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THE DYNAMICAL INVERSE PROBLEM FOR A NON-SELF-ADJOINT STURM–LIOUVILLE OPERATOR

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An approach to the inverse problem (the so-called BC-method) based on boundary-control theory is developed. A procedure of reconstructing a nonsymmetric matrix-function (a potential) given on a semiaxis by a dynamical response operator is described. The results of numerical tests are presented. Bibliography: 6 titles.

0. Introduction

In the present paper, an approach to inverse problems (the so-called BC-method) based on boundary control theory [1, 2] is developed. It also gives a new interpretation of the local approach due to A. S. Blagoveshenskii [3]. The BC-method for a non-self-adjoint Sturm–Liouville operator is stated in [5] (see also [6]). In the present paper, a version of this method most suitable for numerical realization is considered. The results of numerical experiments are discussed.

1. The direct problem. The boundary-control problem

1.1. The direct problem

Let $V(x)$, $x \geq 0$, be a real $N \times N$ matrix-function with continuously differentiable elements. Consider the initial boundary-value problem (Problem 1)

$$u_{tt} - u_{xx} + V(x)u = 0, \quad (x,t) \in \mathbf{R}_+ \times (0,T), \quad T > 0, \tag{1}$$

$$u(x,0) = u_t(x,0) = 0, \tag{2}$$

$$u(0,t) = f(t). \tag{3}$$

The solution of this problem is a vector-function $u = u^f(x,t)$ with values in R^n. Sometimes, when using physical terminology, we call V, f, and u^f a *potential*, a *control*, and a *wave*, respectively.

Let a matrix-function $w(x,t)$ be a solution of the Goursat problem

$$\begin{cases} w_{tt} - w_{xx} + V(x)w = 0, & 0 < x < t < T, \\ w(0,t) = 0, & w(x,x) = -\dfrac{1}{2}\displaystyle\int_0^x V(s)ds. \end{cases} \tag{4}$$

It is known that $w(x,t)$ is twice continuously differentiable in the domain $\{(x,t) : 0 \leq x \leq t \leq T\}$. The following statement is easily verified.

Proposition 1.1.

(a) *If $f \in C^2([0,T]; \mathbf{R}^N)$ and $f(0) = f'(0) = 0$, then Problem 1 has a unique classical solution $u = u^f(x,t)$. In this case, the representation*

$$u^f(x,t) = \begin{cases} f(t-x) + \displaystyle\int_x^t w(x,s) f(t-s)ds, & \text{for} \quad x < t, \\ 0, & \text{for} \quad x \geq t, \end{cases} \tag{5}$$

is valid.

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(b) *For* $f \in L_2([0,T];\mathbf{R}^N)$*, the function* $u^f(x,t)$ *defined by* (5) *satisfies Eq.* (1) *in the sense of distribution theory.*

In the latter case, we regard u^f as a *generalized solution* of Problem 1 for controls of the class$L_2([0,T];\mathbf{R}^N)$. By (5), for any fixed moment $t=\xi$ we have

$$\operatorname{supp}\, u^f(\,\cdot\,,\xi) \subset \Omega^\xi, \quad 0 \le \xi \le T, \tag{6}$$

where $\Omega^\xi := [0,\xi]$ is an interval of the OX axis and the inclusion

$$u^f(\,\cdot\,,T) \in L_2(\Omega^T;\mathbf{R}^N), \qquad f \in L_2([0,T]);\mathbf{R}^N), \tag{7}$$

is valid.

