WEAK UNIQUELY COMPLETABLE SETS FOR FINITE GROUPS

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Abstract

Keedwell has shown that none of the groups of order less than 5 has a weak uniquely completable set. We prove that a weak uniquely completable set exists in a latin square based on a finite group if and only if the group is of order greater than 5.

1. *Introduction*

A *latin square* of order n is an $n \times n$ array with entries chosen from a set N of size n such that each element of N occurs exactly once in each row and column. We shall use $N = \{0, 1, \ldots, n-1\}$, and also label the rows and columns from 0 to $n-1$. We may also represent a latin square by a set of n^2 triples (i, j, k) such that element k appears in row i and column j. A latin square L is *based* on a group G if L can be bordered so as to form the Cayley table of G.

A *partial latin square* of order n is an $n \times n$ array with entries chosen from a set N of size n such that each element of N occurs at most once in each row and column. We shall also use the corresponding set of triples to represent a partial latin square.

A set of triples defining a partial latin square, P, of order n is *uniquely completable* (UC) if there is only one latin square, L, of order n that contains P.

The addition of a triple $t = (i, j, k)$ to a partial latin square P is said to be *forced* if one of the following holds.

(1) For all $h \neq i$, there exists z such that (h, j, z) or (h, z, k) is in P.
(2) For all $h \neq j$, there exists z such that (z, h, k) or (i, h, z) is in P.
(3) For all $h \neq k$, there exists z such that (i, z, h) or (z, j, h) is in P.

A UC set U is *strong* if we can find a sequence of sets of triples $U = S_1 \subset S_2 \subset \cdots \subset S_r = L$ such that each triple $t \in S_{v+1} - S_v$ is forced in S_v. (Some authors have used the term *semi-strong* for this property.) A UC set which is not strong is *weak*. A UC set which contains no smaller UC set is called *critical*. The problem of finding critical sets for latin squares was introduced by Nelder [**3**].

In [**2**] Keedwell showed that weak UC sets do not exist in latin squares of order less than 5, and reported that his research student, David Burgess, has obtained a weak UC set for the latin square of order 5 not based on the cyclic group. Keedwell then made the conjecture, also in [**1**], that no latin square based on a cyclic group has a weak critical set. In this paper we provide counter-examples to this conjecture for all orders greater than 5, by constructing weak UC sets for these groups. (A weak UC set necessarily contains a weak critical set.) Using Sylow's theorems and a consideration of special cases, we then show that all finite groups of order greater

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than 5 contain weak UC sets. A simple computer program has been used to verify that no weak UC set exists for the cyclic group of order 5.

2. *Weak UC sets for cyclic groups*

The set of triples $C_n = \{(i, j, i+j) \mid i = 0, 1, \dots, n-1;\ j = 0, 1, \dots, n-1\}$, where addition is modulo n, defines a latin square based on the cyclic group of order n. Figure 1 displays C_6. Figure 2 shows a strong UC set for C_6 (where • represents an empty cell). Figure 3 shows a weak UC set for C_6. (No entry is forced, but the $(0, 4)$ cell must contain either 4 or 5. The addition of $(0, 4, 5)$ forces the triple $(1, 4, 0)$, which then prohibits any element from appearing in the $(2, 4)$ cell. Conversely, the addition of $(0, 4, 4)$ results in a strong UC set.)

0	1	2	3	4	5
1	2	3	4	5	0
2	3	4	5	0	1
3	4	5	0	1	2
4	5	0	1	2	3
5	0	1	2	3	4

FIG. 1. C_6

0	1	2	3	4	•
1	2	3	4	•	•
2	3	4	•	•	•
3	4	•	•	•	•
4	•	•	•	•	•
•	•	•	•	•	•

FIG. 2. Strong UC set

0	1	2	3	•	•
1	2	3	4	•	•
2	3	4	•	•	1
3	4	•	•	•	•
•	•	•	•	•	3
•	•	•	•	3	•

FIG. 3. Weak UC set

The content of the following lemma is due to Nelder (see [**1**]) but is reproduced in this form as it is referred to in the proof of our Theorem. It states that the type of partial latin square displayed in Fig. 2 is a strong UC set in C_n for all n.

