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# Orthogonality Criteria for Compactly Supported Refinable Functions and Refinable Function Vectors

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*ABSTRACT. A refinable function $\phi(x) : \mathbb{R}^n \to \mathbb{R}$ or, more generally, a refinable function vector $\Phi(x) = [\phi_1(x), \ldots, \phi_r(x)]^T$ is an $L^1$ solution of a system of (vector-valued) refinement equations involving expansion by a dilation matrix A, which is an expanding integer matrix. A refinable function vector is called orthogonal if $\{\phi_j(x-\alpha) : \alpha \in \mathbb{Z}^n, 1 \le j \le r\}$ form an orthogonal set of functions in $L^2(\mathbb{R}^n)$. Compactly supported orthogonal refinable functions and function vectors can be used to construct orthonormal wavelet and multi-wavelet bases of $L^2(\mathbb{R}^n)$. In this paper we give a comprehensive set of necessary and sufficient conditions for the orthogonality of compactly supported refinable functions and refinable function vectors.*

## 1. Introduction

Let $A$ be an expanding matrix in $M_n(\mathbb{Z})$, that is, one with integer entries and all eigenvalues $|\lambda| > 1$. A *refinable function* $\phi(x) : \mathbb{R}^n \to \mathbb{R}$ is a solution to a *refinement equation* with dilation matrix $A$,

$$\phi(x) = \sum_{\alpha \in \mathbb{Z}} c_\alpha \phi(Ax - \alpha) , \tag{1.1}$$

in which $\{c_\alpha : \alpha \in \mathbb{Z}\}$ are real coefficients. More generally, a vector valued function $\Phi(x) = [\phi_1(x), \ldots, \phi_r(x)]^T$ is called a *refinable function vector*, if it satisfies a *vector refinement equation* with dilation $A$,

$$\Phi(x) = \sum_{\alpha \in \mathbb{Z}^n} C_\alpha \Phi(Ax - \alpha) , \tag{1.2}$$

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where each $C_\alpha$ is a matrix in $M_r(\mathbb{R})$. We call $n$ the *dimension* and $r$ the *vector-multiplicity* of the refinable function vector. We only consider the case that such functions and vector-valued functions have all components in $L^1(\mathbb{R}^n)$.

Refinable function vectors are natural generalizations of refinable functions ($r = 1$). The latter have been studied extensively due to their applications in constructing compactly supported orthonormal wavelet bases and in approximation theory, see Daubechies [9], [10]. General constructions are based on multiresolution analysis, for which see Mallat [28] and Jia and Shen [21]. More recently, refinable function vectors have been used to construct orthonormal *multiwavelet bases*, see for example Cohen et al. [4], Donovan et al. [12], Goodman and Lee [14], and Goodman et al. [15]. Multiwavelets can be made to combine smoothness with small support, an advantage that may be important in applications.

In constructing orthonormal wavelet or multiwavelet bases, one requires that all integer translates of refinable functions or function vectors be orthogonal. A fundamental question in constructing orthonormal wavelet or multiwavelet bases is thus: under what conditions does a refinable function or function vector $\Phi(x)$ have the property that all its integer translates $\{\Phi(x-\alpha) : \alpha \in \mathbb{Z}^n\}$ are orthogonal?

This paper addresses the above question by giving a collection of necessary and sufficient conditions for orthogonality, derived in terms of the coefficients of the refinement equations and the dilation matrix $A$. We treat only the case where the vector refinement equation has finitely many nonzero coefficients. In this case, if the equation has a solution in $L^1(\mathbb{R}^n)$, then it must be compactly supported.[1] Also in this case, there is in principle a finite algorithm to determine whether a given vector refinement equation has a nonzero solution which is orthogonal in the sense of Definition 1 below. The criteria of this paper typically do not make sense in the case of infinitely many nonzero coefficients, but some sufficient conditions have been obtained by Conze et al. [7] in the infinite coefficient case.

Various results regarding the orthogonality of compactly supported refinable functions and function vectors are known, especially for $r = 1$ and $n = 1$. Many (but not all) of these results generalize to higher dimensions ($r = 1$ and $n > 1$), and to compactly supported refinable function vectors. However, few of these generalizations have been documented, and even in those papers which discuss higher dimensional cases, the dilation matrix $A$ was usually chosen to be $2I_n$. As we see from Theorems 2 and 3 below, orthogonality conditions vary for different dilation matrices $A$. The object of this paper is to provide a comprehensive set of orthogonality criteria for compactly supported refinable functions and function vectors in the most general setting.

**Definition 1.** Let $\Phi(x)$ be a compactly supported refinable function vector. We say that $\Phi(x)$ is *orthogonal* if $\int_{\mathbb{R}^n} \Phi(x)dx \neq 0$ and

$$\int_{\mathbb{R}^n} \Phi(x-\alpha)\Phi^T(x-\beta)dx = \delta_{\alpha,\beta}\Lambda, \qquad \alpha, \beta \in \mathbb{Z}^n , \tag{1.3}$$

where $\delta_{\alpha,\beta}$ denotes the standard Kronecker symbol, and $\Lambda$ is a diagonal matrix with positive diagonal entries.

The condition $\int_{\mathbb{R}^n} \Phi(x)dx \neq 0$ is necessary[2] for the construction of multiwavelet bases associated to a multiresolution analysis. It is well known that for a compactly supported refinable function vector $\Phi(x)$ to be orthogonal the coefficient matrices $C_\alpha$ of the corresponding vector refinement equation (1.2) must satisfy the necessary conditions encoded in (i) and (ii) of the following definition.

[1] The converse is false, see Strang et al. [33]. Furthermore, a refinement equation with finitely many nonzero coefficients may also have a noncompactly supported $L^2$ solution, see Malone [29].

[2] This condition is automatically fulfilled under the orthogonal coefficients condition, see Lemma 1 (4) below.

**Definition 2.** The vector refinement equation (1.2) with finitely many $C_\alpha \neq 0$ satisfies the *orthogonal coefficients condition* (with respect to $\Lambda$, where $\Lambda$ is a diagonal matrix with positive diagonal entries) if the coefficients $C_\alpha$ satisfy the two properties

(i) 1 is an eigenvalue of the matrix $|\det(A)|^{-1} \sum_{\alpha \in \mathbb{Z}^n} C_\alpha$.

(ii) For $\beta \in \mathbb{Z}^n$,

$$\sum_{\alpha \in \mathbb{Z}^n} C_\alpha \Lambda C^*_{\alpha + A\beta} = \delta_{0,\beta} \, |\det(A)| \, \Lambda \, . \tag{1.4}$$

The necessity of condition (i) for orthogonality follows from Proposition 1 below. A proof of condition (ii) can be found in Flaherty and Wang [13].

Unfortunately, the orthogonal coefficients condition is not sufficient for the orthogonality of the corresponding refinable function vector $\Phi(x)$, even for $r = 1$. The simplest counterexample, which has $r = 1$ and $n = 1$, is the refinement equation

$$\phi(x) = \phi(2x) + \phi(2x - 3) \, .$$

It satisfies the orthogonal coefficients condition, but the solution $\phi(x) = \chi_{[0,3)}(x)$ has non-orthogonal integer translates. To ensure orthogonality of refinable functions and function vectors, additional conditions are needed. In the nonvector case $r = 1$, $n = 1$, such conditions were found by various authors, and the most prominent of these conditions is Cohen's Criterion, due to Cohen [3]. We shall list them in Section 3. It should be pointed out that many of the criteria are given in the contrapositive form as conditions for $\Phi(x)$ *not* being orthogonal.

The contents of this paper are as follows: in Section 2 we state the orthogonality criteria for compactly supported refinable function vectors with arbitrary vector-multiplicity $r$, and in Section 3 we state a larger set of orthogonality criteria that are available for the special case $r = 1$, i.e., for compactly supported refinable functions. These criteria are then proved in Section 4 for arbitrary $r$ and in Section 5 for $r = 1$.

We add a comment on the novelty of the results. Many of the results for compactly supported refinable function vectors stated in Section 2 are new, as is the Generalized Cohen's Criterion stated there. In particular criterion (d) in Theorem 2 is new and (c) is stated for the first time. The proofs extend some of the ideas of the $r = 1$ case stated as Theorem 3 (a)–(d) in Section 3, but have extra complexity arising from products of matrices. The results in Section 3 for $r = 1$ and arbitrary dimension $n$ have not all been stated before, but we do not claim significant novelty in the proofs. The most important idea leading to the criteria in Theorem 3 (e)–(f) is a result on transfer operators due to Cerveau et al. [2]. Other orthogonality criteria for the case $r = 1$ based on this result were derived by Conze et al. [7]. Further remarks on previous results appear at the end of Section 3.

We are greatly indebted to K. Gröchenig for introducing us to this problem. The results and techniques in his paper [16] for the case $r = 1$ and $n = 1$ inspired our results. Several of his proofs generalize to dimension $n > 1$, see the discussion after Theorem 3. We are also indebted to Ingrid Daubechies, Andy Haas, Chris Heil, and Jianao Lian for helpful discussions and references. Finally, we would like to thank the anonymous referee for carefully reading the manuscript and providing valuable comments and suggestions.

## 2. Orthogonality Criteria for Refinable Function Vectors

Throughout this paper we will be concerned with compactly supported refinable function vectors. Therefore, we assume that the vector refinement equation

$$\Phi(x) = \sum_{\alpha \in \mathbb{Z}^n} C_\alpha \Phi(Ax - \alpha) \tag{2.1}$$

where $C_\alpha \in M_r(\mathbb{R})$ has only finitely many nonzero coefficient matrices $C_\alpha$. In this section we state orthogonality criteria; the proofs are given in Section 4.