Let $\mathcal{T}_\xi^T$ be a delay operator:

$$(\mathcal{T}_\xi^T f)(t) := f_{T-\xi}(t) := \begin{cases} 0, & 0 \le t < T-\xi, \\ f(t-(T-\xi)), & T-\xi \le t \le T, \end{cases} \tag{8}$$

where ξ is a parameter, $\xi \in (0,T)$;

$$\mathcal{T}_T^T f := f, \quad \mathcal{T}_0^T f := 0. \tag{9}$$

The independence of the potential $V(\,\cdot\,)$ from time leads to a known property of the solution u^f:

$$u^{f_{T-\xi}}(\,\cdot\,,T) = u^f(\,\cdot\,,\xi). \tag{10}$$

We note another property of the solution u^f, the so-called "localization principle," that is implied by the hyperbolicity of system (1)–(3). For any fixed $\xi \in (0,T/2)$, the values of the solution $u^f(x,t)$ for (x,t), $0 \le x \le \xi$, $x \le t \le 2\xi - x$, are uniquely determined by the values $V\,|_{x \le \xi}$, and they are independent of the behavior of $V\,|_{x>\xi}$.

1.2. The boundary-control problem

The statement of the boundary-control problem is as follows: given $a \in L_2(\Omega^T;\mathbf{R}^N)$, it is required to find $f \in L_2([0,T];\mathbf{R}^N)$ such that

$$u^f(\,\cdot\,,T) = a. \tag{11}$$

This setting naturally follows from relations (6), (7).

Lemma 1.1. *For any* $a \in L_2(\Omega^T;\mathbf{R}^N)$*, there exists a unique solution of problem* (11).

Proof. By (5), the above problem is equivalent to the solution of the equation

$$a(x) = f(T-x) + \int_x^T w(x,s) f(T-s)\,ds, \quad x \in \Omega^T. \tag{12}$$

The latter is the Volterra equation of the second kind with respect to $f(T-x)$. The solvability of this equation implies the solvability of the boundary control problem.

2. A DYNAMICAL SYSTEM

2.1. The control operator

In this section, we endow Problem 1 with the attributes of a dynamical system, namely, with spaces and operators. The space of controls $\mathcal{F}^T := L_2([0,T];\mathbf{R}^N)$ is called the *outer space* of dynamical system (1)–(3). The space $\mathcal{H}^T := L_2(\Omega^T;\mathbf{R}^N)$ is called the *inner one*; at each instant of time $t=\xi$, the wave $u^f(\,\cdot\,,\xi)$ belongs to $\mathcal{H}^T$ (see (6), (7)). The operator $W^T : \mathcal{F}^T \mapsto \mathcal{H}^T$,

$$W^T f = u^f(\,\cdot\,,T), \tag{13}$$

is called the *control operator* of the system.

Lemma 2.1. *For any $T > 0$, the operator W^T is bounded and boundedly invertible (i.e., W^T is an isomorphism).*

The proof follows from Lemma 1.1. From (5) we obtain a representation of the operator W^T:

$$(W^T f)(x) = f(T-x) + \int_x^T w(x,s) f(T-s) ds, \quad x \in \Omega^T. \tag{14}$$

The outer space $\mathcal{F}^T$ contains a family of subspaces $\mathcal{F}^{T,\xi}$ formed by delay controls (see (8)–(10)):

$$\mathcal{F}^{T,\xi} := \mathcal{T}_\xi^T \mathcal{F}^T = \left\{ f \in \mathcal{F}^T : \operatorname{supp} f \subset [T-\xi, T] \right\}, \; 0 \le \xi \le T. \tag{15}$$

The set

$$\mathcal{U}^\xi := W^T \mathcal{F}^{T,\xi} \tag{16}$$

is said to be *reachable* (in an amount of time ξ).

Lemma 2.1 states that $\mathcal{U}^T = \mathcal{H}^T$. It is clear that a similar relation is valid for any instant of time:

$$\mathcal{U}^\xi = \mathcal{H}^\xi, \quad 0 \le \xi \le T, \tag{17}$$

where $\mathcal{H}^\xi$ is a subspace of the space $\mathcal{H}^T$,

$$\mathcal{H}^\xi := \left\{ a \in \mathcal{H}^T : \operatorname{supp} a \subset \Omega^\xi \right\}.$$

2.2. The response operator

The mapping "input–output" in our dynamical system is realized by a *response operator* $R^T : \mathcal{F}^T \mapsto \mathcal{F}^T$;

$$\operatorname{Dom} R^T = \left\{ f \in C^2([0,T]; \mathbf{R}^N) : f(0) = f'(0) = 0 \right\},$$

$$\left(R^T f \right)(t) := u_x^f(0,t), \quad t \in [0,T].$$

It is well defined by Proposition 1.1.

