Applied Mathematics and Mechanics
(English Edition, Vol 23, No 1, Jan 2002)

Published by Shanghai University,
Shanghai, China

Article ID: 0253-4827(2002)01-0013-05

THE TENSOR DENOTATION OF BELTRAMI SPHERICAL VORTICES AND THEIR SYMMETRY ANALYSIS *

HUANG Yong-nian (黄永年), HU Xin (胡 欣)

(Department of Mechanics and Engineering Science, State Key Laboratory for Turbulence Research, Peking University, Beijing 100871, P R China)

(Contributed by HUANG Yong-nian)

Abstract: *One kind of tensor denotation of nth Beltrami axisymmetric and nonaxisymmetric spherical vortices, their classification and symmetries were discussed. Chaotic phenomena will occur in the dynamic system of the nonaxisymmetric Beltrami spherical vortices. From these aspects, it is shown that the tensor denotation has more meaningful characters and nonaxisymmetric Beltrami shperical vortices are various and very complex.*

Key words: Beltrami flow; tensor denotation; symmetry; chaotic phenomena

CLC number: O351.3 **Document code**: A

Introduction

Beltrami flow is a special kind of flow. Its vorticity direction is parallel to its velocity direction everywhere. Moreover, its velocity and vorticity satisfy the linear Helmholtz equation. In fact, it is a kind of relatively simple flow. But there are many complex phenomena hidden in such kind of simple flow that we did not notice before and it is worth while to do further research. In this article, since 1990, we have done much systematic study on the Beltrami flow. This flow satisfies the following equation: $\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = \lambda \boldsymbol{u}$.

We have not only given out the general formula of the exact solutions of this kind of flow, but also given out the definite expressions of many different kinds of vortices of this flow, including spherical vortex, cylindrical vortex, ellipsoidal vortex, and helical vortex[1-4], etc. But they are all the axisymmetric vortices. In the practical flows, the vortices occurred in many situations are the nonaxisymmetric ones, so a kind of Beltrami flow nonaxisymmetric spherical vortices, their structures and symmetries are emphasized in this article. By using the method of the tensor denotation, we can show clearly the complex structures of this kind of flow. Moreover, by using the method of symmetry group, we have classified the vortices' structures and given out the regimentation number of this kind of spherical vortices. Finally, we found that the nonaxisymmetric can be gotten by superposing different axisymmetric vortices which have different axial directions.

---

* **Received date**: 2000-07-18; **Revised date**: 2001-08-10

Foundation item: The Special Funds for Major State Basic Research Project "Some Frontier Problems of Nonlinear Science" (G2000773)

Biography: HUANG Yong-nian (1939 -), Professor

1 The Tensor Denotation of nth Beltrami Flow Spherical Vortices

Firstly, let us consider the expression of axisymmetric nth Beltrami flow spherical vortices. In paper [4], we gave out the velocity expression at the situation of $\lambda = 1$:

$$\begin{cases} u_R = \dfrac{\mathrm{d}R}{\mathrm{d}t} = A_n n(n+1) R^{-3/2} \mathrm{J}_{n+1/2}(R) P_n(\cos\theta), \\ u_\theta = R\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = A_n R^{-1/2}\left[\mathrm{J}_{n-1/2}(R) - \dfrac{n}{R}\mathrm{J}_{n+1/2}(R)\right] P_n^1(\cos\theta), \\ u_\phi = R\sin\theta\dfrac{\mathrm{d}\phi}{\mathrm{d}t} = -A_n R^{-1/2}\mathrm{J}_{n+1/2}(R) P_n^1(\cos\theta). \end{cases} \tag{1}$$

And in paper [5], we pointed out that the velocity field of one kind of nth Beltrami spherical vortex could be written as follows:

$$u_i = \underbrace{\gamma_{ino\cdots qr\cdots st\cdots uv}}\, l_n l_o \cdots l_q m_r \cdots m_s n_t \cdots n_u n_v\,, \tag{2}$$

and we also gave out one kind of the concrete expression of the tensor $\gamma_{ino\cdots qr\cdots st\cdots uv}$. The reason why we call (2) one solution is that its expression is different from the one which is reached by the method of separation of variables (see [6]) at the high order, and they can not substitute each other.