**Definition 3.** For a given vector refinement equation (2.1) we define its *symbol* $\mathfrak{m}(\xi)$ to be

$$\mathfrak{m}(\xi) := |\det(A)|^{-1} \sum_{\alpha \in \mathbb{Z}^n} C_\alpha e^{-2\pi i \langle \alpha, \xi \rangle} . \tag{2.2}$$

The symbol $\mathfrak{m}$ together with the expanding integer matrix $A$ specifies the vector refinement equation uniquely, where we view $\mathfrak{m}$ as a formal object containing all the coefficients $C_\alpha$. However, we also view the symbol as defining a matrix-valued function $\mathfrak{m}(\xi) : \mathbb{R}^n \to M_r(\mathbb{C})$. Suppose that $\Phi(x)$ is a refinable function vector satisfying (2.1). Then the Fourier transform of $\Phi(x)$ satisfies

$$\widehat{\Phi}(\xi) = \mathfrak{m}\left(B^{-1}\xi\right) \widehat{\Phi}\left(B^{-1}\xi\right) , \tag{2.3}$$

where $B := A^T$, and the Fourier transform is applied term-by-term to the vector $\Phi(x)$. Denote

$$L_p^r\left(\mathbb{R}^n\right) := \left\{\Phi(x) = [\phi_1(x), \ldots, \phi_r(x)]^T : \text{ each } \phi_j(x) \in L^p\left(\mathbb{R}^n\right)\right\} . \tag{2.4}$$

The following is a necessary condition for the orthogonality of $\Phi(x)$:

***Proposition 1.***
*Let $\Phi(x)$ be a compactly supported orthogonal refinable function vector satisfying*

$$\Phi(x) = \sum_{\alpha \in \mathbb{Z}^n} C_\alpha \Phi(Ax - \alpha)$$

*with finitely many $C_\alpha \neq 0$. Then 1 is a simple eigenvalue of the $r \times r$ matrix $\mathfrak{m}(0)$, and all other eigenvalues $\lambda$ of $\mathfrak{m}(0)$ satisfy $|\lambda| < 1$.*

Proposition 1 is a corollary of a stronger result of Hogan [20], in which the orthogonality of $\Phi(x)$ is replaced by the weaker condition of stability. We include an independent proof of Proposition 1 in Section 4 for completeness.

To state the general orthogonality criteria we must introduce the transfer operator $\mathcal{C}_\mathfrak{m}$ associated to the symbol $\mathfrak{m}$ and dilation matrix $A$ [and hence to (2.1)]. Let $\Omega_{r\times r}(\mathbb{R}^n)$ denote the linear space of $r \times r$ Hermitian matrices whose entries are trigonometric polynomials with complex coefficients, i.e., functions of the form $g(e^{-2\pi i \xi_1}, \ldots, e^{-2\pi i \xi_n})$ where $g$ is a Laurent polynomial in $n$ variables, with $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$. Note that each $F(\xi) \in \Omega_{r\times r}(\mathbb{R}^n)$ is $\mathbb{Z}^n$-periodic, so we may view $\Omega_{r\times r}(\mathbb{R}^n)$ as a subspace of the Hilbert space $\left(L^2(\mathbb{T}^n)\right)^{r\times r}$. For any trigonometric polynomial $F(\xi) = \sum_{\gamma\in\mathbb{Z}^n} F_\gamma e^{-2\pi i \langle \gamma, \xi\rangle}$ of matrix coefficients we define its support to be

$$\operatorname{supp}(F) = \left\{\gamma \in \mathbb{Z}^n : F_\gamma \neq 0\right\} .$$

**Definition 4.** The *transfer operator* $\mathcal{C}_\mathfrak{m}$ is a linear operator on $\Omega_{r\times r}(\mathbb{R}^n)$ defined by

$$\mathcal{C}_\mathfrak{m} F(\xi) := \sum_{d\in\mathcal{E}} \mathfrak{m}\left(B^{-1}(\xi + d)\right) F\left(B^{-1}(\xi + d)\right) \mathfrak{m}^*\left(B^{-1}(\xi + d)\right) , \tag{2.5}$$

in which $B = A^T$ and $\mathcal{E}$ is a complete set of coset representatives of $\mathbb{Z}^n / B(\mathbb{Z}^n)$.

It is not hard to check, using the computations in Section 4, that $\mathcal{C}_\mathfrak{m}(F) \in \Omega_{r\times r}(\mathbb{R}^n)$ for any $F \in \Omega_{r\times r}(\mathbb{R}^n)$, and it is independent of the choice of the coset representatives $\mathcal{E}$. Furthermore, if (2.1) satisfies the orthogonal coefficients condition with respect to $\Lambda$, then $\mathcal{C}_\mathfrak{m}\Lambda = \Lambda$. The linear space $\Omega_{r\times r}(\mathbb{R}^n)$ is infinite-dimensional, but we will show that when the vector refinement equation with symbol $\mathfrak{m}$ has finitely many nonzero coefficients we can restrict the action of the transfer operator to certain finite dimensional invariant subspaces of $\Omega_{r\times r}(\mathbb{R}^n)$ depending on the symbol $\mathfrak{m}$ and on $A$ which contains the crucial information for orthogonality.

We call a nonempty set $\mathcal{S} \subseteq \mathbb{Z}^n$ $(\mathfrak{m}, A)$*-invariant* if for any $\gamma \notin \mathcal{S}$ the elements $A\gamma+\alpha-\beta \notin \mathcal{S}$ for all $\alpha$, $\beta \in \operatorname{supp}(\mathfrak{m})$. An important $(\mathfrak{m}, A)$-invariant set is

$$\mathcal{S}_{\mathfrak{m},A} := \left\{\gamma \in \mathbb{Z}^n : T_{\mathfrak{m},A} \cap \left(T_{\mathfrak{m},A} + \gamma\right) \neq \emptyset\right\} \tag{2.6}$$

where $T_{\mathfrak{m},A}$ is the attractor of the iterated function system $\{A^{-1}(x+\gamma) : \gamma \in \operatorname{supp}(\mathfrak{m})\}$. Clearly $\mathcal{S}_{\mathfrak{m},A}$ is finite if $\operatorname{supp}(\mathfrak{m})$ is.

***Proposition 2.***

(i) $\mathcal{S}_{\mathfrak{m},A}$ *is* $(\mathfrak{m}, A)$*-invariant.*

(ii) *Let* $\mathcal{S}$ *be a finite* $(\mathfrak{m}, A)$*-invariant set. Then*

$$\Omega_{r\times r}\left(\mathbb{R}^n, \mathcal{S}\right) := \left\{F(\xi) \in \Omega_{r\times r}\left(\mathbb{R}^n\right) : \operatorname{supp}(F) \subseteq \mathcal{S}\right\}.$$

*is a* $\mathbf{C}_{\mathfrak{m}}$*-invariant finite dimensional subspace of* $\Omega_{r\times r}(\mathbb{R}^n)$.

By results of Cohen et al. [5] or Heil and Collela [19], if 1 is a simple eigenvalue of $\mathfrak{m}(0)$ and all other eigenvalues $\lambda$ of $\mathfrak{m}(0)$ have $|\lambda| < 1$, then for $B = A^T$ the infinite (right) product

$$\mathfrak{p}(\xi) := \prod_{j=1}^{\infty} \mathfrak{m}\left(B^{-j}\xi\right) \tag{2.7}$$

converges uniformly on any compact set of $\mathbb{R}^n$. This defines $\mathfrak{p}(\xi) : \mathbb{R}^n \to M_r(\mathbb{C}^n)$. We have:

***Theorem 1.***

*Let* $\Phi(x)$ *be a compactly supported refinable function vector satisfying*

$$\Phi(x) = \sum_{\alpha\in\mathbb{Z}^n} C_\alpha \Phi(Ax-\alpha)$$

*where* $A \in M_n(\mathbb{Z})$ *is expanding and finitely many* $C_\alpha \neq 0$. *Suppose that the vector refinement equation satisfies the orthogonal coefficients condition and that 1 is a simple eigenvalue of* $\mathfrak{m}(0)$ *while all other eigenvalues* $\lambda$ *of* $\mathfrak{m}(0)$ *satisfy* $|\lambda| < 1$. *Then the following statements are equivalent:*

(a) $\Phi(x)$ *is not orthogonal.*

(b) *There exists an* $F(\xi) \in \Omega_{r\times r}(\mathbb{R}^n)$, $F(\xi) \neq a\Lambda$ *for any* $a \in \mathbb{C}$, *such that* $\mathbf{C}_{\mathfrak{m}}F = F$.

(c) *Let* $\mathcal{S}$ *be a finite* $(\mathfrak{m}, A)$*-invariant set containing* $\mathcal{S}_{\mathfrak{m},A}$. *The eigenvalue 1 of* $\mathbf{C}_{\mathfrak{m}}$ *restricted to* $\Omega_{r\times r}(\mathbb{R}^n, \mathcal{S})$ *is a multiple eigenvalue.*

(d) *There exist* $\eta \in \mathbb{R}^n \setminus \mathbb{Z}^n$ *and a nonzero vector* $u_0 \in \mathbb{C}^r$ *such that*

$$u_0^*\mathfrak{p}(\eta+\alpha) = 0, \quad \textit{all}\ \alpha \in \mathbb{Z}^n. \tag{2.8}$$

The equivalence of (a) and (b) in Theorem 1 was established by several authors in one dimension for the dilation 2, see Plonka [31] and Lian [27]. It was established in all dimensions for the dilation matrix $A = 2I_n$ in Shen [32], and his proof should generalize to work for an arbitrary dilation matrix $A$. In addition, it was shown in [32] that under the hypotheses of Theorem 1 the orthogonality of $\Phi(x)$ is equivalent to the stability of $\Phi(x)$ and is equivalent to the $L^2$-convergence of the cascade algorithm. Several variations of criterion (b) were also given in [27].

