hat{x}}(t) = G\hat{x}(t) + B_1u(t) + Ky(t), \quad \hat{x}(0) = x_0 \tag{2.4}$$

are said to define a guaranteed cost state estimator for this system if there exists a constant symmetric matrix $P \geqslant 0$ such that

$$\mathsf{E}\{(x - \hat{x})(x - \hat{x})^T\} \leqslant P, \quad \text{or} \quad \mathsf{E}\{(x - \hat{x})^T(x - \hat{x})\} \leqslant \mathrm{tr}(P) \tag{2.5}$$

for all admissible uncertainties $f_1(t, x(t))$ and $f_2(t, x(t))$.

In this situation, the estimator (2.4) is said to provide *a guaranteed cost matrix P*.

Before ending this section, let us establish the following lemmas which link the relations between linear uncertainty and nonlinear uncertainty.

LEMMA 2.1 For $m \leqslant n$, suppose $v \in \mathbf{R}^n$ with $\|v\| = 1$, and $y \in \Re^m$ with $\|y\| = 1$. Then there exists a matrix $M \in \Re^{n\times m}$ with $\rho(M) = \lambda(M^T M) \leqslant 1$ such that

$$v = My.$$

Proof. By Gram–Schmidt algorithm, together with v and u being unit norm, we may construct orthonormal basis, i.e. $V = (v, v_2, \ldots, v_n)$ and $Y = (y, y_2, \ldots, y_m)$.

It is trivial to show that $V_m \stackrel{\Delta}{=} (v, v_2, \ldots, v_m)$ satisfies $V_m^T V_m = I$. Consequently, $M \stackrel{\Delta}{=} V_m Y^T$ satisfies $M^T M = I$ which implies $\rho(M) \leqslant 1$. Now, from $MY = V_m$, one has $v = My$. This ends the proof.

Denote the admissible uncertainty sets by

$$\Omega_1(t, x(t)) = \{f_1(t, x(t)) : \ \|f_1(t, x(t))\| \leqslant a_1\|x(t)\|\}$$
$$\Omega_2(t, x(t)) = \{f_2(t, x(t)) : \ \|f_2(t, x(t))\| \leqslant a_2\|x(t)\|\}.$$

REMARK 2.1 The matrices $f_1(t, x(t))$ and $f_2(t, x(t))$ contain the uncertain parameters in the state and measurement matrices of the Systems (2.1)–(2.2). The scalars $a_i,\ i = 1, 2$ specify how the uncertain parameters in $f_i(t, x(t)),\ i = 1, 2$ affect the nominal matrices of the Systems (2.1)–(2.2).

Next, we establish the relationship between the sets $\Omega_1(t, x(t))$, $\Omega_2(t, x(t))$ and the sets

$$\Omega_{l1}(t, x(t)) \stackrel{\Delta}{=} \{a_1 M_1 x(t) : \ M_1 \in \mathbf{R}^{n\times n},\ \rho(M_1) \leqslant 1\}$$

and

$$\Omega_{l2}(t, x(t)) \stackrel{\Delta}{=} \{a_2 M_2 x(t) : \ M_2 \in \mathbf{R}^{r\times n},\ \rho(M_2) \leqslant 1\}.$$

LEMMA 2.2 The sets $\Omega_1(t, x(t))$, $\Omega_2(t, x(t))$ and the sets $\Omega_{l1}(t, x(t))$ and $\Omega_{l2}(t, x(t))$ are identical, i.e.

$$\Omega_1(t, x(t)) = \Omega_{l1}(t, x(t))$$
$$\Omega_2(t, x(t)) = \Omega_{l2}(t, x(t)).$$

Proof. It suffices to show only that $\Omega_1(t, x(t)) = \Omega_{l1}(t, x(t))$.