Next we will give out another kind of the tensor denotation of this kind of nth Beltrami spherical vortex $\gamma_{ino\cdots qr\cdots st\cdots uv}$. It can be constructed as follows:

Firstly, we rewrite the solution of (1) in the rectangular coordinate system as:

$$u_i = u_R e_{R_i} + u_\theta e_{\theta_i} + u_\phi e_{\phi_i}\,,$$

here, $e_{R_i} = \dfrac{x_i}{R}$, $e_{\theta_i} = \left(\dfrac{x_i x_j}{R^2} - \delta_{ij}\right)\dfrac{l_j}{\sin\theta}$, $e_{\phi_i} = \dfrac{1}{R\sin\theta}\varepsilon_{ijk} l_j x_k$.

Secondly, we rewrite $P_n(\cos\theta)$, $P_n^1(\cos\theta)\cos\theta$ as the forms of $\cos\theta$, $\sin\theta$; and replace $\cos\theta$ as $\dfrac{x_p l_p}{R}$, we find that the expression will not have the odd terms of $\sin\theta$; and because $\sin^2\theta = 1 - \cos^2\theta$, so we can substitute all the even terms of $\sin\theta$ for $\cos\theta$. After incorporating the same terms, we have the expression of the spherical vortices as follows:

$$u_i = \underbrace{\gamma_{ino\cdots qr\cdots st\cdots uv}}\, l_n l_o \cdots l_q l_r \cdots l_s l_t \cdots l_u l_v\,. \tag{3}$$

Notice that (3) is the tensor denotation of the nth axisymmetric Beltrami spherical vortices in fact. We use the expression of (3) to take the place of the general tensor expression in (2), (nonaxisymmetric and axisymmetric). This is an extrapolation; but we can prove that the solutions got by this extrapolation also obey Navier-Stokes equation; more generally, this construction can ensure the velocity satisfies the properties of the Beltrami flow and the incompressible condition.

In (3), $n, o, \cdots, q, r, \cdots, s, t, \cdots, u, v$ can exchange their positions without changing the expression of this velocity u_i, so at the stipulation of the extrapolation, we will find that the expression $\gamma_{ino\cdots qr\cdots st\cdots uv}$ in (2) will not change with the exchanging of the sub-indices. This is the first property of this kind of tensor denotation.

Let us look at the spherical vortices' tensor denotations when $n = 1$ (expressed as α_{ij}) and $n = 2$ (expressed as β_{imn}):

1) when $n = 1$, $\alpha_{ij} = A^{(1)} x_i x_j + B^{(1)}\delta_{ij} + C^{(1)}\varepsilon_{ijk}x_k$, here, $A^{(1)} = A_1 R^{-5/2}\mathrm{J}_{5/2}(R)$, $B^{(1)} = A_1[R^{-1/2}\mathrm{J}_{1/2}(R) - R^{-3/2}\mathrm{J}_{3/2}(R)]$, $C^{(1)} = A_1 R^{-3/2}\mathrm{J}_{3/2}(R)$;

2) when $n = 2$,

$$\beta_{imn} = A^{(2)} x_i x_m x_n + B^{(2)} x_i \delta_{mn} + C^{(2)} (x_m \delta_{in} + x_n \delta_{im}) + D^{(2)} (x_n \varepsilon_{imp} + x_m \varepsilon_{inp}) x_p ,$$

here, $A^{(2)} = 3A_2 R^{-7/2} J_{7/2}(R)$, $B^{(2)} = -3A_2 R^{-5/2} J_{5/2}(R)$,

$$C^{(2)} = \frac{3}{2} A_2 [R^{-3/2} J_{3/2}(R) - 2R^{-5/2} J_{5/2}(R)], \quad D^{(2)} = \frac{3}{2} A_2 R^{-5/2} J_{5/2}(R).$$

2 The Symmetry Analysis of *n*th Beltrami Spherical Vortex Under the Tensor Denotation

Now we introduce the second property of this kind of nth Beltrami spherical vortex, also the most important property. In (2), $\boldsymbol{l} = (0,0,1)$, $\boldsymbol{m} = (1,0,0)$, $\boldsymbol{n} = (0,1,0)$, we let $n, o, \cdots, q$ take the value of 3; $r, \cdots, s$ the value of 1; and $t, \cdots, u, v$ the value of 2, so $u_i = \gamma_{i33\cdots31\cdots12\cdots22}$ is one kind of nth spherical vortex. We rotate the coordinates, letting the axes alternate in turn.