Proposition 2.1. *For any $T > 0$, the following representation is valid:*

$$\left(R^T f \right)(t) = -f'(t) + \int_0^t r(t-s) f(s) ds, \; 0 < t < T, \tag{18}$$

where $r(t)$ is a continuously differentiable matrix-function for $t > 0$.

This statement follows from Proposition 1.1.

The response operator will play the role of data in the inverse problem.

2.3. The dual system

A dynamical system of the form

$$u_{tt} - u_{xx} + V^{\#}(x) u = 0, \quad (x,t) \in \mathbf{R}_+ \times (0,T), \quad T > 0,$$

$$u(x,0) = u_t(x,0) = 0, \quad u(0,t) = g(t),$$

with matrix-potential $V^{\#}$ transposed to V is said to be *dual* to the initial system (1)–(3). Let $u_{\#}^g$ be a solution of it, and let $W_{\#}^T$ be the corresponding control operator: $W_{\#}^T g = u_{\#}^g(\cdot, T)$. Similarly to W^T, the operator $W_{\#}^T$ is an isomorphism.

The response operator of the dual system also admits a representation of the form (18):

$$(R_{\#}^T g)(t) := \left(u_{\#}^g\right)_x(0,t) = -g'(t) + \int_0^t r_{\#}(t-s) g(s) ds. \tag{19}$$

It can be shown that there is a simple relationship between the response operators of the initial and dual systems.

Proposition 2.2. *The matrix kernels r in (18) and $r_{\#}$ in (19) are mutually transposed:*

$$r^{\#}(t) = r_{\#}(t). \tag{20}$$

2.4. The connecting operator

For arbitrary controls $f,\ g \in \mathcal{F}^T$ and for the corresponding solutions u^f and $u^g_{\#}$ of Problems I and $\mathrm{I}_{\#}$, we have

$$\left(u^f(\cdot,T), u^g_{\#}(\cdot,T)\right)_{\mathcal{H}^T} = (W^T f, W^T_{\#} g)_{\mathcal{H}^T} = \left((W^T_{\#})^* W^T f, g\right)_{\mathcal{F}^T} = (C^T f, g)_{\mathcal{F}^T} \tag{21}$$

with an operator $C^T : \mathcal{F}^T \mapsto \mathcal{F}^T$, where $C^T := (W^T_{\#})^* W^T$. This operator is an isomorphism, because $W^T_{\#}$ and W^T are isomorphisms.

The operator C^T is called the *connecting operator*, because it relates the metrics of the outer and inner spaces.

The following fact is important for the inverse problem: the connecting operator is determined by the response operator.

In order to formulate the results, we introduce auxiliary operators:
the operator of odd extension $S^T : \mathcal{F}^T \mapsto \mathcal{F}^T t$,

$$(S^T f)(t) = \begin{cases} f(t), & 0 \le t \le T, \\ -f(2T - t), & T < t \le 2T; \end{cases}$$

the operator of separation of the odd part $Q^{2T} : \mathcal{F}^T t \mapsto \mathcal{F}^T t$,

$$(Q^{2T} f)(t) = \frac{1}{2}[f(t) - f(2T - t)];$$

the restriction operator $N^{2T} : \mathcal{F}^T t \mapsto \mathcal{F}^T$,

$$N^{2T} f = f\mid_{[0,T]};$$

the integration operator $J^{2T} : \mathcal{F}^T t \mapsto \mathcal{F}^T t$,

$$(J^{2T} f)(t) = \int_0^t f(s)ds, \quad 0 \le t \le 2T.$$

We shall use the relation

$$(S^T)^* := 2M^T Q^{2T}, \tag{22}$$

which is easily verified.