**Remark.** We shall see in Section 4 that the equivalence of (a) and (c) relies only on the orthogonal coefficients condition, not on the assumptions regarding the eigenvalues of $\mathfrak{m}(0)$. The equivalence of (a) and (c) gives rise to an algorithm for checking the orthogonality of a refinable function vector $\Phi(x)$, which is a generalization of the algorithm in Lawton [25] for $n = 1$ and $r = 1$. In fact, all we need is to find a finite $(\mathfrak{m}, A)$-invariant set $\mathcal{S}$ containing $\mathcal{S}_{\mathfrak{m},A}$ and check the multiplicity of the eigenvalue 1 for $\boldsymbol{C}_{\mathfrak{m}}$ restricted to $\Omega_{r\times r}(\mathbb{R}^n, \mathcal{S})$. Such a set is quite easy to find. Since $A$ is expanding, there exists a norm $\|.\|$ on $\mathbb{R}^n$ such that $\|A\| \geq s > 1$. Let $L$ be the diameter of supp($\mathfrak{m}$). One such $\mathcal{S}$ is

$$\mathcal{S} = \left\{\alpha \in \mathbb{Z}^n : \|\alpha\| \leq \frac{L}{s-1}\right\} . \tag{2.9}$$

The drawback with this $\mathcal{S}$ is that it is often much larger than $\mathcal{S}_{\mathfrak{m},A}$, making the dimension of $\Omega_{r\times r}(\mathbb{R}^n, \mathcal{S})$ much larger than necessary. Fortunately there is a simple algorithm to find $\mathcal{S}_{\mathfrak{m},A}$. Here we skip the details; they can be found in Strichartz and Wang [34].

A corollary of Theorem 1 is the following generalization of Cohen's Criterion. Recall that a set $K \subset \mathbb{R}^n$ is a *fundamental domain* of $\mathbb{Z}^n$ if $K$ is congruent to $[0, 1)^n$ modulo $\mathbb{Z}^n$.

***Corollary 1 (Generalized Cohen's Criterion).***

*Under the assumptions of Theorem 1, suppose that for each $u_0 \in \mathbb{C}^r$ there exists a fundamental domain $K_{u_0}$ of $\mathbb{Z}^n$ such that*

$$u_0^*\mathfrak{p}(\xi) \neq 0, \qquad \text{all } \xi \in K_{u_0} .$$

*Then $\Phi(x)$ is orthogonal.*

This corollary differs in appearance from the original Cohen's Criterion in the case $r = 1$. This is due to the occurrence of infinite products of matrices which do not commute in general. For the special case $r = 1$, $u_0^*\mathfrak{p}(\xi) \neq 0$ is equivalent to $\mathfrak{p}(B^{-j}\xi) \neq 0$ for all $j \geq 1$. In this case the condition of Corollary 1 is equivalent to $\mathfrak{p}(B^{-j}\xi) \neq 0$ for all $j \geq 1$, where $B = A^T$, on some fundamental domain of $\mathbb{Z}^n$. This is precisely the original form of Cohen's Criterion, see Cohen [3].

## 3. Orthogonality Criteria for Refinable Functions

More detailed criteria are available for orthogonality in the case $r = 1$, i.e., of refinable functions in $\mathbb{R}^n$. In this section we state such criteria; the proofs are given in Section 5.

The criteria of Theorem 1 can be strengthened for $r = 1$, especially when the dilation matrix $A$ is irreducible over $\mathbb{Z}$. A matrix $A \in M_n(\mathbb{Z})$ is *irreducible over* $\mathbb{Z}$ if its characteristic polynomial $f_A(\lambda)$ is irreducible over $\mathbb{Z}$. In particular, if $A \in M_n(\mathbb{Z})$ is expanding and $|\det(A)|$ is a prime, then $A$ is irreducible over $\mathbb{Z}$.

Note that if $r = 1$, then $\Omega_{r\times r}(\mathbb{R}^n) = \Omega_{1\times 1}(\mathbb{R}^n)$ is the space of all *real* trigonometric polynomials over $\mathbb{R}^n$, and we set $\Omega(\mathbb{R}^n) := \Omega_{1\times 1}(\mathbb{R}^n)$. Let the invariant set $\mathcal{S}_{\mathfrak{m},A}$ be as in (2.6) and set $\Omega(\mathbb{R}^n, \mathcal{S}) := \{F(\xi) \in \Omega(\mathbb{R}^n) : \text{supp}(F) \subseteq \mathcal{S}\}$.

***Theorem 2.***

*Let $A \in M_n(\mathbb{Z})$ be an expanding matrix that is irreducible over $\mathbb{Z}$. Suppose that the compactly supported nontrivial $\phi(x) \in L^2(\mathbb{R}^n)$ satisfies the refinement equation*

$$\phi(x) = \sum_{\alpha\in\mathbb{Z}^n} c_\alpha \phi(Ax - \alpha) ,$$

*which satisfies the orthogonal coefficients condition and has finitely many $c_\alpha \neq 0$. Let $\mathfrak{m}(\xi)$ be its symbol and $B = A^T$. Then the following statements are equivalent:*

(a) *The refinable function $\phi(x)$ is not orthogonal.*

(b) *There exists a nonconstant $f(\xi) \in \Omega(\mathbb{R}^n)$ such that $\boldsymbol{C}_{\mathfrak{m}} f = f$.*

(c) *Let $\mathcal{S}$ be a finite* $(\mathfrak{m}, A)$*-invariant set containing* $\mathcal{S}_{\mathfrak{m},A}$. *The eigenvalue 1 of* $\mathbf{C}_{\mathfrak{m}}$ *restricted to* $\Omega(\mathbb{R}^n, \mathcal{S})$ *is a multiple eigenvalue.*

(d) *There exists* $\eta_0 \in \mathbb{R}^n \setminus \mathbb{Z}^n$ *that has the property: for each* $\alpha \in \mathbb{Z}^n$ *there exists a* $j(\alpha) \geq 1$ *such that* $\mathfrak{m}\big(B^{-j(\alpha)}(\eta_0 + \alpha)\big) = 0$.

(e) *There exists* $\xi_0 \in \mathbb{R}^n \setminus \mathbb{Z}^n$ *such that* $B^N\xi_0 \equiv \xi_0 \pmod{\mathbb{Z}^n}$ *for some* $N > 0$, *and* $\mathfrak{m}\big(B^j\xi_0\big) = 1$ *for all* $j \geq 0$.

(f) *There exists* $\xi_0 \in \mathbb{R}^n \setminus \mathbb{Z}^n$ *such that* $B^N\xi_0 \equiv \xi_0 \pmod{\mathbb{Z}^n}$ *for some* $N > 0$, *and* $\mathfrak{m}\big(B^j\xi_0 + B^{-1}l\big) = 0$ *for all* $j \geq 0$ *and all* $l \in \mathbb{Z}^n \setminus B(\mathbb{Z}^n)$.

We derive Theorem 2 as a special case of a more general result that applies to an arbitrary expanding integer matrix $A$, given below as Theorem 3, which requires a more complicated generalization of (e) and (f). To state it, for each $l \in \mathbb{Z}^n$ we denote

$$\tau_l(\xi) := \left(A^T\right)^{-1}(\xi + l)\,.$$

A *rational subspace* of $\mathbb{R}^n$ is a linear subspace $W$ having a basis consisting of rational vectors $v \in \mathbb{Q}^n$. A set of vectors $\{z_j : 0 \leq j < N\}$ in $\mathbb{R}^n$ is a *periodic orbit* of $A^T$ (mod $\mathbb{Z}^n$) if

$$A^T z_{j+1} \equiv z_j \pmod{\mathbb{Z}^n}, \quad 0 \leq j < N\,,$$

where $z_N := z_0$. We let $\mathcal{E}$ denote an arbitrarily chosen complete set of coset representatives of $\mathbb{Z}^n/A^T(\mathbb{Z}^n)$.

***Theorem 3.***

*Let* $A \in M_n(\mathbb{Z})$ *be an expanding matrix. Suppose that the compactly supported nontrivial* $\phi(x) \in L^2(\mathbb{R}^n)$ *satisfies the refinement equation*

$$\phi(x) = \sum_{\alpha\in\mathbb{Z}^n} c_\alpha \phi(Ax - \alpha)\,,$$

*which satisfies the orthogonal coefficients condition and has finitely many* $c_\alpha \neq 0$. *Let* $\mathfrak{m}(\xi)$ *be its symbol and* $B = A^T$. *Then the following statements are equivalent:*

(a) *The refinable function* $\phi(x)$ *is not orthogonal.*

(b) *There exists a nonconstant* $f(\xi) \in \Omega(\mathbb{R}^n)$ *such that* $\mathbf{C}_{\mathfrak{m}} f = f$.

(c) *Let* $\mathcal{S}$ *be a finite* $(\mathfrak{m}, A)$*-invariant set containing* $\mathcal{S}_{\mathfrak{m},A}$. *The eigenvalue 1 of* $\mathbf{C}_{\mathfrak{m}}$ *restricted to* $\Omega(\mathbb{R}^n, \mathcal{S})$ *is a multiple eigenvalue.*

(d) *There exists* $\eta_0 \in \mathbb{R}^n \setminus \mathbb{Z}^n$ *that has the property: for each* $\alpha \in \mathbb{Z}^n$ *there exists a* $j(\alpha) \geq 1$ *such that* $\mathfrak{m}\big(B^{-j(\alpha)}(\eta_0 + \alpha)\big) = 0$.

