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DELAYED FRACTURE OF AN ORTHOTROPIC BODY SUBJECTED TO TWO-DIMENSIONAL DEFORMATION

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A modified δ_c-model is used to solve for the subcritical growth of macroscopic cracks of a normal cleavage in ageing fiber composite acted on by constant tensile stresses applied at infinity under two-dimensional deformation conditions. A continued fraction expansion is used to interpret the nonrational function of the integral Maslov–Arutyunyan ageing operators. The convergence of the resulting expansion is studied numerically, along with changes in the durability and safe loading conditions for an ageing composite as a function of the filling.

Consider an infinite orthotropic body with a through macroscopic normal-cleavage crack of length $2l_0$. The body consists of a unidirectional fiber composite with a hereditary-ageing binder and an elastic filler and is subject to two-dimensional deformation. The crack lies in the body parallel to the reinforcing fibers and retains its rectilinear shape as it propagates. Tensile stresses p are applied to the body at infinity in a direction perpendicular to the crack line. We shall assume that near the edges of the crack there are small (relative to the crack length) prefracture end zones of length d. We shall model the crack with a modified δ_c-model based on a prefracture zone of length d = const [1]. In order to ensure the condition of rectilinear growth of the crack in the binder, we shall assume that the binder is uniform and has a lower resistance to brittle fracture than the reinforcing fiber.

For estimating the effect of the mechanical parameters and of the age of the initial components of the body on the fracture characteristics, we shall base this study of delayed fracture in this hereditary-ageing body under time-independent subcritical loading on the theory of delayed fracture in viscoelastic materials [1].

We shall model this fiber composite as a uniform orthotropic material with reduced elastic moduli calculated using the formulas of [2]. If the x axis is directed along the reinforcing fiber, then [2]

$$E_{11} = \xi E_a + \eta E, \quad G_{12} = G \cdot \frac{1+\xi+\eta \cdot G/G_a}{\eta + (1+\xi)\cdot G/G_a},$$

$$\nu_{21} = \nu_{31} = \nu - \frac{(\kappa+1)\cdot(\nu-\nu_\alpha)\cdot\xi}{1+\eta+\kappa\xi+\eta\cdot(\kappa_a-1)\cdot G/G_a},$$

$$(E_{22})^{-1} = \frac{(\nu_{21})^2}{E_{11}} + \frac{1-\nu^2}{E}\cdot\left(\frac{2+(\kappa_a-1)\cdot G/G_a}{1+\eta+\kappa\xi+\eta\cdot(\kappa_a-1)\cdot G/G_a} - \frac{2\xi\cdot(1-G/G_a)}{\kappa+\xi+\eta\cdot G/G_a}\right),$$

$$\frac{\nu_{23}}{E_{23}} = -\frac{(\nu_{23})^2}{E_{11}} + \frac{\nu}{2G} + \frac{\kappa+1}{8G}\cdot\xi\cdot\left(\frac{\kappa-1-(\kappa_a-1)\cdot G/G_a}{1+\eta+\kappa\xi+(\kappa_a-1)\cdot\eta\cdot G/G_a} - \frac{2\cdot(1-G/G_a)}{\kappa+\xi+\eta\cdot G/G_a}\right). \tag{1}$$

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Here

$$\kappa = 3 - 4\nu; \quad \kappa_a = 3 - 4\nu_a;$$

E_{11}, E_{22}, G_{12}, ν_{21}, ν_{23}, and ν_{31} are the elastic moduli for the material in the body; ξ, E_a, G_a, and $\eta = 1 - \xi$, E, G are the volume content and elastic constants of the reinforcing and binder substances, respectively; and, ν and ν_a are the Poisson ratios of the reinforcement and binder.

We shall use an operator with a Maslov–Arutyunyan operator [3] core of the type

$$\lambda K(t, \tau) = -E\frac{\partial}{\partial\tau}\left\{\varphi(\tau)\left[1 - e^{-\gamma(t-\tau)}\right]\right\}, \tag{2}$$

where

$$\varphi(\tau) = C_0 + \frac{A_1}{\tau}; \lambda = \gamma E;$$

and C_0, A_1, and γ are constants for the material determined from creep experiments. The elastic-instantaneous value of the elastic modulus E in Eq. (2) is assumed to be constant.