Firstly, it can be easily seen that

$$\Omega_1(t, x(t)) \supseteq \Omega_{l1}(t, x(t)).$$

Also, for any $x(t) \in \mathbf{R}^n$, one has

$$\Omega_1(t, x(t)) \subseteq \Phi \tag{2.6}$$

where

$$\Phi = \{v \in \mathbf{R}^n : \ \|v\| \leqslant a_1 \|x(t)\|\}.$$

Next, assume any nonzero vector $v \in \Phi$. Then there exists a non-negative scalar $\bar{a}_1 \leqslant a_1$ such that

$$\|v\| = \bar{a}_1 \|x(t)\|. \tag{2.7}$$

Without loss of generality, let us assume $x(t) \neq 0$ and define

$$\tilde{v} = \frac{v}{\|v\|}, \quad \tilde{x}(t) = \frac{x(t)}{\|x(t)\|}.$$

Now, by using Lemma 2.1, there exists $M \in \mathbf{R}^{n\times n}$ with $\rho(M) \leqslant 1$ such that

$$\tilde{v} = M\tilde{x}(t)$$

which, by taking into account of (2.7), leads to

$$v = \bar{a}_1 M x(t).$$

Now, we have

$$v = \bar{a}_1 M x(t) \in \Omega_{l1}(t, x(t))$$

which implies that $\Phi \subseteq \Omega_{l1}(t, x(t))$. Bearing in mind (2.6), we conclude that $\Omega_1(t, x(t)) \subseteq \Omega_{l1}(t, x(t))$. Therefore we have $\Omega_1(t, x(t)) = \Omega_{l1}(t, x(t))$.

REMARK 2.2 The advantage of Lemma 2.1 is that, instead of nonlinear uncertainty in (2.3), it suffices to consider only linear uncertainty with structure as in (2.3), while the latter is easier to handle and has been widely used in robust control and filtering (see, for example, Petersen, 1987; Shi *et al.*, 1999; de Souza *et al.*, 1993), although the former represents a large class of physical uncertain systems. Furthermore, many existing results on robust stability and robust control with linear uncertainty as the one given in (2.3) can be extended to the cases involving nonlinear uncertainty.

LEMMA 2.3 Let H, F and E be real matrices of appropriate dimensions. Then, for any scalar $\varepsilon > 0$ and for all matrices F satisfying $F^T F \leqslant I$,

$$HFE + E^T F^T H^T \leqslant \varepsilon E^T E + \frac{1}{\varepsilon} H H^T.$$

Proof. We observe that for any $x \in \mathbf{R}^n$

$$\begin{aligned} 0 &\leqslant x^T(\sqrt{\varepsilon} FE - \frac{1}{\sqrt{\varepsilon}} H^T)^T(\sqrt{\varepsilon} FE - \frac{1}{\sqrt{\varepsilon}} H^T)x \\ &= x^T(\varepsilon E^T F^T FE + \frac{1}{\varepsilon} HH^T - E^T F^T H^T - HFE)x. \end{aligned}$$

Now, the desired result follows immediately from the the above inequality.

3. Guaranteed cost filter design

In this section, we will design, via the Riccati equation approach, a guaranteed cost state estimator of the form (2.4) such that the estimation error covariance of the state $x(t)$ and its estimate $\hat{x}(t)$ satisfies (2.5) for all admissible parameter uncertainties.

From Remark 2.2, to study the guaranteed cost estimation problem of Systems (2.1)–(2.2) with norm-bounded nonlinear uncertainty, it is sufficient to study the same problem for the following system:

$$\dot{x}(t) = Ax(t) + a_1 M_1(t)x(t) + Bu(t) + w(t), \quad x(0) = x_0 \tag{3.1}$$

$$y(t) = Cx(t) + a_2 M_2(t)x(t) + v(t) \tag{3.2}$$

where $x(t) \in \mathbf{R}^n$ is the system state, $u(t) \in \mathbf{R}^m$ is the system input, $y(t) \in \mathbf{R}^r$ is the measurement, $w(t) \in \mathbf{R}^n$ and $v(t) \in \mathbf{R}^r$ are the process and measurement noises, respectively. A, B and C are as in (2.1)–(2.2). $a_1 \geqslant 0$ and $a_2 \geqslant 0$ are known constant numbers. $M_1(t)$ and $M_2(t)$ are unknown matrices which represent time-varying parametric uncertainties and satisfy

$$M_1^T(t)M_1(t) \leqslant I, \; M_2^T(t)M_2(t) \leqslant I, \; \forall t. \tag{3.3}$$

For the simplicity of technique, we adopt the following assumption on $M_1(t)$ and $M_2(t)$.