Let R^{2T} be the response operator corresponding to system (1)–(3) with final instant of time $2T$.

Theorem 2.1. *The following representations are valid:*

$$C^T = -\frac{1}{2}(S^T)^* J^{2T} R^{2T} S^T; \tag{23}$$

$$(C^T f)(t) = f(t) + \int_0^T [p(2T - t - s) - p(\mid t - s \mid)] f(s)ds, \tag{24}$$

where

$$p(t) := \frac{1}{2}\int_0^t r(s)ds.$$

Proof. For arbitrary functions $f,\ g \in C_0^\infty([0,T];\mathbf{R}^N)$, we put $f_- := S^T f$ and introduce the function

$$w^{fg}(s,t) := \left(u^{f_-}(\,\cdot\,,s), u^g_{\#}(\,\cdot\,,t)\right)_{\mathcal{H}^T}; \quad 0 \le s \le 2T,\ 0 \le t \le T.$$

Note that $f_- \in \operatorname{Dom} R^{2T}$. We obtain the relations

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2}\right] w^{fg}(s,t) = \int_{\Omega^T} \left[\langle u^{f_-}(x,s), (u^g_{\#})_{tt}(x,t)\rangle - \langle u^{f_-}_{ss}(x,s), (u^g_{\#})(x,t)\rangle\right] dx$$

$$= \int_{\Omega^T} \left[\langle u^{f_-}(x,s), (u^g_{\#})_{xx} - V^{\#}(x) u^g_{\#}(x,t)\rangle - \langle u^{f_-}_{ss}(x,s) - V(x) u^{f_-}(x,s), u^g_{\#}(x,t)\rangle\right] dx$$

$$= \int_{\Omega^T} \left[\langle u^{f_-}(x,s), (u^g_{\#})_{xx}(x,t)\rangle - \langle u^{f_-}_{xx}(x,s), (u^g_{\#})(x,t)\rangle\right] dx$$

$$= \left[\langle u^{f_-}(x,s), (u^g_{\#})_x(x,t)\rangle - \langle u^{f_-}_x(x,s), (u^g_{\#})(x,t)\rangle\right]_{x=0}^{T} = -\langle f_-(s), (R^T_{\#} g)(t)\rangle + \langle (R^{2T} f_-)(s), g(t)\rangle.$$

In the last relation we use the fact that $(u^g_{\#})_x(T,s) = u^g_{\#}(T,t) = 0$ for $f,\ g \in C_0^\infty([0,T];\mathbf{R}^N)$ (this easily follows from (5)).

Thus, the function w^{fg} satisfies the equation

$$w^{fg}_{tt} - w^{fg}_{ss} = -\langle f_-(s), (R^T_{\#} g)(t)\rangle + \langle (R^{2T} f_-)(s), g(t)\rangle;$$
$$0 \le s \le 2T, \quad 0 \le t \le T, \tag{25}$$

and the condition (see (2))

$$w^{fg}(s,0) = w^{fg}_t(s,0) = 0. \tag{26}$$

Integrating (25), (26) in the domain $(s,t) : 0 \le t \le T,\ t \le s \le 2T - t$ by the d'Alembert formula and putting $t = s = T$, we obtain

$$w^{fg}(T,T) = -\frac{1}{2} \int_0^T d\eta \int_\eta^{2T-\eta} \left[\langle f_-(\xi), (R^T_{\#} g)(\eta)\rangle - \langle (R^{2T} f_-)(\xi), g(\eta)\rangle\right] d\xi.$$

Since $\int_\eta^{2T-\eta} f_-(\xi) d\xi = 0$, we have

$$w^{fg}(T,T) = \frac{1}{2} \int_0^T d\eta \int_\eta^{2T-\eta} \langle (R^{2T} f_-)(\xi), g(\eta)\rangle d\xi. \tag{27}$$

On the other hand,

$$\int_\eta^{2T-\eta} (R^{2T} f_-)(\xi) d\xi = (J^{2T} R^{2T} f_-)(2T - \eta) - (J^{2T} R^{2T} f_-)(\eta) = -2(Q^{2T} J^{2T} R^{2T} f_-)(\eta).$$