LEMMA 1. *The set of triples* $\{(i, j, i+j) \mid i = 0, \dots, n-2;\ j = 0, \dots, n-2-i\}$ *is a strong UC set for* C_n.

Proof. Consider the columns 0 to $n-1$ in order. The triples $(n-1-i, j, j-i-1)$ are forced as i ranges from j to 0.

To help prove that a UC set is weak, we introduce the idea of an array of alternatives.

DEFINITION. Let P be a partial latin square of order n defined on N. Then A_P is an array of alternatives for P if:

(i) A_P is an $n \times n$ array;
(ii) whenever the (i, j) cell of P is filled, the (i, j) cell of A_P is empty;
(iii) whenever the (i, j) cell of P is empty, the (i, j) cell of A_P contains all of the elements of N which do not appear in the ith row or jth column of P.

A UC set that is not a latin square has no forced additional triple if and only if its corresponding array of alternatives is such that each non-empty cell of A_P contains more than one element, and no element occurs exactly once in any row or column.

THEOREM. *The following set of triples is a weak UC set for* C_n:

$$
\begin{aligned}
S = {} & \{(0, j, j) \mid j = 0, \dots, n-3\} \\
& \cup \{(i, j, i+j) \mid i = 1, \dots, n-3;\ j = 0, \dots, n-2-i\} \\
& \cup \{(2, n-1, 1),\ (n-2, n-1, n-3),\ (n-1, n-2, n-3)\}.
\end{aligned}
$$

Proof. We first prove that S is UC. The $(0, n-2)$ cell must contain either $n-2$ or $n-1$. If we adjoin the triple $(0, n-2, n-1)$ to S, then $(1, n-2, 0)$ is forced, but no element may now be placed in the $(2, n-2)$ cell. Therefore we must adjoin the triple $(0, n-2, n-2)$, and then $(n-2, 0, n-2)$ is forced. The set $S \cup \{(0, n-2, n-2), (n-2, 0, n-2)\}$ is contained in C_n and contains the UC set of Lemma 1; it is therefore UC to C_n.

We now prove that S is weak, by showing that there are no forced triples. Consider the array of alternatives A_S.

First, we show that no element occurs exactly once in any row of A_S. This follows since in the ith row of S, at least two of the cells $(i, n-3)$, $(i, n-2)$, $(i, n-1)$ are empty, and the corresponding cells of A_S each contain every one of the elements $0, 2, \ldots, n-4, n-1$ that does not appear in row i of S. This leaves the elements 1 and $n-2$ to consider. (Note that S contains all n occurrences of $n-3$.) If row i of S does not contain a 1, then 1 appears in cells $(i, n-4)$ and $(i, n-3)$ of A_S. The element $n-2$ occurs only in cells $(0, n-2)$, $(0, n-1)$, $(n-2, 0)$, $(n-2, n-2)$, $(n-1, 0)$, $(n-1, n-1)$ of A_S, and so does not appear exactly once in any row of A_S.

Secondly, we show that no element occurs exactly once in any column of A_S. In the jth column of S, at least two of the cells $(n-3, j)$, $(n-2, j)$, $(n-1, j)$ are empty, and the corresponding cells of A_S each contain every one of the elements $0, \ldots, n-4, n-1$ that does not appear in column j of S. This leaves the element $n-2$ but, as noted above, this occurs twice in columns 0, $n-2$ and $n-1$ of A_S, and nowhere else.

Finally, we need to show that each non-empty cell of A_S contains at least two elements. This follows since each non-empty cell of A_S in row 0 or column 0 contains both $n-2$ and $n-1$, and all the other non-empty cells of A_S contain both 0 and $n-1$.

3. *Weak UC sets for non-cyclic groups*

In this section we prove that all finite groups of order greater than 5 possess weak UC sets. We begin with an easy lemma.

