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2*T*-PERIODIC SOLUTION FOR *m* ORDER NEUTRAL TYPE DIFFERENTIAL EQUATIONS WITH TIME DELAYS *

ZHANG Bao-sheng (张保生), ZHU Guang-hui (朱光辉)

(Department of Mathematics, Yunnan Nationalities Institute,
Kunming 650031, P R China)

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Abstract: *Periodic solution of m order linear neutral equations with constant coefficient and time delays was studied. Existence and uniqueness of 2T-periodic solutions for the equation were discussed by using the method of Fourier series. Some new necessary and sufficient conditions of existence and uniqueness of 2T-periodic solutions for the equation are obtained. The main result is used widely. It contains results in some correlation paper for its special case, improves and extends the main results in them. Existence of periodic solution for the equation in larger number of particular case can be checked by using the result, but cannot be checked in another paper. In other words, the main result in this paper is most generalized for (1), the better result cannot be found by using the same method.*

Key words: neutral type equation; 2*T*-periodic solution; Fourier series

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Introduction

It is well-known that neutral type differential equation are widely used in biology, physics and chemistry. The study of the neutral equation has important significance in both theory and application. In this paper, we consider periodic solution of neutral equation as

$$\sum_{i=0}^{m}[a_i x^{(m-i)}(t) + b_j x^{(m-j)}(t-h_j)] = f(t), \tag{1}$$

where $a_0 = 1$, a_i, b_j, $h_j \geqslant 0$ $(j = 0,1,2,\cdots,m)$ are constants. $f(t)$ is m order continuous differentiable function with $2T$-period. We assume that Fourier expansion of $f(t)$ is

$$f(t) = k_0 + \sum_{n=1}^{\infty}[k_n\cos(\alpha nt) + l_n\sin(\alpha nt)], \tag{2}$$

where k_0, k_n, l_n are Fourier coefficients of $f(t)$, $\alpha = \pi/T > 0$.

When $m = 2$, periodic solution for Eq.(1) was discussed under the condition $|b_0| \neq 1$ in

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Biography: ZHANG Bao-sheng (1962 –), Associate Professor

papers [1] - [3], and $|b_0| = 1$ in paper [4]. Necessary and sufficient condition of existence of $2T$-periodic solution for Eq. (1) was obtained under the case $|b_0| < 1/2$ in paper [5]. Necessary and sufficient condition of existence 2π-periodic solution for Eq. (1) with any constant b_0 was given in paper [6]. In this paper, some new necessary and sufficient conditions of existence $2T$-periodic solution for Eq. (1) with any constant b_0 are discussed by using the method of Fourier series. The result for using widely is obtained. It contents, improves and extends the main result in some papers. Existence of periodic solution for the equation in larger number of particular case can be checked by using result in this paper, but cannot be checked in another paper.

1 Main Results and Proofs

Consider algebraic equations

$$\begin{cases}(a_m + b_m)c_0 = k_0, \\ A(n)c_n + B(n)d_n = k_n, \\ -B(n)c_n + A(n)d_n = l_n,\end{cases} \tag{3}$$

where $\alpha = \frac{\pi}{T} > 0$ and

$$A(n) = \sum_{i=0}^{m}(\alpha n)^{(m-i)}\left[a_i\cos\frac{(m-i)\pi}{2} + b_i\cos\left(\frac{(m-i)\pi}{2} - \alpha nh\right)\right],$$

$$B(n) = \sum_{i=0}^{m}(\alpha n)^{(m-i)}\left[a_i\sin\frac{(m-i)\pi}{2} + b_i\sin\left(\frac{(m-i)\pi}{2} - \alpha nh\right)\right].$$

Let $R_{2m}(n) = A^2(n) + B^2(n)$, it is easy to see that $R_{2m}(n)$ is polynomial of degree $2m$ with respect to n, and using finite plus, minus and multiplication opera to derive its coefficients with

$$a_i, b_i, b_0, \cos\frac{(m-i)\pi}{2}, \sin\frac{(m-i)\pi}{2}, \cos\left[\frac{(m-i)\pi}{2} - \alpha nh_i\right], \sin\left[\frac{(m-i)\pi}{2} - \alpha nh_i\right].$$

