A class of discounted models for singular diffusion control

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Abstract A kind of discounted problems for singular diffusion control have been studied. The drift and diffusion coefficients of state process are nonlinear. The class of models has been basically extended from a corresponding one established by Karatzes et al. before. By applying some analysis methods different from earlier works, the sufficient and necessary conditions of the existence of optimal control have been obtained. If an optimal control exists, it is a "transient reflection" of state process for two lines.

Keywords: stochastic control, variational equation, singular stochastic control.

By variational equation and stochastic analysis methods, we discuss a class of singular stochastic control problems, and obtain the sufficient and necessary conditions of existence for optimal control of discounted models in this note.

Stochastic control is a research field covering many interdiscipline branches of science such as Probability, Equation Theory and Functional Analysis, etc. It has been widely applied in aviation spaceflight, communication, manufacture and finance control etc. Conversly, the study of stochastic control has given impetus to the development of these relative subjects.

Based on the fashionable academic opinion, e.g. in the report of American "Future of Control Theory" special group, Future of Control Theory-Math. Forecast, it is pointed out that because singular control sufficiently contains the traditional diffusion control and impulse control, it will be a very important direction in the development of stochastic control theory.

Probably the model of stochastic control initially appeared in ref. [1], and was tentatively extended by Karatzas[2]. The discounted cost models for singular stochastic control was presented as follows.

Let $(\Omega, \mathscr{F}, P)$ be a given probabilty space, $W=\{W_t, t\geqslant 0\}$ be a standard Wiener process on it, $\mathscr{F}_t=\sigma(W_s, 0\leqslant s\leqslant t)$. Denote by $\mathscr{B}$ the totality of $\mathscr{F}_t$-adapted, left-continuous bounded variational processes $\xi=\{\xi_t, t\geqslant 0\}$ with $\xi_0=0$. It is well known that for each $\xi\in\mathscr{B}$, there is a canonical decomposition: $\xi_t=\xi_t^+ + \xi_t^-$, and we denote by $\check{\xi}_t=\xi_t^+ + \xi_t^-$ the total variational process of ξ. Let $\alpha>0$ be some constant, called discounted coefficient. The so-called discounted problem for singular control in ref.[2], means that for any given $x\in R$, find $\xi^*\in\mathscr{B}$ such that

$$E\int_0^{\infty} e^{-\alpha t}\left\{d\check{\xi}^* + h(x+W_t+\xi_t^*)dt\right\} = \min_{\xi\in\mathscr{B}} E\int_0^{\infty} e^{-\alpha t}\{d\xi_t + h(x+W_t+\xi_t)dt\}, \tag{1}$$

where $h(\cdot)$ is twice continuous differentiable even function on R. Clearly, the above Karatzas's model was relatively special both in cost structure and in state process, and should be developed further.

Liu had extended Karatzas's model[3,4]. In the cost structure, he introduced a class of nonnagative even functions such that the model would change in the following form: For any given $x\in$ R, find $\xi^*\in\mathscr{B}$ satisfying

$$E\int_0^{\infty} e^{-\alpha t}\left\{g(x+W_t+\xi_t^*)d\check{\xi}^* + h(x+W_t+\xi_t^*)dt\right\} = \min_{\xi\in\mathscr{B}} E\int_0^{\infty} e^{-\alpha t}\left\{g(x+W_t+\xi_t)d\check{\xi}_t + h(x+W_t+\xi_t)dt\right\}. \tag{2}$$

Consequently this model is an extension of model (1) in cost structure. Later, by a kind of stopping problems models, the author also studied another model similar to (2) but having less restrictions on $h(\cdot)$ and $g(\cdot)$, and got a sufficient condition of existence for optimal control.

Ma[6] had tried to extend Karatzas's model in state process. For this, he built an S.I.E.:

$$X_t = x + \int_0^t \mu(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}W_s + \xi_t, \quad t \geqslant 0, \tag{3}$$

where $x \in \mathbb{R}$ is an initial value, $\xi^* \in \mathscr{B}$, $\mu(\cdot)$ and $\sigma(\cdot)$ are drift and diffusion coefficient respectively. It is well known that if $\mu(\cdot)$ and $\sigma(\cdot)$ satisfy the Lipschitz condition, then there is a unique $\mathscr{F}$-adapted, left-continuous solution of equation (3): $X=\{X_t, t\geqslant 0\}$.