(e) *There exists a proper* $B$*-invariant rational subspace* $W$ *of* $\mathbb{R}^n$ *and a periodic orbit* $\{z_j : 0 \leq j < N\}$ *of* $B$ (mod $\mathbb{Z}^n$) *with every* $z_j \notin W + \mathbb{Z}^n$, *such that*

$$\sum_{\substack{l\in\mathcal{E} \\ \tau_l(\xi)\in z_{j+1}+W+\mathbb{Z}^n}} |\mathfrak{m}(\tau_l(\xi))|^2 = 1 \tag{3.1}$$

*for all* $\xi \in z_j + W$, *where* $0 \leq j < N$ *with* $z_N := z_0$ *and* $\mathcal{E}$ *is a set of complete coset representatives of* $\mathbb{Z}^n/B(\mathbb{Z}^n)$.

(f) *There exists a proper* $B$*-invariant rational subspace* $W$ *of* $\mathbb{R}^n$ *and a periodic orbit* $\{z_j : 0 \leq j < N\}$ *of* $B$ (mod $\mathbb{Z}^n$) *with* $z_j \notin W + \mathbb{Z}^n$, *such that*

$$\mathfrak{m}(\tau_l(\xi)) = 0 \qquad \text{if } l \in \mathbb{Z}^n \text{ and } \tau_l(\xi) \notin z_{j+1} + W + \mathbb{Z}^n$$

*for all* $\xi \in z_j + W$, *where* $0 \leq j < N$ *and* $z_N := z_0$.

**Remark.** A transfer operator applied to wavelet bases apparently first appears in the appendix of Daubechies [9], and such operators were analyzed in Conze and Raugi [8]. The orthogonality criteria in Theorem 3 in dimension $n = 1$ for the case $r = 1$ were found by Cohen [3], Lawton [25], Conze and Raugi [8], and Cohen and Sun [6], and an elegant summary can be found in Gröchenig [16]. The equivalence of (a), (b), and (d) in dimension $n > 1$ is proved here by generalizing the arguments of Gröchenig in one dimension. In higher dimensions, Lawton et al. [26] gave an orthogonality criterion similar to (b), using the wavelet-Galerkin operator defined on $l^2(\mathbb{Z}^n)$ instead of the transfer operator. Criteria (e) and (f) in Theorems 2 and 3 are much harder to prove. The proof given here uses as a principal ingredient a recent result of Cerveau et al. [2] concerning the structure of the set of zeros of eigenfunctions of transfer operators in the multidimensional case. The paper of Conze et al. [7], Section II, applies this result to give various orthogonality criteria, some of which apply even when an infinite number of $c_\alpha \neq 0$ in (1.2).

# 4. Proof of Orthogonality Criteria for Function Vectors

For a given positive definite Hermitian matrix $Q \in M_{r\times r}(\mathbb{C})$ we define the norm $\|.\|_Q$ on $\mathbb{C}^r$ by $\|x\|_Q := \sqrt{x^*Qx}$ where $x^* = \bar{x}^T$. This norm induces a matrix norm in $M_{r\times r}(\mathbb{C})$, which we also denote by $\|.\|_Q$. Throughout this section, $\Lambda$ denotes a diagonal matrix with positive diagonal entries.

***Lemma 1.***

*Suppose that the vector refinement equation (2.1) has finitely many $C_\alpha \neq 0$ and satisfies the orthogonal coefficients condition with respect to the diagonal matrix $\Lambda$.*

(1) *Let $\mathcal{E}$ be any complete set of coset representatives of $\mathbb{Z}^n/B(\mathbb{Z}^n)$ where $B = A^T$. Then $C_\mathfrak{m}\Lambda = \Lambda$.*

(2) $\|\mathfrak{m}^*(\xi)\|_\Lambda \leq 1$ *for all* $\xi \in \mathbb{R}^n$.

(3) *Let $v$ be a left $\lambda$-eigenvector of $\mathfrak{m}(0)$ with $|\lambda| = 1$. Then $v$ is a left $\lambda$-eigenvector of $\Delta_\alpha := \sum_{\beta\in\mathbb{Z}^n} C_{\alpha+A\beta}$ for all $\alpha \in \mathbb{Z}^n$.*

(4) *For any 1-eigenvector $u_0$ of $\mathfrak{m}(0)$, the vector refinement equation (2.1) has a unique compactly supported solution $\Phi(x) \in L_2^r(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \Phi(x)\,dx = u_0$.*

**Proof.** (1) Let $q = |\det(A)|$ and $B = A^T$. Then

$$\begin{aligned} C_\mathfrak{m}\Lambda &= \sum_{d\in\mathcal{E}} \mathfrak{m}\left(\xi + B^{-1}d\right)\Lambda\mathfrak{m}^*\left(\xi + B^{-1}d\right) \\ &= q^{-2}\sum_{d\in\mathcal{E}}\sum_{\alpha\in\mathbb{Z}^n}\sum_{\beta\in\mathbb{Z}^n} C_\alpha\Lambda C_\beta^* e^{-2\pi i\langle\alpha-\beta,\xi+B^{-1}d\rangle} \\ &= q^{-2}\sum_{\alpha\in\mathbb{Z}^n}\sum_{\gamma\in\mathbb{Z}^n} C_\alpha\Lambda C_{\alpha+\gamma}^* \sum_{d\in\mathcal{E}} e^{-2\pi i\langle\gamma,\xi+B^{-1}d\rangle}. \end{aligned}$$

It follows from

$$\sum_{d\in\mathcal{E}} e^{-2\pi i\langle\gamma,\xi+B^{-1}d\rangle} = \begin{cases} qe^{-2\pi i\langle\gamma,\xi\rangle} & \text{if } \gamma \in A\left(\mathbb{Z}^n\right), \\ 0 & \text{otherwise} \end{cases}$$

that

$$\begin{aligned} C_{\mathrm{m}}\Lambda &= q^{-1}\sum_{\alpha\in\mathbb{Z}^n}\sum_{\beta\in\mathbb{Z}^n} C_\alpha \Lambda C^*_{\alpha+A\beta} e^{-2\pi i\langle A\beta,\xi\rangle} \\ &= q^{-1}\sum_{\beta\in\mathbb{Z}^n} e^{-2\pi i\langle A\beta,\xi\rangle}\sum_{\alpha\in\mathbb{Z}^n} C_\alpha\Lambda C^*_{\alpha+A\beta} \\ &= q^{-1}\sum_{\beta\in\mathbb{Z}^n} e^{-2\pi i\langle A\beta,\xi\rangle} q\delta_{0,\beta}\Lambda \\ &= \Lambda\,. \end{aligned}$$

(2) Choose $\mathcal{E}$ so that $0\in\mathcal{E}$. By part (1), for any $v\in\mathbb{C}^r$,

$$\sum_{d\in\mathcal{E}} v^*\mathrm{m}\left(\xi+B^{-1}d\right)\Lambda\mathrm{m}^*\left(\xi+B^{-1}d\right)v = v^*\Lambda v\,.$$

Thus, $\|\mathrm{m}^*(\xi)v\|_\Lambda \le \|v\|_\Lambda$ for all $\xi$ by taking $d=0$, proving (2).

(3) Let $\mathcal{D}$ be a complete set of coset representatives of $\mathbb{Z}^n/A(\mathbb{Z}^n)$. Then $\sum_{\alpha\in\mathcal{D}}\Delta_\alpha = q\mathrm{m}(0)$, and one easily checks that

$$\sum_{\alpha\in\mathcal{D}}\Delta_\alpha\Lambda\Delta^*_\alpha = q\Lambda\,.$$

The above together with the Schwarz inequality yield

$$\left\|\sum_{\alpha\in\mathcal{D}} v\Delta_\alpha\right\|^2_\Lambda \le q\sum_{\alpha\in\mathcal{D}}\|v\Delta_\alpha\|^2_\Lambda = q^2\|v\|^2_\Lambda,$$

and the equality holds if and only if all $v\Delta_\alpha$ are equal. Now

$$\left\|\sum_{\alpha\in\mathcal{D}} v\Delta_\alpha\right\|^2_\Lambda = \|qv\mathrm{m}(0)\|^2_\Lambda = \|q\lambda v\|^2_\Lambda = q^2\|v\|^2_\Lambda\,.$$

So $v\Delta_\alpha = v_0$ for all $\alpha\in\mathcal{D}$, and $\sum_{\alpha\in\mathcal{D}} v\Delta_\alpha = qv\mathrm{m}(0) = q\lambda v$ implies that $v_0=\lambda v$. Finally, for any $\beta\in\mathbb{Z}^n$ there is an $\alpha\in\mathcal{D}$ such that $\Delta_\beta=\Delta_\alpha$. This proves (3).

(4) For $n=1$ and $r=1$ this is a well-known result of Mallat [28]. Mallat's proof generalizes easily to the general case. A proof of this part can be found in Flaherty and Wang [13]. We remark that the solution $\Phi(x)$ is given by $\widehat{\Phi}(\xi) = (\prod_{j=1}^\infty \mathrm{m}(B^{-j}\xi))u_0$. □

A proof of Proposition 1 can be found in Hogan [20]. Here we present a different proof.

**Proof of Proposition1.** Let $\lambda$ be an eigenvalue of $\mathrm{m}(0)$ and $u_0$ be a left $\lambda$-eigenvector of $\mathrm{m}(0)$. By (2) of Lemma 1 we have $|\lambda|\le 1$. Suppose that $|\lambda|=1$. Define $g(x)=\sum_{\alpha\in\mathbb{Z}^n}\left\langle\Phi(x+\alpha),u_0^*\right\rangle$.