Lemma 1 $R_{2m}(n) = \sum_{k=0}^{2m}\beta_{2m-k}(\alpha n)^{2m-k}$, where

$$\beta_{2m-p} = \sum_{i=0}^{p}(-1)^{i+p}[a_ia_{p-i} + a_ib_{p-i}\cos(\alpha nh_{p-i}) + b_ia_{p-i}\cos(\alpha nh_i) + b_ib_{p-i}\cos(h_i - h_{p-i})\alpha n] \quad (p = 0,2,\cdots,2m), \tag{4}$$

$$\beta_{2m-(4q-3)} = \sum_{i=0}^{4q-3}(-1)^i[-a_ib_{p-i}\sin(\alpha nh_{p-i}) + a_{p-i}b_i\sin(\alpha nh_i) - b_ib_{p-i}\sin\alpha n(h_{p-i} - h_i)] \quad \left(p = 4q-3,\ q = 1,2,\cdots,\left[\frac{m+1}{2}\right]\right), \tag{5}$$

$$\beta_{2m-(4q-1)} = \sum_{i=0}^{4q-1}(-1)^i[a_ib_{p-i}\sin(\alpha nh_{p-i}) - a_{p-i}b_i\sin(\alpha nh_i) + b_ib_{p-i}\sin\alpha n(h_{p-i} - h_i)] \quad \left(p = 2q-1,\ q = 1,2,\cdots,\left[\frac{m}{2}\right]\right). \tag{6}$$

($[a]$ represents the integer part of number a)

Theorem 1 If a constant $d > 0$ exists such that $\beta_{2(m-j)} \geqslant d$ for any n and $\beta_{2m-k} \geqslant 0$

$$k = \begin{cases} 0, & \text{when } j = 0, \\ 1,2,\cdots,2j-1, & \text{when } 1 \leqslant j \leqslant m_0 \triangleq \min\left\{\left[\frac{m}{2}\right],\left[\frac{2m-2}{3}\right]\right\}, \end{cases}$$

then there is a m order continuous differentiable $2T$-periodic solution of (1) if and only if algebraic Eqs. (3) have solutions with respect to c_0, c_n, d_n for all natural number n. Where $\beta_{2(m-1)}$ is coefficient of degree $2(m-j)$ for polynomial $R_{2m}(n)$.

Proof Proof of necessity is the same as that of Theorem 1 in [5]. To prove the sufficiency, we assume that Eqs. (3) have solutions. Now consider the following $m+1$ triangular series

$$c_0 + \sum_{n=1}^{\infty}(c_n \cos \alpha nt + d_n \sin \alpha nt), \tag{7}$$

$$\sum_{n=1}^{\infty}\left[(\alpha n)^k\left(c_n \cos\frac{k\pi}{2} + d_n \sin\frac{k\pi}{2}\right)\cos\alpha nt + (\alpha n)^k\left(d_n\cos\frac{k\pi}{2} - c_n \sin\frac{k\pi}{2}\right)\sin\alpha nt \qquad (k = 1,2,\cdots,m).\right. \tag{8}$$

In the following, we shall prove that the series are absolutely convergent and uniformly convergent. According to Lemma 1, we get that

$$A^2(n) + B^2(n) = R_{2m}(n) = \sum_{k=0}^{2m}\beta_{2m-k}(\alpha n)^{2m-k} = \sum_{k=0}^{2m-(2j-1)}\beta_{2m-k}(\alpha n)^{2m-k} + \beta_{2(m-j)}(\alpha n)^{2(m-j)} R_{2(m-j)-1}(n).$$

Since $\beta_{2m-k} \geqslant 0$ $(k = 0,1,2,\cdots,2j-1)$, $\beta_{2(m-j)} \geqslant d > 0$, we have that

$$A^2(n) + B^2(n) \geqslant d(\alpha n)^{2(m-j)} + R_{2(m-j)-1}(n). \tag{9}$$

Hence, there exists a constant $\delta > 0$ and sufficiently large natural number N such that when $n \geqslant N$,

$$A^2(n) + B^2(n) \geqslant \delta^2(\alpha n)^{2(m-j)} \text{ and } \alpha n \geqslant 1. \tag{10}$$