Taking into account (21), relation (27) takes the form

$$w^{fg}(T,T) = -\int_0^T \langle (N^T Q^{2T} J^{2T} R^{2T} S T f)(\eta), g(\eta)\rangle d\eta = -\frac{1}{2}\left((S^T)^* J^{2T} R^{2T} S^T f, g\right)_{\mathcal{F}^T}. \tag{28}$$

By the definition of w^{fg}, we obtain

$$w^{fg}(T,T) = (C^T f, g)_{\mathcal{F}^T}. \tag{29}$$

Comparing the right-hand sides of (28) and (29) and using the arbitrariness of f and g from $C_0^\infty([0,T];\mathbf{R}^N)$, we obtain representation (23). Taking into account (18), we derive (24) from (23). The theorem is proved.

3. The inverse problem

3.1. The Gelfand–Levitan equation

We consider the boundary-control problem (see Sec. 1.2) with a special right-hand side, namely, as a we take the function y^T that is the restriction of the solution of the Cauchy problem to Ω^T:

$$-y''(x) + V(x)y(x) = 0, \;\; x > 0, \tag{30}$$

$$y(0) = \alpha, \;\; y'(0) = \beta, \tag{31}$$

where α and β are arbitrary vectors from $\mathbf{R}^N$. Denote by z^T a solution of the problem $z^T := \left(W^T\right)^{-1} y^T$. We prove that the function $z^T(\,\cdot\,)$ satisfies a linear equation both sides of which are expressed in terms of the data of the inverse problem.

Put $\kappa^T(t) := T - t$, $0 \le t \le T$, and consider elements of the space of controls of the form $\kappa^T\alpha$, $\kappa^T\beta$.

Theorem 3.1. *The function $z^T \in \mathcal{F}^T$ is a unique solution of the equation*

$$C^T z^T = \kappa^T\beta - (R^T_{\#})^* \kappa^T\alpha. \tag{32}$$

Proof. For any $g \in C_0^\infty([0,T];\mathbf{R}^N)$, we have

$$\begin{aligned}
\left(C^T z^T, g\right)_{\mathcal{F}^T} &= \left(W^T z^T, W^T_{\#} g\right)_{\mathcal{H}^T} = \int_{\Omega^T} \langle y(x), u^g_{\#}(x,T)\rangle dx = \int_0^T (T-t)dt \int_{\Omega^T} \langle y(x), \left(u^g_{\#}\right)_{tt}(x,t)\rangle dx \\
&= \int_0^T (T-t)dt \int_{\Omega^T} \langle y(x), \left(u^g_{\#}\right)_{xx}(x,t) - V^{\#}(x) u^g_{\#}(x,t)\rangle dx \\
&= \int_0^T (T-t)\langle y(x), [(u^g_{\#}(x,t)\rangle - \langle y'(x), u^g_{\#}(x,t)\rangle]_{x=0}^{x=T} dt \\
&= -\int_0^T (T-t)[\langle \alpha, (R_{\#}g)(t)\rangle - \langle \beta, g(t)\rangle] dt \\
&= \int_0^T [\langle \kappa^T(t)\beta, g(t)\rangle - \langle \kappa^T(t)\alpha (R_{\#}g)(t)\rangle] dt = \left(\kappa^T\beta - [R^T_{\#}]^* \kappa^T\alpha, g\right)_{\mathcal{F}^T}.
\end{aligned}$$

Comparing the beginning and the end of the above string of equalities, we obtain (32). Using Lemma 1.1, we complete the proof of the theorem.