The operator λK^* contains a resolvent operator $\lambda R^*(\lambda)$with a core of the form [3]

$$R(\lambda, t, \tau) = \varphi(\tau) - \left(\gamma \cdot \varphi(\tau) + \lambda \cdot \varphi^2(\tau) + \varphi'(\tau)\right)\int_\tau^t \left(\frac{\tau}{s}\right)^{\lambda \cdot A_1} \cdot e^{(\gamma + \lambda \cdot C_0)\cdot(\tau - s)}ds, \ \lambda \geq 0. \tag{3}$$

To simplify the solution, we shall assume that the Poisson ratio ν is time independent. Then the hereditary properties of the binder will be described by the operator

$$\frac{1}{E^*} = \frac{1}{E}\cdot\left(1 + \lambda K^*\right)$$

or

$$E^* = E\cdot\left(1 - \lambda R^*(\lambda)\right). \tag{4}$$

In accounting for the hereditary deformation properties of a fibrous orthotropic body, we shall neglect creep along the reinforcing fiber, i.e., we shall assume that $E_{11}^* = E_{11}$ = const and $\nu_{ij}^* = \nu_{ij}$ = const. Then, applying the algebra of resolvent operators and neglecting the terms with $G/G_a \ll 1$, from Eqs. (1) we obtain

$$\frac{1}{E_{22}^*} = \frac{1}{E_{22}}\cdot\left(1 + \chi_1 R^*(0)\right), \quad \frac{1}{G_{12}^*} = \frac{1}{G_{12}}\cdot\left(1 + \chi_2 R^*(0)\right), \tag{5}$$

where

$$\chi_1 = 2\lambda\cdot\left(1 - \nu^2\right)\cdot\frac{E_{22}}{E}\cdot\frac{\eta\cdot(\kappa + \xi\kappa - \xi)}{(1 + \eta + \xi\kappa)\cdot(\xi + \kappa)},$$

$$\chi_2 = \lambda\cdot\frac{\eta\cdot(1+\xi) + 2\eta^2\cdot G/G_a}{\eta\cdot(1+\xi) + \left[(1+\xi)^2 + \eta^2\right]\cdot G/G_a}. \tag{6}$$

A slowly growing crack in this model satisfies the Volterra principle [1]. Then the equation of the crack contour can be written in the form

$$\delta(x, t, \tau_1) = T^*\delta_0(x, l(t)). \tag{7}$$

Here $\delta_0(x, l(t))$ is a function of the force and geometric parameters and T^* is a linear Volterra integral operator of the second kind for which we must develop an expression.

For the given macroscopic crack ($d \ll l_0$) with $l(t) \le x \le L(t)$, the function $\delta_0(x, l(t))$ can be written in the form [5]

$$\delta_0(x, l(t)) = \frac{2}{\pi} \cdot \frac{\sqrt{\pi} K_1(t)}{\sqrt{8d}} \left(2\sqrt{d \cdot [d-(x-l(t))]} + (x-l(t)) \cdot \ln \frac{\sqrt{d} - \sqrt{d-(x-l(t))}}{\sqrt{d} + \sqrt{d-(x-l(t))}} \right), \tag{8}$$

where $K_1(t) = p\sqrt{\pi l(t)}$.

The operator T^* for this problem will have the form [6]

$$T^* = \sqrt{\frac{1-(\nu_{23})^2}{E_{11}E_{22}^*}} \cdot \sqrt{2 \cdot \left[\sqrt{(1-\nu_{12}\nu_{21}) \cdot \left(1-(\nu_{23})^2\right) \frac{E_{11}}{E_{22}^*}} - \nu_{21} \cdot (1+\nu_{23}) \right] + \frac{E_{11}}{G_{12}^*}}. \tag{9}$$