LEMMA 2. *Let H be a subgroup of G. If H has a weak UC set, then so does G.*

Proof. The Cayley table of G contains the Cayley table of H as a subsquare. On replacing this subsquare by the partial latin square corresponding to a weak UC set for H, we obtain a weak UC set for G.

By Cauchy's theorem, a group whose order is divisible by a prime p greater than 5 has a cyclic subgroup of order p and so, by Lemma 2 and the Theorem, weak UC sets exist for all such groups. This leaves only groups of orders $n = 2^a 3^b 5^c$, where $n > 5$, to consider. By the Sylow theorems, such a group has subgroups of orders 2^a, 3^b and 5^c. We show below that the groups of orders 2^3, 3^2 and 5^2 all have weak UC sets, so this leaves only the groups of orders $2^a 3^b 5^c$ with $a \in \{0, 1, 2\}$, $b, c \in \{0, 1\}$, to consider. Of these groups, only D_3, D_5, the dihedral groups of orders 6 and 10, A_4, the alternating group of order 12, and the group $\mathrm{Hol}(C_5)$ do not have subgroups of the kinds already considered or which are cyclic of composite order greater than 5. (For the groups up to order 30, see [**5**].) For a group of order 60, we may argue as follows. If G is a group of order 60, then let P be a Sylow 5-subgroup of G. If P is normal in G, then $|G/P| = 12$ and hence G/P has a subgroup K/P of order $m = 4$ or 6, where K is a subgroup of G and $|K| = 5m = 20$ or 30. If P is not normal in G, then G has exactly six Sylow 5-subgroups, and hence the normaliser of P has index 6 in G and is of order 10. A weak UC set for D_3 was found by Sittampalam [**4**], and examples for D_5 and A_4 are included below. Hol(C_5) contains D_5 as a subgroup and therefore has a weak UC set by Lemma 2.

For each of the groups (except $C_5 \times C_5$) considered, we present a weak UC set in the form of a partial latin square (entries in bold) which is superimposed onto its array of alternatives. The examples given are not critical, but have the property that there is no forced triple. As an aid to verifying that a set S has the UC property, we identify a cell, say (i, j), such that the (i, j) cell of A_S contains only two elements, say k_1 and k_2. The (i, j) cell is such that: (i) on adding the triple (i, j, k_1) to S, where k_1 is the element occurring in the (i, j) cell of the group-based square, we obtain a strong UC set; (ii) on adding the triple (i, j, k_2) to S, we obtain a sequence of forced triples leading to a cell, say (x, y), which cannot be filled (indicated by the triple $(x, y, ?)$).

The group $C_5 \times C_5$ is the largest of the groups, and its weak UC set is the most difficult to check. We have not presented its full array of alternatives, but for every non-empty cell we give enough alternatives to verify that there is no forced triple. The partial latin square has been partitioned into 5×5 blocks; when checking that the presented set is UC, we observe that each subsquare of the array is a strong UC set over the elements which have already appeared in the given block. It follows that when we have adjoined enough triples so that one of the blocks is filled correctly, this set is UC. Below the weak set for $C_5 \times C_5$, we present two sequences of triples: one beginning $(0, 15, 16)$, which ends in a contradiction, and another beginning $(0, 15, 15)$, which leads to a complete block. Since the $(0, 15)$ cell of the array of alternatives contains 15 and 16 only, it follows that the set is UC.