The model in ref. [6] takes the following form: For any given $x \in R$, find $\xi^* \in \mathscr{B}$ such that

$$\begin{aligned} &E\int_0^\infty \mathrm{e}^{-\alpha t}\left\{\tilde{C}\mathrm{d}\check{\xi}^* + h(X_t^*)\mathrm{d}t\right\} \\ &= \min_{\xi\in\mathscr{B}} E\int_0^\infty \mathrm{e}^{-\alpha t}\left\{\tilde{C}\mathrm{d}\check{\xi}_t + h(X_t)\mathrm{d}t\right\}, \end{aligned} \tag{4}$$

where $X^* = \{X_t^*, t\geqslant 0\}$ satisfies S.I.E.

$$X_t^* = x + \int_0^t \mu(X_s^*)\mathrm{d}s + \int_0^t \sigma(X_s^*)\mathrm{d}W_s + \xi_t^*, \qquad t\geqslant 0, \tag{5}$$

$\tilde{C}$ is some positive constant, for short it is taken: $\tilde{C} \equiv 1$[6]. Clearly, the model in ref. [6] has no basical extension in cost structure of Karatzas's model.

Besides, a series of restrictions were made[6]: first, $\mu(\cdot)$ was supposed a linear function, i.e. $\mu(x) = ax+b$, $\forall x \in \mathbb{R}$, and $\sigma(\cdot)$ was desired as a positive (or negative) value function satisfying $|\sigma'(x)| + |[\sigma^2(x)]''| \leqslant K$ for any $x \in \mathbb{R}$ and some $K>0$. Secondarily, the discounted coefficient α had to satisfy $\alpha > \frac{1}{2}\sup_{x\in\mathbb{R}} |[\sigma^2(x)]''| + 2|a|$. Thirdly, for function $h(\cdot)$, suppose that it is a twice continuous differentable function such that $K' \leqslant h''(x) \leqslant K''$ for any $x \in \mathbb{R}$ and $K', K'' > 0$, also there is $\bar{x} \in \mathbb{R}$ such that $h'(\bar{x}) = 0$, $(x-\bar{x})h'(x) \geqslant 0$ for any $x \in \mathbb{R}$. Clearly, too many extra restraints were brought to bear on the model[6].

Recently, Takser[7] has studied the discounted problem for singular control from another angle. He has discussed the problem of applying linear programming model to approximate singular control model, this problem is essentially proposing the relation between linear programming model and singular control model. This approach is similar as in ref. [5].

In this note, we have made more extension for Karatzas's model, and obtained a sufficient and necessary condition of existence of optimal control. First, we have kept the extension in cost structure of model (2), then extended the state process from $X = \{x + W_t + \xi_t, t \geqslant 0\}$ to the solution process of a class of S.I.E.: $X = \{X_t, t \geqslant 0\}$. Also, we cancel the restraint put on model (3) in ref. [6]: $\mu(\cdot)$ only was linear.

As for the analystical approach, this article is very different from earlier works. Because the model was subject to many special restrictions in earlier works, the relative method was no longer suitable for the model in this note. Therefore, we first discuss a corresponding variational equation problem and prove the existence of its solution. Besides, in this note, applying some special stochastic analysis mothods, we also prove the non-existence of optimal control under some conditions and consequently obtain the sufficient and necessary conditions of optimal control. Clearly, this work is more difficult than the earlier ones in which the existence was only considered.

Next, we shall present some lemmas and main results.

Assume that $\mu(x)$ is a continuous differentiable odd function on $\mathbb{R}$ and $\sigma(x)$ is a continuous even function,

$$\sigma(0) > 0, \tag{6}$$

also nondecreasing on $\mathbb{R}^+$, they all satisfy Lipschitz condition;

$g(x)$ is a continuous even function, $g'(x)$ exists for $x>0$, continuous, nonnegative and nondecreasing,

$$g(0) > 0. \tag{7}$$

$h(x)$ is a continuous differentable, nonnegative even function, $h''(x)$ exists for $x>0$ and nonnegative, also there are two positive constants M, N such that

$$h''(x) \geqslant N \quad \text{for } x > M. \tag{8}$$

Lemma 1. Suppose that $\mu(\cdot)$ and $\sigma(\cdot)$ are real value functions satisfying Lipschitz condition, then for any given $x \in \mathbb{R}$, $\xi \in \mathscr{B}$, there exists a unique solution of S.I.E. (3): $X = \{X_t, t \geqslant 0\}$ and it is $\mathscr{F}_t$-adapted, left-continuous.