As can be seen from Eq. (9), the operator T^* is a complicated nonrational function of the hereditary-ageing operators E_{22}^* and G_{12}^*. Equation (9) for T^* will have to be expanded in order to solve the problem. To do this, we expand the operator T^* in a continued fraction using the Thiele formula [7]. The technique for expanding expressions of this type is discussed in detail in [1] and [8]. The major point in expanding expression (9) is to represent the square root of one resolvent operator or a linear combination of them as a continued fraction. Limiting ourselves to a finite number of terms in the continued fraction and summing it in accordance with the rules of the algebra of resolvent operators, we expand the square root as a linear combination of resolvent operators:

$$\sqrt{1+\sum_{i=1}^{M_1} \alpha_i R^*(\beta_i)} = 1 + 2 \cdot \cfrac{0{,}25 \cdot \sum_{i=1}^{M_1} \alpha_i R^*(\beta_i)}{1 + \cfrac{0{,}25 \cdot \sum_{i=1}^{M_1} \alpha_i R^*(\beta_i)}{1 + \cfrac{0{,}25 \cdot \sum_{i=1}^{M_1} \alpha_i R^*(\beta_i)}{1 +}}} \overset{M}{\approx} 1 + \sum_{j=1}^{\left[\frac{M+1}{2}\right]} \sum_{i=1}^{M_1} \alpha_{ij}^{\prime M} R^*\left(\beta_{ij}^{\prime M}\right). \tag{10}$$

This expansion technique with the retention of M terms in the expansions for the operator T^* yields the expression

$$T^* = T_0 \cdot \left(1 + \sum_{i=1}^{N} \mu_i R^*(\zeta) \right). \tag{11}$$

In the following we discuss the rate of convergence of the continued fraction representation of the operator T^* for a specific example.

Subcritical crack growth in viscoelastic bodies can be broken up into incubation, transition, and main periods [1]. According to the criterion for critical opening, the following inequality is satisfied at the tip of a crack during the incubation period:

$$T_0 \delta_0(l_0) \cdot \left(1 + \int_{\tau_1}^{t} \sum_{i=1}^{N} \mu_i R(\zeta_i, t, \tau) d\tau \right) \le \delta_c, \tag{12}$$

where, based on Eq. (8), we have

$$\delta_0(l_0) = \delta_0(l_0, l_0) = \sqrt{\frac{2d}{\pi}} K_{10}, \quad K_{10} = p\sqrt{\pi l_0}.$$

TABLE 1

Number of interactions	Age of material τ_1, days		
	7	21	120
1	5.2231	3.9258	3.3770
2	3.4361	2.7570	2.4813
3	3.8153	3.0003	2.6633
4	3.7178	2.9386	2.6183
10	3.7368	2.9506	2.6269

TABLE 2

Number of interactions	Age of material τ_1, days		
	7	21	120
1	4.9670	3.7507	3.2365
2	3.3302	2.6799	2.4159
3	3.6684	2.8970	2.5782
4	3.5840	2.8435	2.5391
10	3.6001	2.8537	2.5464

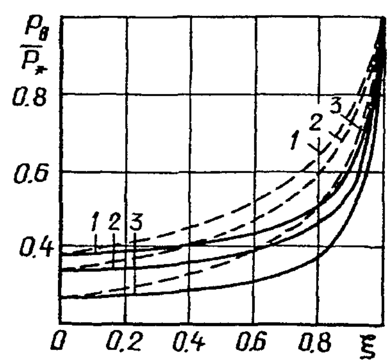

Fig. 1

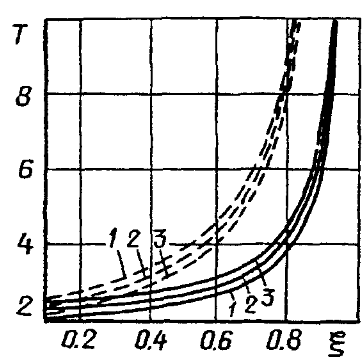

Fig. 2

Here equality is reached in Eq. (12) at the moment the crack moves.

The equation for determining the time t_* at which the incubation period ends is

$$1+\int_{\tau_1}^{t_*}\sum_{i=1}^{N}\mu_i R(\zeta_i, t_*, \tau)d\tau=\frac{K_{1c}}{K_{10}}. \tag{13}$$

Here K_{1c} is the critical stress intensity factor for two-dimensional deformation. The time $(t_* - \tau_1)$ determines the duration of the incubation period of crack growth.

Given the finiteness of the operator $R^*(\lambda)$, on letting t_* go to infinity, according to Eq. (13) for the definition of safe loading p_b at which the opening of the crack will never attain its critical value δ_c, we obtain

$$\frac{K_{1c}}{K_1^b}=1+\lim_{t\to\infty}\int_{\tau_1}^{t}\sum_{i=1}^{N}\mu_i R(\zeta_i, t, \tau)d\tau, \tag{14}$$

where K_1^b is the safe stress intensity factor.