ASSUMPTION 3.1 There exists a known constant matrix H such that $M_2(t) = HM_1(t)$ for all t.

We also assume that System (3.1) is quadratically stable (Khargonekar *et al.*, 1990), that is, there exists a symmetric positive definite matrix P such that

$$[A + a_1 M_1(t)]^T P + P[A + a_1 M_1(t)] < 0$$

for all uncertainties $M_1(t)$ satisfying (2.3).

To begin with the study of the robust state estimation problem, let us first define the estimation error

$$e(t) = x(t) - \hat{x}(t). \tag{3.4}$$

Then from Systems (2.1)–(2.2) and estimator (2.4), $e(t)$ satisfies the following dynamics

$$\begin{aligned}\dot{e}(t) = Ge(t) + (A - G - KC)x(t) + [\Delta A(t) - K\Delta C(t)]x(t) \\ + (B - B_1)u(t) + w(t) - Kv(t),\end{aligned} \tag{3.5}$$

where $\Delta A(t) \overset{\Delta}{=} a_1 M_1(t)$, and $\Delta C(t) \overset{\Delta}{=} a_2 H M_1(t)$.

Now, we have the argumented system of (2.1)–(2.2) and (3.5)

$$\dot{\bar{x}}(t) = [\bar{A} + \bar{H}M_1(t)E]\bar{x}(t) + \bar{B}u(t) + F\xi(t) \tag{3.6}$$

$$e(t) = [0 \quad I]\bar{x}(t) \tag{3.7}$$

where

$$\bar{x}(t) = \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ x(t) - \hat{x}(t) \end{bmatrix}$$

$\xi(t)$ is a white noise process with an identity covariance matrix and the matrices $\bar{A}$, $\bar{B}$, $\bar{H}$, F and E are defined by

$$\bar{A} = \begin{bmatrix} A & 0 \\ A - G - KC & G \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B \\ B - B_1 \end{bmatrix}, \quad FF^T = \begin{bmatrix} W & W \\ W & W + KVK^T \end{bmatrix} \tag{3.8}$$

$$\bar{H} = \begin{bmatrix} a_1 I \\ a_1 I - a_2 KH \end{bmatrix}, \quad E = [I \quad 0]. \tag{3.9}$$

DEFINITION 3.1 Given Systems (2.1)–(2.2), the state estimator (2.4) is said to be a stable quadratic state estimator if there exists a symmetric nonnegative matrix $Q \geqslant 0$ such that

$$[\bar{A} + \bar{H}M_1(t)E]Q + Q[\bar{A} + \bar{H}M_1(t)E]^T + FF^T \leqslant 0 \tag{3.10}$$

for all admissible uncertainties $M_1(t)$.

We now show that a stable quadratic state estimator is a guaranteed cost state estimator.

THEOREM 3.1 Assume that Systems (3.1)–(3.2) satisfy (3.3) and Assumptions (2.1) and (3.1), and is quadratically stable. Also, suppose that (2.4) is a stable quadratic state estimator for Systems (3.1)–(3.2) and let the symmetric nonnegative matrix Q be as defined in (3.10). Then, (2.4) is a guaranteed cost state estimator for Systems (3.1)–(3.2), and

$$\mathsf{E}\{(x - \hat{x})(x - \hat{x})^T\} \leqslant Q_{22}, \quad \text{or } \mathsf{E}\{(x - \hat{x})^T(x - \hat{x})\} \leqslant \mathrm{tr}(Q_{22})$$

is satisfied with Q_{22} being the 2–2 block of the matrix Q.