Taking into account Proposition 2.2, it is easy to verify the relation

$$\left([R^T_{\#}]^* g\right)(t) = g'(t) + \int_t^T r(s-t) g(s) ds,$$

whence

$$\left([R^T_{\#}]^* \kappa^T\alpha\right)(t) = -\alpha + \int_t^T r(s-t)(T-s)\alpha ds, \;\; 0 \le t \le T.$$

In addition to (34), we find a relation that will be used for the solution of the inverse problem. Putting $f = z^T$ in (14), we have

$$\left(W^T z^T\right)(x) = z^T(T-x) + \int_x^T w(x,s) z^T(T-s) ds = y(x), \;\; x \in \Omega.$$

As $x \to T - 0$, we obtain

$$z^T(+0) = y(T). \tag{33}$$

3.2. Solution of the inverse problem

The statement of the inverse problem is as follows: given a function $r(t)$, $0 \le t \le 2T$ (or, equivalently, a response operator R^{2T}, see (18)), it is required to reconstruct a potential $V(x)$, $x \in \Omega^T$, by using $r(t)$.

Describe the process of solution of the inverse problem:

(a) by the matrix-function $r(t)$, $0 < t \le 2T$, using (18)–(20) and (23), we find the operators R^τ, $R^\tau_\#$, $0 < \tau \le 2T$, and C^τ, $0 < \tau \le T$.

(b) Form a family of equations:

$$C^\tau z^\tau = \kappa^\tau \beta - (R^\tau_\#)^* \kappa^\tau \alpha, \tag{34}$$

where α, $\beta \in \mathbf{R}^N$, and the index τ is a parameter, $\tau \in (0, T]$. Each of Eqs. (34) is uniquely solvable in the respective space $\mathcal{F}^\tau$, and the solution z^τ of it is connected with the solution y of the Cauchy problem (30), (31) with given α and β by the relation $z^\tau = (W^\tau)^{-1} y^\tau$. We fix τ and consider N equations of the form (34), where α_j and β_j are chosen so that the vectors-columns

$$\begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix}, \quad j = 1, \dots, N,$$

are linearly independent in $\mathbf{R}^{2N}$. Find solutions of these equations $z^\tau_j(t)$, $t \in [0, \tau]$.

(c) Using (33), by $z^\tau_j(t)$ $(0 \le t \le \tau \le T)$ we reconstruct a solution $y_j(x)$ $(x \in \Omega^T)$ of the Cauchy problem (30), (31) with given α_j and β_j:

$$y_j(x) = z^x_j(+0), \quad 0 \le x \le T, \quad j = 1, \dots, N. \tag{35}$$

Having obtained the solution $y_j(x)$ for any $x \in \Omega^T$, we find the second derivatives

$$y''_j(x) = \frac{d^2}{dx^2} z^x_j, \quad 0 \le x \le T, \quad j = 1, \dots, N. \tag{36}$$

(d) Fix $x \in \Omega^T$ and form the matrices $Y(x)$ and $Y''(x)$ with vectors (35) and (36) as columns. If the matrix $Y(x)$ is nonsingular, we reconstruct the potential $V(x)$ by the formula $V(x) = Y''(x) Y^{-1}(x)$. At a finite number of points of singularity of $Y(x)$, we define $V(x)$ by continuity. The inverse problem is solved.

4. A NUMERICAL EXPERIMENT

4.1. An algorithm

The Gelfand–Levitan equation (32) and relation (33) underlie a numerical algorithm for solving the inverse problem:

(a) for given $T > 0$, we specify a partition of the time interval $[0, T]$ by a system of points $0 = \xi_0 < \xi_1 < \dots < \xi_M = T$. Consider the family of equations obtained from (32) for $\alpha = \mathrm{col}\{0, \dots, 0\}$, $\beta = \mathrm{col}\{1, \dots, 1\}$:

$$C^\xi Z = \kappa^\xi E \tag{37}$$

($\xi = \xi_1, \dots, \xi_M$; E is the identity matrix, $\kappa^\xi = \xi - t$). Approximate solutions $\widetilde{Z}$ of these equations are sought in the form of matrix polynomials

$$\widetilde{Z}(t) = \sum_{k=0}^{q} (\xi - t)^k \cdot A^\xi_k \quad (\xi = \xi_1, \dots, \xi_M) \tag{38}$$

with unknown A^ξ_k.