According to Eq. (3), it is easy to see that

$$[A^2(n) + B^2(n)]c_n = A(n)k_n - B(n)l_n, \tag{11}$$

$$[A^2(n) + B^2(n)]d_n = B(n)k_n + A(n)l_n. \tag{12}$$

From (10) – (12), when $n \geqslant N$, we see that

$$\delta(\alpha n)^k(|c_n|+|d_n|) \leqslant \delta(\alpha n)^{m-j}(|c_n|+|d_n|) \leqslant \sqrt{A^2(n)+B^2(n)}(|c_n|+|d_n|) = \frac{|A(n)k_n - B(n)l_n|}{\sqrt{A^2(n)+B^2(n)}} + \frac{|B(n)k_n - A(n)l_n|}{\sqrt{A^2(n)+B^2(n)}} \leqslant 2(|k_n|+|l_n|)$$
$$(k = 0,1,2,\cdots,m-j,\ 0 \leqslant j \leqslant m_0).$$

Notice that

$$|k_n|+|l_n| = \frac{1}{(\alpha n)^{(m-j)}}(|\alpha n|^{(m-j)}|k_n| + |\alpha n|^{(m-j)}|l_n|) \leqslant \frac{1}{2(\alpha n)^{2(m-j)}} + [(|\alpha n|^{(m-j)}|k_n|)^2 + (|\alpha n|^{(m-j)}|l_n|)^2].$$

We arrive at

$$(an)^k(|c_n|+|d_n|) \leqslant (an)^{(m-j)}(|c_n|+|d_n|) \leqslant$$
$$\frac{1}{\delta(an)^{2(m-j)}} + \frac{2}{\delta}[|(an)^{(m-j)}k_n|^2 + |(an)^{(m-j)}l_n|^2]$$
$$(k = 0,1,2,\cdots,m-j). \tag{13}$$

Multiplying inequality (13) by $(an)^j$, we have

$$(an)^k(|c_n|+|d_n|) \leqslant (an)^m(|c_n|+|d_n|) \leqslant$$
$$\frac{1}{\delta(an)^{(2m-3j)}} + \frac{2}{\delta}[|(an)^{(m-j)}k_n|^2 + |(an)^{(m-j)}l_n|^2](an)^j \leqslant$$
$$\frac{1}{\delta(an)^{(2m-3j)}} + \frac{2}{\delta}[|(an)^m k_n|^2 + |(an)^m l_n|^2]$$
$$(k = j, j+1,\cdots,m,\ 1 \leqslant j \leqslant m_0). \tag{14}$$

Notice that $j \leqslant m_0 \leqslant \left[\frac{m}{2}\right] \leqslant \frac{m}{2}$, i.e. $m - j \geqslant j$, therefore, one of (13) and (14) holds at least.

As $0 \leqslant j \leqslant \left[\frac{2m-2}{3}\right] \leqslant \frac{2m-2}{3}$, i.e. $2m - 3j \geqslant 2$, so series $\sum_{n=N}^{\infty} \frac{1}{\delta(n\alpha)^{2m-3j}}$ are convergent. Since $(n\alpha)^k\left(k_n \cos\frac{k\pi}{2} + l_n \sin\frac{k\pi}{2}\right)$ and $(n\alpha)^k\left(l_n \cos\frac{k\pi}{2} - k_n \sin\frac{k\pi}{2}\right)$ are Fourier coefficient of continuous function $f^{(k)}(t)$ $(k = 1,2,\cdots,m)$. It implies from Bessel inequality that

$$\sum_{n=N}^{\infty}[|(an)^k k_n|^2 + |(an)^k l_n|^2] = \sum_{n=N}^{\infty}\left[\left|(an)^k\left(k_n\cos\frac{k\pi}{2} + l_n \sin\frac{k\pi}{2}\right)\right|^2 + \right.$$
$$\left.\left|(an)^k\left(l_n\cos\frac{k\pi}{2} - k_n\sin\frac{k\pi}{2}\right)\right|^2 \leqslant \frac{1}{T}\int_{-T}^{T}|f^{(k)}(t)|^2 dt \quad (k = 1,2,\cdots,m).$$