0	**1**	**2**	3, 7	**4**	3, 5, 6	6, 7	5, 7
1	**0**	**3**	2, 7	2, 5	4, 5	4, 7	**6**
2	3, 5	**0**	1, 3	1, 3, 5, 6	**7**	4, 6	4, 5
3, 4	2, 3	1, 4, 6	0, 1, 2, 3	**7**	3, 4, 6	**5**	0, 4
4, 5, 7	5, 7	4, 6, 7	0, 7	0, 5, 6	**1**	**2**	**3**
5, 7	**4**	1, 7	**6**	1, 5	**0**	**3**	**2**
6	2, 3, 7	4, 7	**5**	2, 3	3, 4	**0**	**1**
3, 7	**6**	**5**	**4**	0, 3	**2**	**1**	0, 7

Weak UC set for $C_2 \times C_2 \times C_2$: $(6, 4, 3)$, $(6, 1, 2)$, $(6, 2, 7)$, $(3, 1, 3)$, $(3, 0, 4)$, $(3, 5, 6)$, $(3, 2, 1)$, $(5, 2, ?)$.

0	1, 6	1, 2, 5	**3**	**4**	2, 5, 6	2, 6	**7**
1	**2**	**3**	**0**	5, 6	5, 6	**7**	**4**
2, 6	**3**	**0**	1, 2	5, 6	**7**	**4**	1, 2, 5
3	0, 1	1, 2	1, 2, 7	0, 7	**4**	**5**	**6**
2, 4	**5**	**6**	2, 4, 7	0, 7	**1**	0, 2, 3	2, 3
5	4, 6	**7**	2, 4	**1**	2, 3, 6	2, 3, 6	**0**
4, 6	**7**	1, 4	**5**	**2**	0, 3, 6	0, 3, 6	1, 3
7	0, 4	2, 4, 5	**6**	**3**	0, 2, 5	**1**	2, 5

Weak UC set for $C_2 \times C_4$: $(4, 7, 2)$, $(4, 0, 4)$, $(2, 0, 2)$, $(2, 7, 5)$, $(7, 7, ?)$.

0	**1**	2,3,6,7	2,3,5	**4**	2,5	6,7	5,6,7
1	**2**	3,7	0,3	**5**	**6**	0,7	**4**
2,3,4,5	3,5	**0**	1,2,3,5	1,6	**7**	4,6	5,6
2,3	0,3	1,2,3,6	0,1,2,3	**7**	**4**	**5**	0,6
4,5	**7**	4,6	0,5	0,6	**3**	**2**	**1**
5,7	**4**	1,7	**6**	0,1	0,1,5	**3**	**2**
6	0,5	1,4	**7**	**2**	0,1,5	0,4	**3**
2,7	**6**	**5**	**4**	**3**	0,2	**1**	0,7

Weak UC set for D_4: $(7,7,7)$, $(7,0,2)$, $(3,7,0)$, $(3,1,3)$, $(3,0,?)$.

0	**1**	**2**	**3**	**4**	**5**	6,7	6,7
1	**2**	**3**	**0**	**5**	6,7	6,7	**4**
2	**3**	**0**	**1**	**6**	4,7	4,7	**5**
3,4,7	0,7	1,7	2,4,7	2,7	1,2,4,6,7	**5**	0,1,2,3,6,7
3,4,7	0,7	**6**	**5**	2,7	1,2,4,7	0,1,2,4,7	0,1,2,3,7
5	**4**	1,7	**6**	**3**	1,2,7	0,1,2,7	0,1,2,7
6	**5**	**4**	2,7	**0**	**3**	1,2,7	1,2,7
4,7	**6**	**5**	2,4,7	**1**	**0**	**3**	2,7

Weak UC set for Q_3: $(0,6,7)$, $(1,6,6)$, $(1,5,7)$, $(2,5,4)$, $(2,6,?)$.

0,2	**1**	2,7	3,5	**4**	2,5	**6**	0,3,7	**8**
1	**2**	**0**	4,5,6	5,8	**3**	**7**	4,8	5,6
2,8	**0**	**1**	3,5	3,5,7,8	**4**	5,8	**6**	2,5,7
3	4,8	**5**	1,4,6	7,8	2,6,8	**0**	1,4,7,8	1,2,6,7
0,4,6,8	**5**	**3**	**7**	0,8	6,8	1,4,8	**2**	0,1,6
5	**3**	**4**	**8**	**6**	**7**	**2**	0,1	0,1
2,4,6,8	**7**	2,8	**0**	**1**	2,5,6,8	**3**	4,8	2,5,6
7	4,8	**6**	1,4	**2**	**0**	1,4,8	**5**	**3**
0,8	**6**	7,8	**2**	0,3,5,7,8	**1**	5,8	0,3,7,8	**4**

Weak UC set for $C_3 \times C_3$: $(0,0,2)$, $(0,5,5)$, $(0,3,3)$, $(2,0,8)$, $(2,6,5)$, $(2,3,?)$.