We view $g(x)$ as a function in $L^1(\mathbb{T}^n)$. Observe that

$$\begin{aligned} g(x) &= \sum_{\alpha\in\mathbb{Z}^n}\sum_{\beta\in\mathbb{Z}^n}\left\langle C_\beta \Phi(Ax+A\alpha-\beta), u_0^*\right\rangle \\ &= \sum_{\alpha\in\mathbb{Z}^n}\sum_{\gamma\in\mathbb{Z}^n}\left\langle C_{A\alpha-\gamma}\Phi(Ax+\gamma), u_0^*\right\rangle \\ &= \sum_{\gamma\in\mathbb{Z}^n}\left\langle \Delta_{-\gamma}\Phi(Ax+\gamma), u_0^*\right\rangle \\ &= \sum_{\gamma\in\mathbb{Z}^n}\left\langle \Phi(Ax+\gamma), \Delta_{-\gamma}^* u_0^*\right\rangle \\ &= \bar{\lambda} g(Ax)\,, \end{aligned}$$

where $\Delta_{-\gamma} = \sum_{\alpha\in\mathbb{Z}^n} C_{A\alpha-\gamma}$ and $\Delta_{-\gamma}^* u_0^* = \lambda u_0^*$ by (3) of Lemma 1. So $|g(x)| = |\lambda||g(Ax)|$. It follows from the ergodicity of $A$ on $\mathbb{T}^n$ that $|g(x)| = c$ for some constant $c$, so $g(x) \in L^2(\mathbb{T}^n)$. Consider the Fourier expansion of $g(x) = \sum_{\alpha\in\mathbb{Z}^n} b_\alpha e^{2\pi i\langle\alpha,x\rangle}$. The equality $g(x) = \lambda g(Ax)$ yields $b_\alpha = 0$ for all $\alpha \neq 0$ and $b_0 = 0$ if $\lambda \neq 1$, by comparing the Fourier coefficients of $g(x)$ and $\lambda g(Ax)$. If $\lambda \neq 1$, then $g(x) = 0$ almost everywhere. But this is impossible because $\Phi(x)$ is orthogonal. So $\lambda = 1$. In this case, the ergodicity of $A$ on $\mathbb{T}^n$ implies that $g(x) = c$ almost everywhere for some constant $c$.

We show that 1 is a simple eigenvalue of $\mathfrak{m}(0)$. If not, because $\|\mathfrak{m}(0)\|_\Lambda \leq 1$ for some positive definite diagonal matrix $\Lambda$, $\mathfrak{m}(0)$ must have two independent left 1-eigenvectors $u_1,\ u_2 \in \mathbb{C}^r$. Therefore, there exists a nonzero linear combination $u$ of $u_1,\ u_2$ such that

$$\sum_{\alpha\in\mathbb{Z}^n}\left\langle \Phi(x-\alpha), u^*\right\rangle = 0 \quad \text{a.e.}\,.$$

Again this contradicts the orthogonality of $\Phi(x)$. □

**Proof of Proposition 2.** (i) By definition $A(T_{\mathfrak{m},A}) = T_{\mathfrak{m},A} + \operatorname{supp}(\mathfrak{m})$. For any $\gamma \notin \mathcal{S}_{\mathfrak{m},A}$ we have

$$\begin{aligned} \emptyset &= A\left(T_{\mathfrak{m},A} \cap \left(T_{\mathfrak{m},A}+\gamma\right)\right) \\ &= \left(T_{\mathfrak{m},A} + \operatorname{supp}(\mathfrak{m})\right) \cap \left(T_{\mathfrak{m},A} + A\gamma + \operatorname{supp}(\mathfrak{m})\right) \\ &= \bigcup_{\alpha,\beta\in\operatorname{supp}(\mathfrak{m})} \left(T_{\mathfrak{m},A} \cap \left(T_{\mathfrak{m},A} + A\gamma + \alpha - \beta\right)\right) + \beta\,. \end{aligned}$$

So $A\gamma + \alpha - \beta \notin \mathcal{S}_{\mathfrak{m},A}$ for all $\alpha,\ \beta \in \operatorname{supp}(\mathfrak{m})$. Therefore, $\mathcal{S}_{\mathfrak{m},A}$ is $(\mathfrak{m}, A)$-invariant.

(ii) Let $F(\xi) = \sum_{\gamma\in\mathcal{S}} F_\gamma e^{-2\pi i\langle\gamma,\xi\rangle} \in \Omega_{r\times r}(\mathbb{R}^n, \mathcal{S})$. It is straightforward to check that

$$(C_{\mathfrak{m}}F)(\xi) = \sum_{\gamma\in\mathbb{Z}^n} G_\gamma e^{-2\pi i\langle\gamma,\xi\rangle}, \quad \text{where } G_\gamma = \sum_{\alpha,\beta\in\mathbb{Z}^n} C_\alpha F_{A\gamma+\beta-\alpha} C_\beta^*\,.$$

Suppose that $G_\gamma \neq 0$. Then there exist $\alpha,\ \beta \in \mathbb{Z}^n$ such that $C_\alpha F_{A\gamma+\beta-\alpha} C_\beta^* \neq 0$, so $\alpha,\ \beta \in \operatorname{supp}(\mathfrak{m})$ and $A\gamma + \beta - \alpha \in \mathcal{S}$. It follows that $\gamma \in \mathcal{S}$. Hence $C_{\mathfrak{m}}F \in \Omega_{r\times r}(\mathbb{R}^n, \mathcal{S})$. □

We now prove the orthogonality criteria for refinable function vectors. We first introduce some notation to simplify our exposition. For any $k > 0$ we let $\mathfrak{m}_k(\xi)$ denote the (right) product

$$\mathfrak{m}_k(\xi) = \prod_{j=1}^{k} \mathfrak{m}\left(B^{k-j}\xi\right) := \mathfrak{m}\left(B^{k-1}\xi\right)\mathfrak{m}\left(B^{k-2}\xi\right)\cdots\mathfrak{m}\left(B^{0}\xi\right)$$

where $B = A^T$. Given a complete set of coset representatives $\mathcal{E}$ of $\mathbb{Z}^n/B(\mathbb{Z}^n)$ let

$$\mathcal{E}_{B,k} := \mathcal{E} + B\mathcal{E} + \cdots + B^{k-1}\mathcal{E} .$$

Observe that

$$C_{\mathfrak{m}}^k F(\xi) = \sum_{d \in \mathcal{E}_{B,k}} \mathfrak{m}_k\left(B^{-k}(\xi + d)\right) F\left(B^{-k}(\xi + d)\right) \mathfrak{m}_k\left(B^{-k}(\xi + d)\right) . \quad (4.1)$$

**Proof of Theorem 1.** The standard Possion Summation Formula gives

$$\sum_{\alpha \in \mathbb{Z}^n} \left( \int_{\mathbb{R}^n} \Phi(x)\Phi^*(x + \alpha)\, dx \right) e^{2\pi i \langle \alpha, \xi \rangle} = \sum_{\alpha \in \mathbb{Z}^n} \widehat{\Phi}(\xi + \alpha)\widehat{\Phi}^*(\xi + \alpha) . \quad (4.2)$$

(a) $\Rightarrow$ (b). The proof here is a generalization of the proof in Gröchenig [16] for the case $n = 1$, $r = 1$. Suppose that $\Phi(x)$ is not orthogonal. Then

$$F(\xi) := \sum_{\alpha \in \mathbb{Z}^n} \left( \int_{\mathbb{R}^n} \Phi(x)\Phi^*(x + \alpha)\, dx \right) e^{2\pi i \langle \alpha, \xi \rangle}$$

is in $\Omega_{r\times r}(\mathbb{R}^n)$ and $F(\xi) \neq a\Lambda$ for any $a \in \mathbb{C}$. We show that $C_{\mathfrak{m}} F = F$. Let $\mathcal{E}$ be any complete set of coset representatives for $\mathbb{Z}^n/B(\mathbb{Z}^n)$. Denote $\xi_d := B^{-1}(\xi + d)$. Then

$$\begin{aligned}
C_{\mathfrak{m}} F(\xi) &= \sum_{d \in \mathcal{E}} \mathfrak{m}(\xi_d)\, F(\xi_d)\, \mathfrak{m}^*(\xi_d) \\
&= \sum_{d \in \mathcal{E}} \sum_{\alpha \in \mathbb{Z}^n} \mathfrak{m}(\xi_d)\, \widehat{\Phi}(\xi_d + \alpha)\, \widehat{\Phi}^*(\xi_d + \alpha)\, \mathfrak{m}^*(\xi_d) \\
&= \sum_{d \in \mathcal{E}} \sum_{\alpha \in \mathbb{Z}^n} \mathfrak{m}(\xi_d + \alpha)\, \widehat{\Phi}(\xi_d + \alpha)\, \widehat{\Phi}^*(\xi_d + \alpha)\, \mathfrak{m}^*(\xi_d + \alpha) \\
&= \sum_{d \in \mathcal{E}} \sum_{\alpha \in \mathbb{Z}^n} \widehat{\Phi}(\xi + d + B\alpha)\widehat{\Phi}^*(\xi + d + B\alpha) \\
&= \sum_{\alpha \in \mathbb{Z}^n} \widehat{\Phi}(\xi + \alpha)\widehat{\Phi}^*(\xi + \alpha) \\
&= F(\xi) .
\end{aligned}$$

(b) $\Rightarrow$ (c). Since $\operatorname{supp}(\Phi) \subseteq T_{\mathfrak{m},A}$, we see that $\operatorname{supp}(F) \subseteq \mathcal{S}_{\mathfrak{m},A}$. Therefore, $F \in \Omega_{r\times r}(\mathbb{R}^n, \mathcal{S})$ since $\mathcal{S}$ contains $\mathcal{S}_{\mathfrak{m},A}$. Observe that $0 \in \mathcal{S}_{\mathfrak{m},A}$, so $G(\xi) := \Lambda \in \Omega_{r\times r}(\mathbb{R}^n, \mathcal{S})$, and is also a 1-eigenvector of $C_{\mathfrak{m}}$. So 1 is a multiple eigenvalue of $C_{\mathfrak{m}}$ restricted to $\Omega_{r\times r}(\mathbb{R}^n, \mathcal{S})$, proving (c).