As Eq. (14) shows, the safe load depends on the parameters of the kernel $R(\lambda, t, \tau)$, but also on the age of the material, τ_1. As the age τ_1 increases, K_1^b and p_b also increase. If $K_{10} > K_1^b$, then after a time t_* the crack begins to grow slowly. The equation for the transitional period describes the initial growth of the crack from length l_0 to $l(t_1) = l_0 + d$ and is given by

$$\frac{K_{1c}}{K_1(t)}=1+\frac{\delta_0(l(t), l_0)}{\delta_0(l(t))}\int_{\tau_1}^{t_*}\sum_{i=1}^{N}\mu_i R(\zeta_i, t, \tau)d\tau+\frac{1}{\delta_0(l(t))}\int_{t_*}^{t}\sum_{i=1}^{N}\mu_i R(\zeta_i, t, \tau)\delta_0(l(t), l(\tau))d\tau. \tag{15}$$

The time t_1 when the crack tip passes the initial end zone d will be the end of the transitional period. Since the function $\delta_0(l(t), l_0)$ in Eq. (15) equals zero in the first integral for $l(t) = l_0 + d$, Eq. (15) transforms to

$$\frac{K_{1c}}{K_1(t_1)}=1+\int_{t_*}^{t_1}\sum_{i=1}^{N}\mu_i R(\zeta_i, t_1, \tau)\delta_0(l_0+d, l(\tau))d\tau. \tag{16}$$

The equation for crack growth during the main period [1] is

$$\frac{K_{1c}}{K_1(t)}=1+\frac{1}{\delta_0(l(t))}\int_{t'}^{t}\sum_{i=1}^{N}\mu_i R(\zeta_i, t, \tau)\delta_0(l(t), l(\tau))d\tau. \tag{17}$$

Here the time t' is defined by the equation $l(t) - l(t') = d$.

It is evident from Eq. (17) that slow crack growth takes place as long as $K_1(t) < K_{1c}$. The time t_2 at which $K_1(t_2) = K_{1c}$ will be the end of the main period of crack growth, after which it enters a dynamic propagation regime. Thus, the longevity of the material under a given external load will be determined by the sum of the durations of each of the subcritical crack growth periods.

We now give some examples of the numerical solution of these equations for subcritical crack growth when the binder in the composite material is concrete and the filler material is steel reinforcement or glass fiber, i.e., we shall examine (steel) reinforced concrete and glass reinforced concrete. The mechanical parameters of these materials are:

- for concrete: $E = 2\cdot10^{10}$ Pa, $\nu = 0.167$, $A = 4.918\cdot10^{-10}$ day/Pa, $C_0 = 0.918\cdot10^{-10}$ Pa^{-1}, $\gamma = 0.026$ day^{-1};
- for steel reinforcement: $E = 2\cdot10^{11}$ Pa, $\nu = 0.3$; and
- for glass fiber: $E = 0.7\cdot10^{11}$ Pa, $\nu = 0.2$.

Tables 1 and 2 show the rate of convergence of the continued fraction representation of the operator T^* for a volume concentration ξ = 10% of the filler for steel and glass fiber reinforcements, respectively. The tables show that the rate of convergence of the continued fraction representation of T^* is quite high. The norm $\| T^* \|_E$ of the operator in the positive function E space yields an error of less than 10% by the second iteration, with the norm's approaching the exact value from below.

Figure 1 shows plots of the variation in the safe load with the volume content of reinforcing fibers. The dashed curves are for glass reinforced concrete and the smooth curves, for (steel) reinforced concrete. Curves *1*, *2*, and *3* correspond to materials with ages of 7, 21, and 120 days. These graphs show that glass reinforced concrete is more stable against cracks than steel reinforced concrete.

Figure 2 shows plots of the longevity of the body as a function of the volume content of reinforcing fibers for $p/p_* = 0.8$ and a prefracture zone size given by $d/l_0 = 0.05$. The numbers on the curves correspond to the same age data for the material as in Fig. 1. These curves show that glass reinforced concrete is more durable than steel reinforced concrete.

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