(b) The coefficients A^ξ_k are determined with the help of the optimization procedure

$$\min_{A^\xi_0, \dots, A^\xi_q} \Phi^\xi(A^\xi_0, \dots, A^\xi_q) = \min \|C^\xi \widetilde{Z} - \kappa^\xi E\|^2.$$

In the ordinary way we obtain a system of equations

$$\frac{\partial \Phi^{\xi}}{\partial A_k^{\xi}} = 0 \quad (k = 0, \dots, q).$$

(c) By the methods of numerical differentiation, approximating values of the potential

$$\widetilde{V}(\xi) = \left(\frac{d^2}{d\xi^2}\widetilde{Z}^{\xi}\right)\bigg|_{\xi=+0} \cdot \left(\widetilde{Z}^{\xi}(+0)\right)^{-1} \cdot \quad (\xi = \xi_1, \dots, \xi_M)$$

are found.

The algorithm has the following input parameters:

N is the dimension of the matrix problem;
T is a finite moment of time;
M is the number of points $\xi \in [0, T]$;
h is a step of partition, $h = \xi_i - \xi_{i-1} = T/M$;
q is the dimension of an approximating polynomial;
$r(t)$ is an $N \times N$ matrix-function given for $t \in [0, 2T]$.

4.2. Problems for testing

As tests we considered the following problems in which the Gelfand–Levitan equations are solved in explicit form:

1. $N = 2$; $T = 0.5$; $M = 20$; $h = 0.025$; $q = 3$;

$$r(t) = \begin{pmatrix} 2t, & 0 \\ 0, & 2t \end{pmatrix}, \quad t \in [0, 1];$$

(the diagonal case)

2. $N = 2$; $T = 0.5$; $M = 20$; $h = 0.025$; $q = 3$;

$$r(t) = \begin{pmatrix} 3t, & t \\ t, & 3t \end{pmatrix}, \quad t \in [0, 1];$$

(the nondiagonal symmetric case)

3. $N = 2$; $T = 0.5$; $M = 20$; $h = 0.025$; $q = 4$;

$$r(t) = \begin{pmatrix} \frac{A_1}{\Delta}\varphi_b(t) - \frac{A_2}{\Delta}\varphi_a(t), & \frac{1}{\Delta}\varphi_a(t) - \frac{1}{\Delta}\varphi_b(t) \\ \frac{a_{21}}{a_{11}\Delta}[\varphi_b(t) - \varphi_a(t)], & \frac{A_1}{\Delta}\varphi_a(t) - \frac{A_2}{\Delta}\varphi_b(t) \end{pmatrix}, \quad t \in [0, 1],$$

where

$$a_{11} = 6; \quad a_{12} = 3; \quad a_{21} = 5; \quad a_{22} = 8;$$

$$A_1 = \frac{a}{a_{12}}; \quad A_2 = \frac{b}{a_{12}}; \quad \Delta = A_2 - A_1;$$

$$a = \frac{a_{11} + a_{12}}{2} + \sqrt{a_{12} \cdot a_{21} - \left(\frac{a_{22} - a_{11}}{2}\right)^2};$$

$$b = \frac{a_{11} + a_{12}}{2} - \sqrt{a_{12} \cdot a_{21} - \left(\frac{a_{22} - a_{11}}{2}\right)^2};$$

$$\varphi_a(t) = \begin{cases} \dfrac{\alpha}{t} J_1(\alpha t), & \alpha = \sqrt{a}, \quad a > 0, \\ -\dfrac{\alpha}{t} I_1(\alpha t), & \alpha = \sqrt{|a|}, \quad a < 0; \end{cases}$$

$$\varphi_b(t) = \begin{cases} \dfrac{\beta}{t} J_1(\beta t), & \beta = \sqrt{b}, \quad b > 0, \\ -\dfrac{\beta}{t} I_1(\beta t), & \beta = \sqrt{|b|}, \quad b < 0, \end{cases}$$

(the nondiagonal asymmetric case).