0	**1**	**2**	3,9	**4**	**5**	3,6	**7**	6,8,9	6,8,9
1	2,3,9	3,8,9	**4**	0,2,3	**6**	**7**	0,2,5,8	0,8,9	0,5,8,9
2,6	2,3	**4**	0,2,3	**1**	**7**	**8**	**9**	**5**	0,6
3	2,4	**0**	1,2,5	2,6	1,8	**9**	2,4,5,8	1,4,6,8	**7**
2,4	**0**	**1**	2,3	2,3,7	**9**	**5**	**6**	4,7,8	4,8
5	4,9	8,9	**7**	0,6	0,8	0,4,6	**3**	**2**	**1**
4,6,9	**5**	7,9	**8**	0,6,7	0,1	0,1,4,6	0,4	**3**	**2**
7	**6**	**5**	0,1,9	**8**	**2**	0,1,4	0,4	0,1,4,9	**3**
8	**7**	**6**	0,5	**9**	**3**	**2**	**1**	0,4	0,4,5
2,9	**8**	3,7,9	**6**	**5**	**4**	0,1,3	0,2	0,1,7,9	0,9

Weak UC set for D_5: $(9,7,0)$, $(7,7,4)$, $(6,7,?)$.

0	**1**	2, 6	**3**	**4**	**5**	2, 6, 10	**7**	**8**	**9**	6, 10	**11**
1	**0**	3, 6	**2**	5, 9	**4**	**7**	3, 6	5, 9	**8**	**11**	**10**
2	**3**	**0**	1, 4	**6**	**7**	4, 10	**5**	1, 10	**11**	**8**	**9**
3	**2**	**1**	**0**	5, 7	**6**	5, 8	**4**	**11**	**10**	**9**	5, 7, 8
4	**7**	**5**	1, 6, 11	**8**	1, 2, 11	**9**	3, 6, 10	**0**	3, 6	1, 6, 10	2, 3
5	**6**	3, 4	1, 4, 7	7, 9	**10**	4, 8	**11**	1, 9	**2**	0, 1, 4	0, 3, 7, 8
6, 9	4, 5, 8, 9	**7**	4, 6	2, 5, 9, 10	2, 8, 9	**11**	0, 6, 8, 10	2, 5, 9, 10	**1**	**3**	0, 2, 5, 8
7	4, 8	4, 6, 11	**5**	10, 11	8, 11	4, 6, 8, 10	**9**	**3**	**0**	**2**	**1**
8	**10**	2, 6, 11	**9**	**0**	2, 11	**3**	**1**	**4**	5, 6	**7**	2, 5
6, 9	**11**	**10**	**8**	**1**	**3**	2, 5, 6	0, 6	2, 5, 9	**7**	0, 6	**4**
10	4, 8	**9**	4, 7, 11	2, 7, 11	**0**	**1**	3, 8	**6**	3, 4	**5**	2, 3, 7, 8
11	4, 5, 9	**8**	**10**	**3**	1, 9	**0**	**2**	**7**	4, 5	1, 4	**6**

Weak UC set for A_4: (9, 0, 6), (9, 7, 0), (9, 10, ?).