(c) $\Rightarrow$ (b). Since 1 is a multiple eigenvalue of $C_{\mathfrak{m}}$ restricted to $\Omega_{r\times r}(\mathbb{R}^n, \mathcal{S})$, either $C_{\mathfrak{m}}$ has two independent 1-eigenvectors in $\Omega_{r\times r}(\mathbb{R}^n, \mathcal{S})$, in which case we complete our proof, or $C_{\mathfrak{m}}^k$ is unbounded in $\Omega_{r\times r}(\mathbb{R}^n, \mathcal{S})$ as $k \to \infty$. We show that the latter is impossible. Assume that it did, then there exists a $F(\xi) \in \Omega_{r\times r}(\mathbb{R}^n, \mathcal{S})$ such that $C^k F$ is unbounded as $k \to \infty$. By adding $a\Lambda$ to $F$ for a sufficiently large $a > 0$ we may without loss of generality assume that $F(\xi)$ is positive definite for all $\xi$. Let $\Gamma$ be the positive definite diagonal matrix $\Gamma := \sqrt{\Lambda}$. Then for any $\xi \in \mathbb{R}^n$ and

$u \in \mathbb{C}^r$,

$$\begin{aligned}
(\Gamma u)^* \Gamma^{-1} (C_{\mathfrak{m}} F)(\xi) \Gamma^{-1} (\Gamma u) &= u^* (C_{\mathfrak{m}} F)(\xi) u \\
&= \sum_{d \in \mathcal{E}} u^* \mathfrak{m}(\xi_d) F(\xi_d) \mathfrak{m}^*(\xi_d) u \\
&= \sum_{d \in \mathcal{E}} u^* \mathfrak{m}(\xi_d) \Gamma \left( \Gamma^{-1} F(\xi_d) \Gamma^{-1} \right) \Gamma \mathfrak{m}^*(\xi_d) u \\
&\leq \rho_\Gamma(F) \sum_{d \in \mathcal{E}} u^* \mathfrak{m}(\xi_d) \Gamma \Gamma \mathfrak{m}^*(\xi_d) u \\
&= \rho_\Gamma(F) u^* \Lambda u \\
&= \rho_\Gamma(F) (\Gamma u)^* (\Gamma u),
\end{aligned}$$

where $\xi_d := B^{-1}(\xi + d)$ and $\rho_\Gamma(F)$ is the supremum over all $\xi$ of the spectral radii $\rho(\Gamma^{-1} F(\xi) \Gamma^{-1})$. Therefore, the spectral radius of $\Gamma^{-1}(C_{\mathfrak{m}} F)(\xi)\Gamma^{-1}$ is bounded by $\rho_\Gamma(F)$. This implies that for all $k$ the spectral radius of $\Gamma^{-1}(C_{\mathfrak{m}}^k F)(\xi)\Gamma^{-1}$ is bounded by $\rho_\Gamma(F)$. But this would mean that $\Gamma^{-1}(C_{\mathfrak{m}}^k F)(\xi)\Gamma^{-1}$ is bounded for all $\xi$ and $k$ because it is Hermitian. This is a contradiction.

(b) $\Rightarrow$ (d). Since $F(\xi)$ is bounded and periodic (mod $\mathbb{Z}^n$), there exist $a_+, a_- \in \mathbb{R}$ such that

$$\begin{aligned}
a_+ &= \inf \left\{ a \in \mathbb{R} : a\Lambda - F \text{ is positive definite for all } \xi \in \mathbb{R}^n \right\}, \\
a_- &= \sup \left\{ a \in \mathbb{R} : F - a\Lambda \text{ is positive definite for all } \xi \in \mathbb{R}^n \right\}.
\end{aligned}$$

Let $F_+(\xi) = a_+\Lambda - F(\xi)$ and $F_-(\xi) = F(\xi) - a_-\Lambda$. Then both $F_+$ and $F_-$ are nonnegative definite but neither is positive definite for all $\xi \in \mathbb{R}^n$. To simplify our notation we let $\Delta := \mathfrak{m}(0)$. The hypotheses of the theorem implies that $\Delta^\infty := \lim_{k \to \infty} \Delta^k$ exists and is a rank one matrix whose columns are 1-eigenvectors of $\Delta$.

***Claim 1.***

*Suppose that $F_+(\xi)$ (resp., $F_-(\xi)$) is singular for $\xi \in \mathbb{Z}^n$ only. Then $F_+(0)v_0 = 0$ (resp., $F_-(0)v_0 = 0$) where $v_0 \neq 0$ is a 1-eigenvector of $\mathfrak{m}^*(0)$.*

**Proof of Claim 1.** We prove the claim for $F_+(\xi)$, the proof is identical for $F_-(\xi)$. Let $v \in \mathbb{C}^r$ such that $\|v\|_\Lambda = 1$, $v^* F_+(0) = 0$. Then it follows from $C_{\mathfrak{m}}^k F_+ = F_+$ that

$$0 = v^* F_+(0) v = \sum_{d \in \mathcal{E}_{B,k}} v^* \mathfrak{m}_k \left( B^{-k} d \right) F_+ \left( B^{-k} d \right) \mathfrak{m}_k^* \left( B^{-k} d \right) v . \tag{4.3}$$

Since $C_{\mathfrak{m}}$ is independent of the choice of $\mathcal{E}$ we choose $0 \in \mathcal{E}$. Now all $F_+(B^{-k}d)$ are positive definite unless $B^{-k}d \in \mathbb{Z}^n$, which holds only for $d = 0$. We thus have $\mathfrak{m}_k^*(B^{-k}d)v = 0$ for all $d \in \mathcal{E}_{B,k}$, $d \neq 0$. Note that the orthogonal coefficients condition gives

$$\sum_{d \in \mathcal{E}_{B,k}} \left\| \mathfrak{m}_k^* \left( B^{-k} d \right) v \right\|_\Lambda^2 = \|v\|_\Lambda^2 = 1 .$$

Hence $\|\mathfrak{m}_k^*(0)v\|_\Lambda = \|(\Delta^*)^k v\|_\Lambda = 1$. It follows by letting $k \to \infty$ that $v_0 := (\Delta^\infty)^* v \neq 0$. Clearly $v_0$ is the unique (up to scalar multiples) 1-eigenvector of $\Delta^*$. By (4.3) $v_0^* F_+(0) v_0 = 0$, and hence $F_+(0)v_0 = 0$ by the nonnegative definiteness of $F_+(0)$, proving the claim. □

***Claim 2.***

*Either $G(\xi) = F_+(\xi)$ or $G(\xi) = F_-(\xi)$ has the property that $\Delta^\infty G(0) \neq 0$ and $G(\eta)$ is singular for some $\eta \in \mathbb{R}^n \setminus \mathbb{Z}^n$.*

**Proof of Claim 2.** First we observe that $F_+(\xi)+F_-(\xi)=(a_+-a_-)\Lambda$ is always nonsingular, so Claim 1 implies that at least one of $F_+(\xi)$ and $F_-(\xi)$ is singular for some $\eta\in\mathbb{R}^n\setminus\mathbb{Z}^n$. Assume that Claim 2 is false. Then either $\Delta^\infty G(0)=0$ or $G(\eta)$ is nonsingular for all $\eta\in\mathbb{R}^n\setminus\mathbb{Z}^n$, where $G(\xi)$ is either $F_+(\xi)$ or $F_-(\xi)$. Now $\Delta^\infty(F_+(0)+F_-(0))\neq 0$ because $F_+(0)+F_-(0)$ is nonsingular, so either $\Delta^\infty F_+(0)\neq 0$ or $\Delta^\infty F_-(0)\neq 0$. If both are nonzero then we have a contradiction. So without loss of generality we assume that $\Delta^\infty F_+(0)=0$ and thus $F_-(\eta)$ is nonsingular for all $\eta\in\mathbb{R}^n\setminus\mathbb{Z}^n$. By Claim 1 we have $F_-(0)v_0=0$, where $v_0$ is a 1-eigenvector of $\Delta^*$. Now, $v_0^*\Delta^\infty=v_0^*$. So

$$v_0^*\left(F_+(0)+F_-(0)\right)v_0=v_0^*\Delta^\infty\left(F_+(0)+F_-(0)\right)v_0=0\,.$$

This contradicts the positive definiteness of $F_+(0)+F_-(0)$, proving Claim 2. □

To finish proving (b) ⇒ (d), let $G(\xi)$ be $F_+(\xi)$ or $F_-(\xi)$ such that $\Delta^\infty G(0)\neq 0$ and $G(\eta)$ is singular for some $\eta\in\mathbb{R}^n\setminus\mathbb{Z}^n$. Let $G(\eta)u_0=0$ for some nonzero $u_0\in\mathbb{C}^r$. We show that $u_0^*\mathfrak{p}(\eta+\alpha)=0$ for all $\alpha\in\mathbb{Z}^n$. For a given $\alpha\in\mathbb{Z}^n$, we write $\alpha=B^l\beta$ for some $\beta\in\mathbb{Z}^n\setminus B(\mathbb{Z}^n)$. Choose $\mathcal{E}$ so that $0,\ \beta\in\mathcal{E}$. Then for all $k>l$ we have $\alpha\in\mathcal{E}_{B,k}$. It follows from $C_{\mathfrak{m}}^kG=G$ that

$$0=u_0^*G(\eta)u_0=\sum_{d\in\mathcal{E}_{B,k}}u_0^*\mathfrak{m}_k\left(B^{-k}(\eta+d)\right)G\left(B^{-k}(\eta+d)\right)\mathfrak{m}_k^*\left(B^{-k}(\eta+d)\right)u_0\,.$$

In particular we have

$$u_0^*\mathfrak{m}_k\left(B^{-k}(\eta+\alpha)\right)G\left(B^{-k}(\eta+\alpha)\right)\mathfrak{m}_k^*\left(B^{-k}(\eta+\alpha)\right)u_0=0\,.$$

It follows by letting $k\to\infty$ that

$$u_0^*\mathfrak{p}(\eta+\alpha)G(0)\mathfrak{p}^*(\eta+\alpha)u_0=0\,,$$

and the nonnegative definiteness of $G(0)$ yields

$$u_0^*\mathfrak{p}(\eta+\alpha)G(0)=0\,.$$

Observe that $\mathfrak{p}(\xi)=\mathfrak{p}(\xi)\Delta^\infty$. So $\mathfrak{p}(\xi)G(0)=\mathfrak{p}(\xi)\Delta^\infty G(0)$. Since $\Delta^\infty G(0)\neq 0$ and $\Delta^\infty$ has rank 1, there exists a nonzero column $v_1$ in $\Delta^\infty G(0)$, which is clearly a 1-eigenvector of $\Delta$. Hence all columns of $\mathfrak{p}(\xi)$ are scalar multiples of $\mathfrak{p}(\xi)v_1$. Thus $u_0^*\mathfrak{p}(\eta+\alpha)=0$.