(Here $J_1(\cdot)$, $I_1(\cdot)$ are the Bessel functions of the first and second kind, respectively.)

4.3. Results

The results of numerical solution of the test inverse problems are shown in Figs. 1–3. The exact values of $V(x)$ are drawn by a dotted line; the reconstructed values of $\widetilde{V}(x)$ are drawn by a solid line.

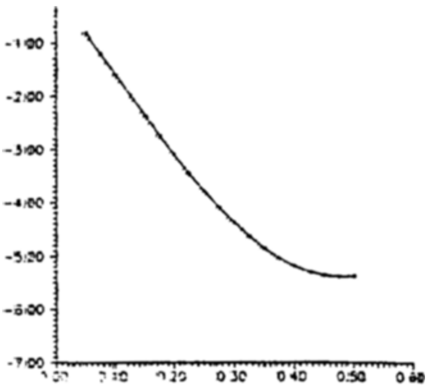

FIG. 1

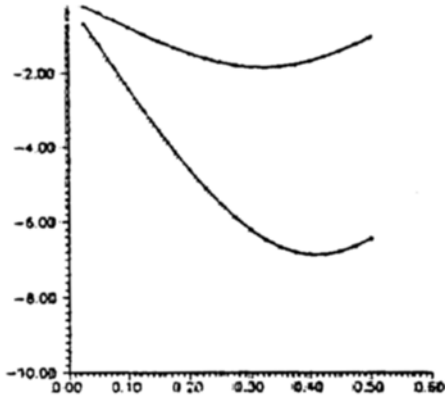

FIG. 2

In the process of reconstructing the potential, the determination of the matrix-function $Y(x) = \{ y_1(x), \dots, y_N(x) \}$ (see (30), (31)) is of independent interest. As was mentioned above, this is concerned with the solution of a respective boundary-control problem (8). The values of $Y(\cdot)$ determined by the algorithm are shown in Figs. 4–6.

These results allow us to claim that the algorithm works successfully in the case of sufficiently smooth matrix-functions $r(t)$. Note that the precision of the reconstruction of $V(x)$ depends on the dimension q of the approximating polynomial (38).

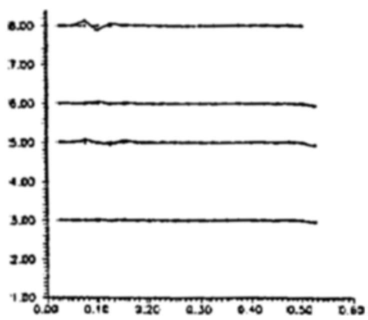

FIG. 3

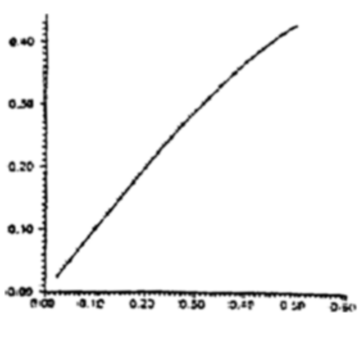

FIG. 4

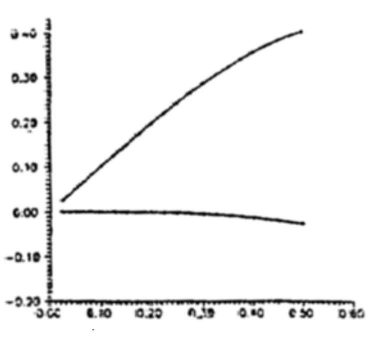

FIG. 5

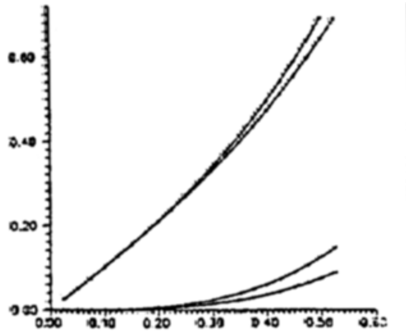

FIG. 6

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