Proof. Let $\mathsf{E}\{\bar{x}(t)\bar{x}^T(t)\} = Q_F(t)$. From (3.6), by the result in Grimble & Johnson (1988), one has

$$\dot{Q}_F(t) + [\bar{A} + \bar{H}M_1(t)E]Q_F(t) + Q_F(t)[\bar{A} + \bar{H}M_1(t)E]^T + FF^T = 0. \tag{3.11}$$

Define $S(t) = Q - Q_F(t)$. One obtains from (3.10) and (3.11)

$$\dot{S}(t) + [\bar{A} + \bar{H}M_1(t)E]S(t) + S(t)[\bar{A} + \bar{H}M_1(t)E]^T + FF^T \leqslant 0. \tag{3.12}$$

Bearing in mind the fact that System (3.1) is quadratically stable and (2.4) is a stable quadratic estimator, it can be shown that $\bar{A} + \bar{H}M_1(t)E$ is exponentially stable (Bolzern *et al.*, 1996). On the other hand, note that System (3.1) is assumed to the quadratically stable (implying this system is *stable*) and the definitions of $S(t)$ and $Q_F(t)$, together with (2.4) is a stable quadratic state estimator and the initial time $t_0 \to -\infty$, it follows that System (3.1) is under steady state, which implies from (3.12) that $S(t) \geqslant 0$, $\forall t \geqslant 0$. That is, $Q_F(t) \leqslant Q$, $\forall t \geqslant 0$. Finally, from (3.7) we have

$$\mathsf{E}\{e(t)e^T(t)\} = [0 \quad I]Q_F(t)\begin{bmatrix} 0 \\ I \end{bmatrix} \leqslant Q_{22},$$

which implies that the estimator (2.4) provides a guaranteed cost matrix Q_{22} for System (3.1)–(3.2), and the proof is complete.

Now, let us design a robust filter for Systems (3.1)–(3.2), and show that the covariance of the estimation error will be guaranteed within a certain level for all admissible uncertainty. To this end, we first introduce the following concept of *stabilizing solution* of Riccati equation.

DEFINITION 3.2 Let A, W and R be known constant matrices of appropriate dimensions with W and R being symmetric. Then, a solution P of ARE

$$A^T P + PA + PWP + R = 0$$

is said to be stabilizing if the matrix $A + WP$ is stable.

Now, we are in a position to present our main results of this paper.

THEOREM 3.2 Assume that Systems (3.1)–(3.2) satisfy (3.3) and Assumptions (2.1) and (3.1), and are quadratically stable. Suppose there exists an $\varepsilon > 0$ such that the following conditions hold:

(a) There exists a stabilizing solution P to the ARE

$$AP + PA^T + \frac{1}{\varepsilon}P^2 + \varepsilon a_1^2 I + W = 0. \tag{3.13}$$

(b) There exists a stabilizing solution Q to the ARE

$$\begin{aligned} AQ + QA^T + \frac{1}{\varepsilon}QE^T EQ - (QC^T + \varepsilon a_1 a_2 H^T)(V + \varepsilon a_2^2 HH^T)^{-1} \\ \times (QC^T + \varepsilon a_1 a_2 H^T)^T + \varepsilon a_1^2 I + W = 0. \end{aligned} \tag{3.14}$$

Then, the estimator given by

$$\dot{\hat{x}}(t) = G\hat{x}(t) + B_1 u(t) + Ky(t) \tag{3.15}$$

where

$$\begin{aligned} G &= A + \frac{1}{\varepsilon}QE^T E - (QC^T + \varepsilon a_1 a_2 H^T)(V + \varepsilon a_2^2 HH^T)^{-1}C \\ K &= (QC^T + \varepsilon a_1 a_2 H^T)(V + \varepsilon a_2^2 HH^T)^{-1} \end{aligned}$$

is a stable quadratic estimator with guaranteed cost

$$\mathsf{E}\{(x - \hat{x})^T (x - \hat{x})\} \leqslant \operatorname{tr}(P - Q).$$