Proof. See ref. [8].

Lemma 2. Suppose that conditions (6)—(8) hold, also $\inf_{x\in \mathbb{R}} \mu'(x) \geqslant \alpha$, then there exist a twice continuous differentiable, positive value even function $v(x)$ and a constant $c>0$ such that

$$\frac{1}{2}\sigma^2(x)v''(x) + \mu(x)v'(x) + h(x) \geqslant \alpha v(x), \quad v''(x) \geqslant 0, \quad x \in \mathbb{R}; \tag{9}$$

$$\frac{1}{2}\sigma^2(x)v''(x) + \mu(x)v'(x) + h(x) = \alpha v(x) \quad \text{for } |x| \leqslant c,$$

$$\text{if } 0 \leqslant x < c, \text{ then } 0 \leqslant v'(x) \leqslant g(x);$$

$$\text{if } x \geqslant c, \text{ then } v'(x) = g(x); \text{ if } -c \leqslant x \leqslant 0, \text{ then } -g(x) \leqslant v(x) \leqslant 0; \tag{10}$$

$$\text{if } x \leqslant -c, \text{ then } v'(x) = -g(x).$$

When $|x| > c$ and $g'(x) \neq 0$, we have

$$\frac{1}{2}\sigma^2(x)v''(x) + \mu(x)v'(x) + h(x) > \alpha v(x). \tag{11}$$

Besides, for any given $x_0 \in \mathbb{R}$, we can construct a twice continuous differentiable, nonnegative even function $V(x;x_0)$ and constant $d > |x_0|$ such that

$$V(x;x_0) = v(x) \quad \text{for } |x| \leqslant x_0,$$

$$\text{for any } x \in \mathbb{R}, \quad |V'(x;x_0)| \leqslant g(x), \quad \frac{1}{2}\sigma^2(x)V''(x;x_0) +$$

$$\mu(x)V'(x;x_0) + h(x) \geqslant \alpha V(x;x_0), \text{ also } V''(x;x_0) = 0 \quad \text{for } |x| \geqslant d. \tag{12}$$

Lemma 3. Let $-\infty < c_1 < c_2 < +\infty$ be two real numbers. Then for any given $x \in [c_1, c_2]$, there exist a unique pair of monotone nondecreasing continuous processes $\theta^{\pm} = \{\theta_t^{\pm}, t \geqslant 0\}$ and a continuous semimartingale $X^* = \{X_t^*, t \geqslant 0 =$ such that

(i) $X_t^* = x + \int_0^t \mu(X_s^*)\mathrm{d}s + \int_0^t \sigma(X_s^*)\mathrm{d}W_s + \theta_t^+ - \theta_t^-$, $t \geqslant 0$ and for any $t \geqslant 0$, $c_1 \leqslant X_t^* \leqslant c_2$.

(ii) θ^+ has increment only at $X_t^* = c_1$, otherwise its paths are all straight.

(iii) θ^- has increment only at $X_t^* = c_2$, otherwise its paths are all straight.

Proof. See ref. [8].

Theorem. Suppose that the conditions of Lemma 2 all are satisfied, $v(\cdot)$ and c are the function and constant respectively determined by Lemme 2, then for any given $x \in \mathbb{R}$, we have

$$v(x)=\inf_{\xi\in\mathscr{B}} E\int_0^\infty \mathrm{e}^{-\alpha t}\left\{g(X_t)\mathrm{d}\check{\xi}_t+h(X_t)\mathrm{d}t\right\}, \tag{13}$$

where $X=\{X_t,\ t\geqslant 0\}$ is solution process of S. I. E. (3) corresponding to x and $\xi=\{\xi_t,\ t\geqslant 0\}$. If $|x|\leqslant c$ or $|x|>c$ and $g'(x)=0$, then there is an optimal control $\xi^*\in\mathscr{B}$, i.e. we have

$$v(x)=E\int_0^\infty \mathrm{e}^{-\alpha t}\left\{g(X_t^*)\mathrm{d}\check{\xi}_t^*+h(X_t^*)\mathrm{d}t\right\}, \tag{14}$$

where $X^*=\{X_t^*,\ t\geqslant 0\}$ is the solution process of S.I.E. (4) corresponding to x and $\xi^*=\{\xi_t^*,\ t\geqslant 0\}$. However, if $|x|>c$ and $g'(x)\neq 0$, then the optimal control does not exist.

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