A) Firstly, axis 1→axis 3, axis 2→axis 1, axis 3→axis 2; It means that axis 1 turns to the position of the old axis 3, axis 2 turns to the position of the old axis 1, axis 3 turns to the position of the old axis 2, as shown in Fig. 1; What we get is another flow u_i', because the expression is different now. Comparing the components of u_i and u_i', we have

$$\begin{cases} u_1(x_1, x_2, x_3) = u_2'(x_1', x_2', x_3'), \\ u_2(x_1, x_2, x_3) = u_3'(x_1', x_2', x_3'), \\ u_3(x_1, x_2, x_3) = u_1'(x_1', x_2', x_3'). \end{cases} \quad (4)$$

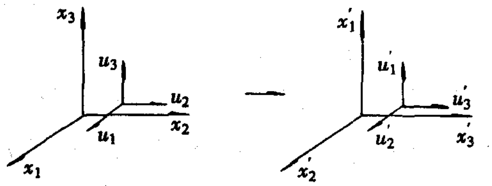

Fig. 1 The rotation symmetries of nth spherical vortex

Next we look for one kind of more general method which can stand for this property: let us see the structure of one spherical vortex $\gamma_{i\underbrace{1\cdots1}\underbrace{2\cdots2}\underbrace{3\cdots3}}$. It means that the number of sub-index 1 is a, the number of sub-index 2 is b, the number of sub-index 3 is c. So after rotating them in turn, we will have the following equations:

$$\begin{cases} \gamma_{1\underbrace{1\cdots1}\underbrace{2\cdots2}\underbrace{3\cdots3}}(x_1, x_2, x_3) = \gamma_{2\underbrace{1\cdots1}\underbrace{2\cdots2}\underbrace{3\cdots3}}(x_3, x_2, x_1), \\ \gamma_{2\underbrace{1\cdots1}\underbrace{2\cdots2}\underbrace{3\cdots3}}(x_1, x_2, x_3) = \gamma_{3\underbrace{1\cdots1}\underbrace{2\cdots2}\underbrace{3\cdots3}}(x_3, x_1, x_2), \\ \gamma_{3\underbrace{1\cdots1}\underbrace{2\cdots2}\underbrace{3\cdots3}}(x_1, x_2, x_3) = \gamma_{1\underbrace{1\cdots1}\underbrace{2\cdots2}\underbrace{3\cdots3}}(x_3, x_2, x_1). \end{cases} \quad (5)$$

B) Secondly, axis 1→axis 2, axis 2→axis 3, axis 3→axis 1. It means that axis 1 turns to the position of the old axis 2, axis 2 turns to the position of the old axis 3, axis 3 turns to the position of the old axis 1. We can also get the expression of the tensor transform as follows:

$$\begin{cases} \gamma_{1\underbrace{1\cdots1}\underbrace{2\cdots2}\underbrace{3\cdots3}}(x_1, x_2, x_3) = \gamma_{3\underbrace{1\cdots1}\underbrace{2\cdots2}\underbrace{3\cdots3}}(x_2, x_3, x_1), \\ \gamma_{2\underbrace{1\cdots1}\underbrace{2\cdots2}\underbrace{3\cdots3}}(x_1, x_2, x_3) = \gamma_{1\underbrace{1\cdots1}\underbrace{2\cdots2}\underbrace{3\cdots3}}(x_2, x_3, x_1), \\ \gamma_{3\underbrace{1\cdots1}\underbrace{2\cdots2}\underbrace{3\cdots3}}(x_1, x_2, x_3) = \gamma_{2\underbrace{1\cdots1}\underbrace{2\cdots2}\underbrace{3\cdots3}}(x_2, x_3, x_1). \end{cases} \quad (6)$$