Proof. Define a matrix

$$X = \begin{bmatrix} P & P - Q \\ P - Q & P - Q \end{bmatrix},$$

where P and Q are the stabilizing solutions to (3.13) and (3.14), respectively. By standard matrix manipulations, it can be shown that the matrix X satisfies the following ARE

$$\bar{A}X + X\bar{A}^T + \frac{1}{\varepsilon}XE^T EX + \varepsilon \bar{H}\bar{H}^T + FF^T = 0, \tag{3.16}$$

where $\bar{A}$, $\bar{H}$, E and F are as in (3.8) and (3.9).

By applying Lemma 2.3 to (3.16), one obtains

$$[\bar{A} + \bar{H}M_1(t)E]X + X[\bar{A} + \bar{H}M_1(t)E]^T + FF^T \leqslant 0$$

for all $M_1(t)$ satisfying $M_1^T(t)M_1(t) \leqslant I$, $\forall t$. Therefore, from Theorem 3.1, it follows that (3.15) is a stable quadratic estimator and

$$\mathsf{E}\{(x - \hat{x})^T(x - \hat{x})\} \leqslant \mathrm{tr}(P - Q),$$

and the proof ends.

REMARK 3.1 It should be noted that the the positive semi-definiteness of the matrix $P - Q$ is ensured by the fact of, as in (3.16), X is symmetric and $\bar{A}$ is stable. In addition, if the pair (A, W) is controllable, it can be shown that $P - Q$ is positive definite, see, for example, Bolzern *et al.* (1996).

REMARK 3.2 Theorem 3.2 presents a sufficient condition for the solvability of the robust filtering problem of Systems (3.1)–(3.2) (consequently, (2.1)–(2.2)). If for a fixed $\varepsilon > 0$, the AREs (3.13) and (3.14) do admit stabilizing solutions P_s and Q_s, then, such P_s and Q_s are unique and turn out to be minimal, i.e. $P_s \leqslant P$ and $Q_s \leqslant Q$, where P and Q are any solutions of (3.13) and (3.14), respectively. Hence, P_s and Q_s provides the tightest upper bound of $\mathsf{E}\{(x - \hat{x})^T(x - \hat{x})\}$ for all admissible uncertainty. The problem of optimizing the bound with respect to ε has been investigated by Bolzern *et al.* (1994). It is shown that the above minimization problem is convex for trace functions of P_s and Q_s, which allows us to develop efficient numerical procedures for the computation of the minimal upper bound. Furthermore, the optimization problem of finding minimal upper bound of $\mathrm{tr}(P - Q)$ for covariance of the estimation error $\mathsf{E}\{(x - \hat{x})^T(x - \hat{x})\}$ may be solved by the linear matrix inequality technique Boyd *et al.* (1994), that is,

$$\begin{aligned} &\text{minimize } \mathrm{tr}(P - Q) \\ &\text{Subject to } \varepsilon > 0,\ Y \leqslant 0 \text{ and } Z \leqslant 0, \end{aligned}$$

where Y and Z stand for the left hand sides of (3.13) and (3.14), respectively.

Acknowledgements

This work was supported in part by the Australian Research Council under Grant A49532206 and in part by the Faculty Research Grant, Faculty of Information Technology, The University of South Australia.

REFERENCES

ANDERSON, B. D. O. & MOORE, J. B. 1979 *Optimal Filtering*. Englewood Cliffs, NJ: Prentice–Hall.

BERNSTEIN, D. S. & HADDAD, W. M. 1989 Steady-state Kalman filtering with an H_∞ error bound. *System & Control Lett.* **12**, 9–16.

BOLZERN, P., COLANERI, P., & DE NICOLAO, G. 1994 Optimal robust filtering with time-varying parameter uncertainty. *IEEE Trans Automat. Control* **39**, 623–626.

BOLZERN, P., COLANERI, P., & DE NICOLAO, G. 1996 Optimal robust filtering with time-varying parameter uncertainty. *Int. J. Control* **63**, 557–576.