This implies that $\sum_{n=N}^{\infty}[|(an)^k k_n|^2 + |(an)^k l_n|^2]$ are convergent $(k = 1,2,\cdots,m)$. Therefore, from (13), (14), we see that $\sum_{n=N}^{\infty}(an)^k(|c_n|+|d_n|)$ are convergent $(k = 0,1,2,\cdots,m)$. Thus series $\sum_{n=1}^{\infty}(an)^k(|c_n|+|d_n|)$ $(k = 0,1,2,\cdots,m)$ are convergent. We note

$$|c_n\cos(ant) + d_n\sin(ant)| \leqslant |c_n| + |d_n|,$$
$$\left|(an)^k\left(c_n\cos\frac{k\pi}{2} + d_n\sin\frac{k\pi}{2}\right)\cos ant + (an)^k\left(d_n\cos\frac{k\pi}{2} - c_n\sin\frac{k\pi}{2}\right)\sin ant\right| =$$
$$|(an)|^2\left|c_n\cos\left(\frac{k\pi}{2} + ant\right) + d_n\sin\left(\frac{k\pi}{2}ant\right)\right| \leqslant |(an)|^k(|c_n|+|d_n|)$$
$$(k = 1,2,\cdots,m).$$

So we conclude that triangular series (7), (8) are absolutely convergent and uniformly convergent. Now we define

$$x(t) = c_0 + \sum_{n=1}^{\infty}(c_n\cos ant + d_n\sin ant).$$

Then it is easy to prove that

$$x^{(k)}(t) = \sum_{n=1}^{\infty}\left\{(an)^k\left[\left(c_n\cos\frac{k\pi}{2} + d_n\sin\frac{k\pi}{2}\right)\cos ant + \left(d_n\cos\frac{k\pi}{2} - c_n\sin\frac{k\pi}{2}\right)\sin ant\right]\right\}$$
$$(k = 1,2,\cdots,m)$$

are continuous. We easily check that $x(t)$ satisfies Eq.(1). Therefore, we show that $x(t)$ is an m order continuous differentiable $2T$-periodic solution of Eq.(1). This completes the proof.

Corollary 1 Assume $|b_0| \neq 1$ or $b_0 \neq -1$ and $h_0 = 2kT$, k is a natural number. Then necessary and sufficient conditions of existence of m order continuous differentiable $2T$-periodic solution for (1) is the same as that in Theorem 1.

Proof According to Lemma 1, when $|b_0| \neq 1$, $\beta_{2m} = (1 + 2b_0\cos anh_0 + b_0^2) \geqslant (1 - |b_0|)^2 = d > 0$; when $b_0 \neq -1$ and $h_0 = 2Tk$, $\beta_{2m} = (1 + 2b_0\cos 2k\pi + b_0^2) = (1 + b_0)^2 = d > 0$. According to Theorem 1, we prove that Corollary 1 holds. This completes the proof.

Corollary 2 Let $m \geqslant 3$, $h_0 = h_1 = \frac{p}{q}T$, (p, q are unreduced natural number). If one of conditions as follows is satisfied:

(i) $(a_1 \pm b_1)^2 > 4(|a_2| + |b_2|)$;

(ii) $a_2 \leqslant 0$, $b_2 b_0 \leqslant 0$, $|a_1| \neq |b_1|$, $h_0 = h_2$.

Then necessary and sufficient conditions of existence of m order continuous differentiable $2T$-periodic solution for (1) is the same as that in Theorem 1.

Proof If $|b_0| \neq 1$, we can conclude Corollary 1 holds from Theorem 1. Now, we assume that $|b_0| = 1$, and consider two cases of delays.

① When $anh_0 = anh_1 = k\pi$, k is a natural number.

According to Lemma 1, we see that

$$R_{2m}(n) = \beta_{2m}(an)^{2m} + \beta_{2m-1}(an)^{2m-1} + \beta_{2(m-1)}(an)^{2(m-1)} + R_{2m-3}(n),$$

where $\beta_{2m} = 1 + 2b_0\cos k\pi + b_0^2 = (1 \pm b_0)^2 \geqslant 0$.

$$\beta_{2m-1} = \sum_{i=0}^{1}(-1)^i[a_i b_{1-i}\sin(anh_{1-i}) - b_i a_{1-i}\sin(anh_i) + b_i b_{1-i}\sin(an(h_{k-i}h_i))] = 0,$$

$$\begin{aligned}\beta_{2(m-1)} = &\sum_{i=0}^{2}(-1)^{i+1}[a_i a_{k-i} + a_i b_{k-i}\cos(anh_{k-i}) + b_i a_{k-i}\cos(anh_i) + \\ &b_i a_{k-i}\cos(an(h_i - h_{k-i}))] = [a_1^2 + b_1^2 + 2a_1 b_1\cos k\pi - \\ &2(a_2 + a_2 b_0\cos k\pi + b_0 b_2\cos(an(h_2 - h_0)) + b_2\cos(anh_2))].\end{aligned}$$