(d) ⇒ (a). It follows from $\widehat{\Phi}(\xi)=\mathfrak{p}(\xi)\widehat{\Phi}(0)$ that $u_0^*\widehat{\Phi}(\eta+\alpha)=0$ for all $\alpha\in\mathbb{Z}^n$. Hence by the Poisson Summation Formula,

$$\sum_{\alpha\in\mathbb{Z}^n}u_0^*\left(\int_{\mathbb{R}^n}\Phi(x)\Phi^*(x-\alpha)\,dx\right)u_0e^{2\pi i\langle\alpha,\eta\rangle}=\sum_{\alpha\in\mathbb{Z}^n}u_0^*\widehat{\Phi}(\eta+\alpha)\widehat{\Phi}^*(\eta+\alpha)u_0=0$$

Therefore,

$$\sum_{\alpha\in\mathbb{Z}^n}\left(\int_{\mathbb{R}^n}\Phi(x)\Phi^*(x-\alpha)\,dx\right)e^{2\pi i\langle\alpha,\eta\rangle}\neq\tilde{\Lambda}$$

for any diagonal matrix $\tilde{\Lambda}$ with positive diagonal entries, and so $\Phi(x)$ cannot be orthogonal. □

**Proof of Corollary 1.** It follows easily from the fact that for any fundamental domain $K$ one of $\eta+\alpha$ in (d) of Theorem 1 is in $K$. □

# 5. Proof of Orthogonality Criteria for Refinable Functions

Let $\mathbb{T}^n$ be the $n$-dimensional torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$, and $\pi_n : \mathbb{R}^n \to \mathbb{T}^n$ be the canonical covering map.

***Lemma 2.***

*Let $V$ be a subspace of $\mathbb{R}^n$. Then $\pi_n(V)$ is closed in $\mathbb{T}^n$ if and only if $V$ is a rational subspace of $\mathbb{R}^n$.*

**Proof.** We first show that if $V$ is a rational subspace of $\mathbb{R}^n$, then $\pi_n(V)$ is closed in $\mathbb{T}^n$. Let $w_1, w_2, \ldots, w_r \in \mathbb{Z}^n$ form a basis of $V$. Suppose that $z^* \in \mathbb{T}^n$ is in the closure of $\pi_n(V)$. Then we may find a sequence $\{x_j\}$ in $V$ such that $\lim_{j\to\infty} \pi_n(x_j) = z^*$. Write

$$x_j = \sum_{k=1}^{r} b_{j,k} w_k \ .$$

Since all $w_k \in \mathbb{Z}^n$, we may choose all $b_{j,k} \in [0, 1)$. Therefore we can find a subsequence $\{j_m\}$ of $\{j\}$ such that

$$\lim_{m\to\infty} b_{j_m,k} = b_k^*, \qquad \text{all } 1 \le k \le r \ .$$

Let $x^* = \sum_{k=1}^{r} b_k^* w_k$. Clearly, $\pi_n(x^*) = z^*$. Hence $z^* \in \pi_n(W)$. Therefore, $\pi_n(V)$ is closed in $\mathbb{T}^n$.

We next prove the following fact: If $v \in \mathbb{R}^n$, then the closure of $\pi_n(\mathbb{R}v)$ in $\mathbb{T}^n$ is a rational subspace. To see this, let $v = [\beta_1, \ldots, \beta_n]^T$. Without loss of generality we assume that $\beta_1, \ldots, \beta_r$ are linearly independent over $\mathbb{Q}$ while $\beta_k = \sum_{j=1}^{r} a_{k,j}\beta_j$ with $a_{k,j} \in \mathbb{Q}$ for all $1 \le k \le n$. The set

$$\left\{ m \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_r \end{bmatrix} \pmod{\mathbb{Z}^r} \ : \ m \in \mathbb{Z} \right\}$$

is dense in $\mathbb{T}^r$ (see Cassels [1], Theorem I, p. 64). Now let $V_0 = \{Ax : x \in \mathbb{R}^r\}$ where $A = [a_{k,j}]$. Then $V_0$ is a rational subspace of $\mathbb{R}^n$, and $\pi_n(V_0)$ is contained in the closure of $\pi_n(\mathbb{R}v)$. But $\pi_n(V_0)$ is closed and $V_0 \supseteq \mathbb{R}v$. Hence the closure of $\pi_n(\mathbb{R}v)$ is $\pi_n(V_0)$, proving the fact.

Finally, let $v_1, \ldots, v_r$ be a basis of $V$. Suppose that $\bar{W}_j$ is the closure of $\pi_n(\mathbb{R}v_j)$ in $\mathbb{T}^n$. Then the closure of $\pi_n(V)$ contains $\bar{W}_1 + \cdots + \bar{W}_r$. But $\bar{W}_1 + \cdots + \bar{W}_r$ is closed in $\mathbb{T}^n$ because it is a rational subspace, and it contains $\pi_n(V)$. Hence the closure of $\pi_n(V)$ is $\bar{W}_1 + \cdots + \bar{W}_r$, proving the lemma. □

***Corollary 2.***

*Let $f : \mathbb{R}^n \to \mathbb{C}$ be continuous and periodic (mod $\mathbb{Z}^n$) and $V$ be a subspace of $\mathbb{R}^n$. If $v_0 + V$ is contained in the zero set of $f(x)$ for some $v_0 \in \mathbb{R}^n$, then so is $v_0 + W$ where $W$ is the smallest rational subspace of $\mathbb{R}^n$ containing $V$.*

**Proof.** First, let $\{V_\alpha\}$ be a set of rational subspaces of $\mathbb{R}^n$. Then $\pi_n\left(\bigcap_\alpha V_\alpha\right) = \bigcap_\alpha \pi_n(V_\alpha)$ is closed in $\mathbb{T}^n$, so $\bigcap_\alpha V_\alpha$ must be a rational subspace of $\mathbb{R}^n$. This implies that the minimal rational subspace $W$ containing $V$ exists. Since $f(x)$ is periodic (mod $\mathbb{Z}^n$) we may view it as a continuous function defined on $\mathbb{T}^n$. Now, $\pi_n(v_0) + \pi_n(W)$ is the closure of $\pi_n(v_0) + \pi_n(V)$ in $\mathbb{T}^n$. Hence $\pi_n(v_0) + \pi_n(W)$ is in the zero set of $f : \mathbb{T}^n \to \mathbb{C}$. Thus, $v_0 + W \subseteq Z_f$. □

We derive the following key lemma from a result of Cerveau et al. [2]. First, we define the notion of *$\tau$-invariance* in $\mathbb{R}^n$. Let $\mathrm{m}(x)$ be the symbol of a given dilation equation that satisfies the orthogonal coefficients condition. Let $\mathcal{E}$ be a given complete set of coset representatives of

$\mathbb{Z}^n/A^T(\mathbb{Z}^n)$. A closed set $Y \subseteq \mathbb{R}^n$ is $\tau$-*invariant* if for any $l \in \mathcal{E}$,

$$\omega \in Y \quad \text{and} \quad |\mathfrak{m}(\tau_l(\omega))| > 0 \quad \Longrightarrow \quad \tau_l(\omega) \in Y . \tag{5.1}$$

A compact $\tau$-invariant set is *minimal* if it contains no smaller nonempty compact $\tau$-invariant set.

***Proposition 3.***

*Let $f(\xi) \in \Omega(\mathbb{R}^n)$ and let $Y$ be a minimal compact $\tau$-invariant set contained in the zero set of $f(\xi)$. Then there exist a subspace $V$ of $\mathbb{R}^n$ and a periodic orbit $\{z_j : 0 \le j \le N-1\}$ of $A^T$ (mod $\mathbb{Z}^n$) such that*

$$Y \subseteq \bigcup_{j=0}^{N-1} (z_j + V) .$$

**Proof.** This is Theorem 2.8 of Cerveau et al. [2]. The theorem of Cerveau et al. is actually valid in a more general setting, where $f(\xi)$ and $\mathfrak{m}(\xi)$ are allowed to be any real analytic functions. □

***Lemma 3.***

*Let $f(\xi) \in \Omega(\mathbb{R}^n)$ such that $C_\mathfrak{m} f = f$, and let $E_f^- := \{\xi \in \mathbb{R}^n : f(\xi) = \inf_{\omega \in \mathbb{R}^n} f(\omega)\}$. Then there exist a rational subspace $W$ and a periodic orbit $\{z_j : 0 \le j < N\}$ of $A^T$ (mod $\mathbb{Z}^n$) such that $F := \bigcup_{j=0}^{N-1}(z_j + W) \subseteq E_f^-$ and $F$ is $\tau$-invariant.*

**Proof.** We first observe that $E_f^-$ is $\tau$-invariant. This follows from

$$C_\mathfrak{m} f(\xi) = \sum_{l \in \mathcal{E}} |\mathfrak{m}(\tau_l(\xi))|^2 f(\tau_l(\xi)) = f(\xi) .$$

Since $\sum_{l\in\mathcal{E}} |\mathfrak{m}(\tau_l(\xi))|^2 = 1$, if $\xi \in E_f^-$ then all $f(\tau_l(\xi)) \ge f(\xi)$ so equality can hold only if $\tau_l(\xi) \in E_f^-$ whenever $|\mathfrak{m}(\tau_l(\xi))| > 0$.