BOYD, S., GHAOUI, L. E., FERON, E., & BALAKRISHNAN, V. 1994 *Linear Matrix Inequalities in System and Control Theory*. Philadelphia: SIAM.

GRIMBLE, M. J. & JOHNSON, M. A. 1988 *Optimal Control and Stochastic Estimation*. New York: John Wiley & Sons.

GROSSMAN, W. 1991 Bearings-only tracking: a hybrid coordinate system approach. *Proc. 30th IEEE Conf. on Decision & Control*. Brighton, England pp. 2032–2037.

HELFERTY, J. P. & MUDGETT, D. R. 1993 Optimal observer trajectories for bearings-only tracking by minimizing the trace of the cramér-rao lower bound. *Proc. 32nd IEEE Conf. on Decision & Control*. San Antonio, Texas pp. 936–939.

KHARGONEKAR, P. P., PETERSEN, I. R., & ZHOU, K. 1990 Robust stabilization of uncertain linear system: quadratic stabilizability and H_∞ control theory. *IEEE Trans Automat. Control* **35**, 356–361.

LE CADRE, J. P. & LAURENT-MICHEL, S. 1997 Optimizing the receiver maneuvers for berings-only tracking. *Technical report. Preprint.*

LÉVINE, J. & MARINO, R. 1992 Constant-speed target tracking via bearings-only measurements. *IEEE Trans Automat. Control* **28**, 174–182.

NARDONE, S., LINDGREN, A. G., & GONG, K. F. 1984 Fundamental properties and performance of conventional bearings-only target motion analysis. *IEEE Trans Automat. Control* **29**, 775–787.

PETERSEN, I. R. 1987 A stabilization algorithm for a class of uncertain linear systems. *System & Control Lett.* **8**, 351–357.

PETERSEN, I. R. & MCFARLANE, D. C. 1991 Robust state estimation for uncertain systems. *Proc. 30th IEEE Conf. Decision & Control*. Brighton, England.

SHI, P., DE SOUZA, C. E., & XIE, L. 1993 Robust H_∞ filtering for uncertain systems with sampled-data measurements. *Proc. 32nd IEEE Conf. Decision & Control*. San Antonio, Texas, USA pp. 793–798.

SHI, P. 1996 Robust filtering for uncertain systems with sampled measurements. *Int. J. Syst. Sci.* **27**, 1403–1415.

SHI, P., WANG, Y., & XIE, L. 1997 Robust filtering for interconnected uncertain systems under sampled measurements. *J. Dynamic Systems, Measurement, and Control, Transactions of the ASME* **119**, 337–340.

SHI, P. 1998 Filtering on sampled-data systems with parametric uncertainty. *IEEE Trans Automat. Control* **43**, 1022–1027.

SHI, P., BOUKAS, E. K., & AGARWAL, R. K. 1999 Kalman filtering for continuous-time uncertain systems with Markovian jumping parameters. *IEEE Trans Automat. Control* **44**, 1592–1597.

DE SOUZA, C. E., FU, M., & XIE, L. 1993 H_∞ analysis and synthesis of discrete-time systems with time-varying uncertainty. *IEEE Trans Automat. Control* **38**, 459–462.

SUN, W., NAPAL, K. M., & KHARGONEKAR, P. P. 1991 H_∞ control and filtering with sampled-data measurements. *Proc. 1991 American Control Conf.*. Boston, MA pp. 1652–1657.

XIE, L., DE SOUZA, C. E., & FRAGOSO, M. D. 1991 H_∞ filtering for linear periodic systems with parameter uncertainty. *System & Control Lett.* **17**, 343–350.

XIE, L. & SOH, Y. C. 1994 Robust Kalman filtering for uncertain systems. *System & Control Lett.* **22**, 123–129.

XIE, L., SOH, Y. C., & DE SOUZA, C. E. 1994 Robust Kalman filtering for uncertain discrete-time systems. *IEEE Trans Automat. Control* **39**, 1310–1314.