Using condition (i), we get that

$$\beta_{2(m-1)} \geqslant [(a_1 \pm b_1)^2 - 4(|a_2| + |b_2|)] = d_1 > 0, \text{ for all } n \quad (j = 1 \leqslant m_0, m \geqslant 3).$$

Using condition (ii), we get that

$$\begin{aligned}\beta_{2(m-1)} \geqslant &(a_1 \pm b_1)^2 - 2(a_2 + a_2 b_0\cos k\pi + b_0 b_2\cos(an(h_2 - h_0)) + b_2\cos(anh_2)) \geqslant \\ &(a_1 \pm b_1)^2 - 2a_2(1 + b_0\cos k\pi) - 2b_2 b_0[1 + b_0\cos k\pi] \geqslant \\ &(a_1 \pm b_1)^2 = d_2, \text{ for all } n \quad (j = 1 \leqslant m_0, m \geqslant 3).\end{aligned}$$

② $anh_0 = anh_1 = k\pi + \frac{\lambda}{p}\pi$ ($\lambda = 1,2,\cdots,p-1$), let

$$d = \max\left\{\left|\cos\frac{\pi}{p}\right|, \left|\cos\frac{2\pi}{p}\right|, \cdots, \left|\cos\frac{(p-1)\pi}{p}\right|\right\}, \quad \text{then } 0 < d < 1,$$

$\beta_{2m} = (1 + 2b_0\cos k\pi + b_0^2) = (2 + 2b_0\cos k\pi) \geqslant 2(1 - d) = d_3 > 0$, for all n. Thus condition of Theorem 1 is satisfied. Therefore, Corollary 2 holds. This completes the proof.

Corollary 3 Let $m \geqslant 4$; $h_0 = h_4 = \frac{p}{q}T$ (p, q are unreduced natural number). If $a_1 = b_1 = 0$, $a_2 \leqslant 0$, $b_2 b_0 \leqslant 0$, $|a_2| \neq |b_2|$, $a_4 \geqslant 0$, $b_4 b_0 \geqslant 0$, then necessary and sufficient conditions of existence of m order continuous differentiable $2T$-periodic solution for (1) is the same as that in Theorem 1.

Proof It is the same as Corollary 2.

We can prove that algebraic Eqs.(3) have unique solution if and only if

$$a_m + b_m \neq 0 \text{ and } A^2(n) + B^2(n) \neq 0 \quad (n = 1,2,\cdots). \tag{15}$$

Theorem 2 Assume correlation conditions in Theorem 1 or Corollaries 1 – 3 are satisfied, then there exists unique m order continuous differentiable $2T$-periodic solution of (1) if and only if (15) holds.

Theorem 3 Assume $b_0 \neq -1$, $h_j = 2Tk_j$, k_j are natural number ($j = 0,1,2,\cdots,m$). If $a_m + b_m \neq 0$ and one of conditions as follows is satisfied:

(i) $(a_{2j-1} + b_{2j-1})(a_{2j+1} + b_{2j+1}) \leqslant 0$, $j = 1,2,\cdots,m_1$, and there exists at least one natural number j_0 ($1 \leqslant j_0 \leqslant m_1 + 1$) such that $(a_{2j_0-1} + b_{2j_0-1}) \neq 0$.

(ii) $(a_{2(j-1)} + b_{2(j-1)})(a_{2j} + b_{2j}) \leqslant 0$, $j = 1,2,\cdots,m_2$, and there exists at least one natural number j_0 ($1 \leqslant j_0 \leqslant m_2 + 1$) such that $(a_{2(j_0-1)} + b_{2(j_0-1)}) \neq 0$.

Then Eq.(1) has unique m order continuous differentiable $2T$-periodic solution. Where

$$m_1 = \begin{cases} \dfrac{m-2}{2}, & m \text{ is even}, \\ \dfrac{m-1}{2}, & m \text{ is odd}, \end{cases} \qquad m_2 = \begin{cases} \dfrac{m}{2}, & m \text{ is even}, \\ \dfrac{m-1}{2}, & m \text{ is odd}. \end{cases}$$

Proof Using Corollary 2 and Theorem 2 $b_0 \neq -1$, $h_0 = 2k_0 T$, we see that Eq.(1) have unique m order continuous differentiable $2T$-periodic solution if and only if (15) holds.