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	•	•	•	•	19	20	21	22	23	24
1	2	3	4	0	6	7	8	9	5	11	12	13	14	10	•	17	18	19	15	•	22	23	•	•
2	3	4	0	1	7	8	9	5	6	12	13	14	10	11	17	•	19	•	16	•	23	•	•	21
3	4	0	1	2	8	9	5	6	7	13	14	10	11	12	18	•	•	•	17	23	•	20	21	22
4	0	1	2	3	9	5	6	7	8	14	10	11	12	13	19	15	16	•	•	•	•	•	22	•
5	6	7	8	9	10	11	12	13	14	15	16	•	18	19	20	•	•	23	24	0	1	2	3	4
6	7	8	9	5	11	12	13	14	10	16	•	•	•	•	•	•	23	24	20	1	2	3	4	0
7	8	9	5	6	12	13	14	10	11	17	•	•	15	16	22	23	•	•	21	2	3	4	0	1
8	9	5	6	7	13	14	10	11	12	•	•	15	•	17	23	24	•	•	•	3	4	0	1	2
9	5	6	7	8	14	10	11	12	13	19	15	•	17	18	24	20	21	•	•	4	0	1	2	3
10	11	12	13	14	15	•	17	18	•	20	21	22	23	24	0	1	2	3	4	5	6	7	8	9
11	12	13	14	10	16	17	18	19	•	21	•	•	24	•	1	2	3	4	0	6	7	8	9	5
12	13	14	10	11	•	18	•	15	•	•	•	•	•	•	2	3	4	0	1	7	8	9	5	6
13	14	10	11	12	18	•	•	16	17	23	24	•	•	22	3	4	0	1	2	8	9	5	6	7
14	10	11	12	13	19	•	16	•	•	•	20	•	22	23	4	0	1	2	3	9	5	6	7	8
15	16	17	•	19	20	21	22	23	•	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
16	17	18	19	15	21	22	23	•	•	1	2	3	4	0	6	7	8	9	5	11	12	13	14	10
17	18	19	•	16	22	23	•	•	21	2	3	4	0	1	7	8	9	5	6	12	13	14	10	11
18	19	•	•	•	23	•	20	•	•	3	4	0	1	2	8	9	5	6	7	13	14	10	11	12
•	15	16	•	18	•	20	•	22	23	4	0	1	2	3	9	5	6	7	8	14	10	11	12	13
20	21	22	•	•	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
•	•	•	•	20	1	2	3	4	0	6	7	8	9	5	11	12	13	14	10	16	•	•	•	15
•	•	•	•	21	2	3	4	0	1	7	8	9	5	6	12	13	14	10	11	•	18	•	15	•
23	24	•	•	22	3	4	0	1	2	8	9	5	6	7	13	14	10	11	12	18	•	•	16	17
24	20	21	•	•	4	0	1	2	3	9	5	6	7	8	14	10	11	12	13	•	15	16	•	18

Weak UC set for $C_5 \times C_5$: cell (0, 15) contains either 15 or 16.

(0, 15, 16)
(0, 16, 18), no alternative
(1, 15, 21), no alternative
(1, 20, 24), no alternative
(2, 16, 22), no alternative
(2, 20, ?)

(0, 15, 15)
(0, 17, 17), no alternative
(1, 15, 16), 16 must appear in $c15$
(1, 20, 21), 21 must appear in $r1$
(1, 23, 24), 24 must appear in $r1$
(1, 24, 20), 20 must appear in $r1$
(4, 18, 17), 17 must appear in $c18$
(4, 20, 24), no alternative

(4, 21, 20), no alternative
(4, 22, 21), 21 must appear in $r4$
(4, 24, 23), no alternative
(2, 20, 22), no alternative
(2, 22, 24), 24 must appear in $r2$
(2, 23, 20), no alternative
(3, 17, 15), 15 must appear in $c17$
(3, 21, 24), 24 must appear in $r3$

															15 16	16 18	15 17	17 18						
															16 21					21 24			20 24	16 20
																18 22		15 20		22 24		15 18	20 24	
																16 19	15 24	15 16			19 24			
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Partial array of alternatives of the weak UC set for $C_5 \times C_5$.

In conclusion, we conjecture that all but a finite number of latin squares contain a weak UC set.

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