We construct a nonempty minimal compact $\tau$-invariant set $Y$ in $E_f^-$ as follows. Take any point $\xi_0 \in E_f^-$ and set $X_0 = \{\xi_0\}$ and recursively define the finite sets $\{X_j : j \ge 0\}$ by letting $X_j$ consist of all points $\xi_j$ such that $\xi_j = \tau_l(\xi_{j-1})$ with $\xi_{j-1} \in X_{j-1}$ and $l \in \mathcal{E}$ such that $|\mathfrak{m}(\xi_j)| > 0$. Then the $\tau$-invariance of $E_f^-$ gives $X_j \subseteq E_f^-$ for all $j \ge 0$. The set $\bigcup_{j=0}^{\infty} X_j$ lies in a bounded region in $\mathbb{R}^n$ because the mappings $\tau_l$ are uniformly contracting with respect to a suitable norm in $\mathbb{R}^n$ (cf. Lagarias and Wang [23], Section 3). Thus the closure $Y_0$ of $\bigcup_{j=0}^{\infty} X_j$ is compact, and $Y_0 \subseteq E_f^-$ because $E_f^-$ is a closed set. We show that $Y_0$ is $\tau$-invariant. If $\omega \in Y_0$ and $|g(\tau_l(\omega))| > 0$ where $l \in \mathcal{E}$, take a subsequence $\xi_{j_k} \in X_{j_k}$ that converges to $\omega$, so that $\tau_l(\xi_{j_k}) \to \tau_l(\omega)$. Now $|\mathfrak{m}(\tau_l(\xi_{j_k}))| > 0$ for $k$ sufficiently large, hence $\tau_l(\xi_{j_k}) \in X_{j_k+1}$; so we may construct a sequence having $\tau_l(\omega)$ as a cluster point, proving $\tau_l(\omega) \in Y_0$. The existence of a nonempty minimal compact $\tau$-invariant set $Y$ contained in $Y_0$ follows by Zorn's Lemma argument.

It follows now from Proposition 3 that there exist an $A^T$-invariant subspace $V$ and a periodic orbit $\{z_j \in Y : 0 \le j < N\}$ such that

$$Y \subseteq \bigcup_{j=0}^{N-1} (z_j + V) \subseteq E_f^- ,$$

with the property that the set $\bigcup_{j=0}^{N-1}(z_j + V)$ is $\tau$-invariant. Now let $W$ be the smallest rational subspace of $\mathbb{R}^n$ containing $V$. Since $A^T(W)$ is also a rational subspace containing $V$ and it has the same dimension as $W$, $A^T(W) = W$. Because $E_f^-$ is the zero set of $\tilde{f}(\xi) := f(\xi) - \inf_\omega f(\omega)$, Corollary 2 applies to $\tilde{f}$ to give

$$Y \subseteq \bigcup_{j=0}^{N-1} (z_j + W) \subseteq E_f^- .$$

Finally, since $\pi_n\big(\bigcup_{j=0}^{N-1}(z_j + W)\big)$ is the closure of $\pi_n\big(\bigcup_{j=0}^{N-1}(z_j + V)\big)$ in $\mathbb{T}^n$, we conclude that $\bigcup_{j=0}^{N-1}(z_j + W)$ is $\tau$-invariant. □

**Proof of Theorem 3.** Observe that for $r = 1$ criterion (d) of Theorem 1 is equivalent to criterion (d) of Theorem 3. Therefore, the equivalence of (a)–(d) of this theorem has already been established in Theorem 1.

(b) $\Rightarrow$ (e). Let the nonconstant $f(\xi) \in \Omega(\mathbb{R}^n)$ satisfy $\mathbf{C}_{\mathrm{m}} f = f$. Without loss of generality we assume that $f(0) \neq \min_\omega f(\omega)$, or else we can replace $f(\xi)$ by $-f(\xi)$. By Lemma 3 there exists an $A^T$-invariant rational subspace $W$ and a periodic orbit $\{z_j : 0 \leq j < N\}$ of $A^T$ (mod $\mathbb{Z}^n$) such that $\cup_{j=0}^{N-1}(z_j + W) \subseteq E_f^-$ is $\tau$-invariant. We prove the following claim: Let $\xi \in z_j + W$. Suppose that $|\mathrm{m}(\tau_l(\xi))| > 0$ for some $l \in \mathbb{Z}^n$. Then $\tau_l(\xi) \in z_{j+1} + W + \mathbb{Z}^n$, where $z_N := z_0$.

Assume that the claim is false. Then the $\tau$-invariance of $\bigcup_{j=0}^{N-1}(z_j + W)$ implies that $\tau_l(\xi) \in z_{k+1} + W + \mathbb{Z}^n \neq z_{j+1} + W + \mathbb{Z}^n$. Hence $\xi \in A^T(z_{k+1} + W) + \mathbb{Z}^n = z_k + W + \mathbb{Z}^n$. But this could happen only if

$$z_k + W + \mathbb{Z}^n = z_j + W + \mathbb{Z}^n .$$

Applying the operator $(A^T)^{N-1}$ to both sides of the above equality yields

$$z_{k+1} + W + \left(A^T\right)^{N-1}\left(\mathbb{Z}^n\right) = z_{j+1} + W + \left(A^T\right)^{N-1}\left(\mathbb{Z}^n\right) ,$$

and adding $\mathbb{Z}^n$ to both sides then gives

$$z_{k+1} + W + \mathbb{Z}^n = z_{j+1} + W + \mathbb{Z}^n ,$$

which is a contradiction.

It now follows from the claim that for any $\xi \in z_j + W$,

$$1 = \sum_{l \in \mathcal{E}} |\mathrm{m}(\tau_l(\xi))|^2 = \sum_{\substack{l \in \mathcal{E} \\ \tau_l(\xi) \in z_{j+1}+W+\mathbb{Z}^n}} |\mathrm{m}(\tau_l(\xi))|^2 .$$

Finally, $z_j \notin W + \mathbb{Z}^n$ because otherwise we would have $z_j + W + \mathbb{Z}^n = W + \mathbb{Z}^n \subseteq E_f^-$, contradicting $0 \notin E_f^-$.

(e) $\Rightarrow$ (f). It follows from (e) that $\mathrm{m}(\tau_l(\xi)) = 0$ for $\xi \in z_j + W$ and $l \in \mathcal{E}$ such that $\tau_l(\xi) \notin z_{j+1} + W + \mathbb{Z}^n$, where $z_N := z_0$. Now for any $l \in \mathbb{Z}^n$ there exists an $l' \in \mathcal{E}$ such that $\tau_l(\xi) \equiv \tau_{l'}(\xi)$ (mod $\mathbb{Z}^n$); hence (f) follows.

(f) $\Rightarrow$ (d). Choose any $\eta \in z_0 + W$. Then $\eta \notin \mathbb{Z}^n$ because $z_0 \notin W + \mathbb{Z}^n$. For any $\alpha \in \mathbb{Z}^n$ consider the sequence

$$\omega_k = \left(A^T\right)^{-k}(\eta + \alpha), \quad k = 0, 1, 2, \cdots .$$

Then $\lim_{k\to\infty} \omega_k = 0$. Since $\bigcup_{j=0}^{N-1}(z_j + W) + \mathbb{Z}^n$ is locally compact and is disjoint from $\mathbb{Z}^n$, for sufficiently large $k$ we must have $\omega_k \notin \bigcup_{j=0}^{N-1}(z_j + W) + \mathbb{Z}^n$. Now $\omega_0 = \eta + \alpha \in z_0 + W$, so there exists a $k_0 > 0$ such that $\omega_{k_0-1} \in \bigcup_{j=0}^{N-1}(z_j + W) + \mathbb{Z}^n$ but $\omega_{k_0} \notin \bigcup_{j=0}^{N-1}(z_j + W) + \mathbb{Z}^n$.

We show that $\mathrm{m}(\omega_{k_0}) = 0$. Assume that $\omega_{k_0-1} \in z_j + W + \mathbb{Z}^n$. So $\omega_{k_0-1} = \xi_0 + l$ for some $\xi_0 \in z_j + W$ and $l \in \mathbb{Z}^n$. Now

$$\omega_{k_0} = \left(A^T\right)^{-1} \omega_{k_0-1} = \left(A^T\right)^{-1} (\xi_0 + l) = \tau_l(\xi_0) .$$

But $\omega_{k_0} = \tau_l(\xi_0) \notin z_{j+1} + W + \mathbb{Z}^n$, where $z_N := z_0$. So $\mathrm{m}(\omega_{k_0}) = 0$ by (f), proving (d). □

**Proof of Theorem 2.** $A^T$ is irreducible because it has the same characteristic polynomial as $A$ does. So the only $A^T$-invariant rational subspace $W$ of $\mathbb{R}^n$ with $\dim(W) < n$ is $W = \{0\}$, see Theorem III.12 of Newman [30]. Theorem 2 now follows immediately from Theorem 3. □

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