When $m = 2k$, $k = 1,2,\cdots$, we have that

$$A(n) = (an)^{2k}(1 + b_0)(-1)^k + (an)^{2(k-1)}(a_2 + b_2)(-1)^{k-1} + \cdots + (a_{2k} + b_{2k})(-1)^0,$$

$$B(n) = (an)^{2k-1}(a_1 + b_1)(-1)^{k-1} + (an)^{2k-3}(a_3 + b_3)(-1)^{k-2} + \cdots + (an)(a_{2k-1} + b_{2k-1}).$$

When $m = 2k + 1$, $k = 0,1,2,\cdots$, we have that

$$A(n) = (an)^{2k}(a_1 + b_1)(-1)^{k+1} + (an)^{2(k-1)}(a_3 + b_3)(-1)^{k-2} + \cdots + (a_{2k-1} + b_{2k-1}),$$

$$B(n) = (an)^{2k+1}(1 + b_0)(-1)^k + (an)^{2k-1}(a_2 + b_2)(-1)^{k-1} + \cdots + (a_{2k} + b_{2k})(an).$$

Thus, we see that $B(n) \neq 0$, as m is even, and $A(n) \neq 0$ as m is odd, since condition (i) holds; $A(n) \neq 0$ as m is even, and $B(n) \neq 0$ and m is odd, since condition (ii) holds. Therefore, (15) holds for any n, this completes the proof.

Remark 1 For Eq.(1), the results in Theorem 1 in this paper is most generalized, the better result than that using the same method, i.e., the work for studying Eq.(1) is over by using the method of Fourier series. It cannot have new exposition.

Remark 2 The result in Theorem 1 in this paper is widely used. It contains results in Corollaries 1 – 3 and Theorem 3 for special case, improves and extends the main results in papers [1 – 6]. Existence of periodic solution for Eq.(1) in larger number of particular cases can be proved by using it, but cannot be proved in another paper.

Example 1 Neutral equation

$$x^{(10)}(t)+x^{(7)}(t)+8x^{(6)}(t)+x^{(10)}\left(t-\frac{\pi}{3}\right)+x^{(7)}\left(t-\frac{5}{6}\pi\right)-x^{(6)}\left(t-\frac{7}{6}\pi\right)+x\left(t-\frac{\pi}{6}\right)=\sin(12t).$$

This is a particular case of Eq.(1) existence of periodic solution for it can't be judged in papers [5,6], but we see it exists unique m order continuous differentiable $\frac{\pi}{6}$-periodic solution.

Example 2 Neutral equation

$$x''(t)-12x(t)+x''(t-\sqrt{2})+6x(t-3\sqrt{3})=2\cos\sqrt{2}\pi t,$$

where $b_0=1$, $a_1=b_1=0$, $h_0=\sqrt{2}$, $h_2=3\sqrt{2}$, $2T=\sqrt{2}$. This is a particular case of Eq.(1), conditions in papers [1 - 6] are not satisfied. But condition of Theorems 1 - 3 in this paper is satisfied. Hence there exist unique second-order continuous differentiable $\sqrt{2}$-periodic solution.

Example 3 Neutral equation

$$x^{(5)}(t)+4x^{(3)}(t)+x^{(2)}(t)+7x'(t)+2x(t)-x^{(5)}\left(t-\frac{2}{3}\right)+2x^{(3)}(t-\sqrt{3})-x^{(2)}(t-4)-x'\left(t-\frac{2}{3}\right)+x(t-8)=\sin\pi t.$$

This is a particular case of Eq.(1) condition satisfied in papers [5,6]. But since $m=5$, $h_0=h_4=\frac{2\times 2}{2\times 3}=\frac{2}{3}$, $2T=2$, it easily checks conditions (i) and (ii) on Corollary 3 that holds and $B(n)=(\pi n)^5\left(1-\cos\frac{2n}{3}\pi\right)+(\pi n)^3(4-2\cos\sqrt{3}\pi n)+\pi n\left(7-\cos\frac{2n}{3}\pi\right)>0$.

Thus there exists unique fifth-order continuous differentiable 2-periodic solution from Theorem 2.

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