In order to generalize the results of the two equations above, we use a simple notation, that is $\begin{pmatrix} 1 & a \\ 2 & b \\ 3 & c \end{pmatrix}$. It has two physical meanings: First, 1,2,3 in the first vertical row stand for the components of the velocity; Second, the second vertical row stands for the numbers of subindices 1,2,3 are a, b, c, respectively. So from (5) and (6), we get the result as follows:

$$\begin{pmatrix} 1 & a \\ 2 & b \\ 3 & c \end{pmatrix} \overset{\text{Law}}{=} \begin{pmatrix} 2 & c \\ 3 & a \\ 1 & b \end{pmatrix} \overset{\text{Law}}{=} \begin{pmatrix} 3 & b \\ 1 & c \\ 2 & a \end{pmatrix}. \tag{7}$$

We also introduce the third property, this is the property of exchanging. If we take the number of direction dyads a, b, c as a group, we can carry the transform of exchanging for them. For example, we can relate $\gamma_{i\underbrace{11\cdots11}_{a}\underbrace{22\cdots22}_{b}\underbrace{33\cdots33}_{c}}$ with $\gamma_{i\underbrace{11\cdots11}_{b}\underbrace{22\cdots22}_{a}\underbrace{33\cdots33}_{c}}$, and we get:

$$\begin{cases} \gamma_{1\underbrace{11\cdots1}_{a}\underbrace{22\cdots2}_{b}\underbrace{33\cdots3}_{c}}(x_1, x_2, x_3) = \gamma_{2\underbrace{11\cdots1}_{b}\underbrace{22\cdots2}_{a}\underbrace{33\cdots3}_{c}}(x_2, x_1, x_3), \\ \gamma_{2\underbrace{11\cdots1}_{a}\underbrace{22\cdots2}_{b}\underbrace{33\cdots3}_{c}}(x_1, x_2, x_3) = \gamma_{1\underbrace{11\cdots1}_{b}\underbrace{22\cdots2}_{a}\underbrace{33\cdots3}_{c}}(x_2, x_1, x_3), \\ \gamma_{3\underbrace{11\cdots1}_{a}\underbrace{22\cdots2}_{b}\underbrace{33\cdots3}_{c}}(x_1, x_2, x_3) = \gamma_{3\underbrace{11\cdots1}_{b}\underbrace{22\cdots2}_{a}\underbrace{33\cdots3}_{c}}(x_2, x_1, x_3). \end{cases} \tag{8}$$

So we add a new symmetries, this is the property of the coordinates exchanging, also the new property is gotten by the method of extrapolation.

3 The Regimentation Number of nth Beltrami Spherical Vortices

From the symmetry analysis in paper [5], we know that nth Beltrami spherical vortices have 3^n kinds; but from the three properties above, it can be reduced by transforming, because many kinds of spherical vortices are the same kind in fact.

Firstly, from the first property, we konw that if a, b, c, the number of the sub-index 1, 2, 3, are the same, those kinds of spherical vortices belong to the same kind, so they can be expressed as $\gamma_{i\underbrace{1\cdots1}_{a}\underbrace{2\cdots2}_{b}\underbrace{3\cdots3}_{c}}$;

Secondly, from the second property, $\gamma_{i\underbrace{1\cdots1}_{a}\underbrace{2\cdots2}_{b}\underbrace{3\cdots3}_{c}}$, $\gamma_{i\underbrace{1\cdots1}_{b}\underbrace{2\cdots2}_{c}\underbrace{3\cdots3}_{a}}$ and $\gamma_{i\underbrace{1\cdots1}_{c}\underbrace{2\cdots2}_{a}\underbrace{3\cdots3}_{b}}$ are aslo the same kind of spherical vortex.

Thirdly, from the third property, $\gamma_{i\underbrace{1\cdots1}_{a}\underbrace{2\cdots2}_{b}\underbrace{3\cdots3}_{c}}$, $\gamma_{i\underbrace{1\cdots1}_{b}\underbrace{2\cdots2}_{a}\underbrace{3\cdots3}_{c}}$, $\gamma_{i\underbrace{1\cdots1}_{c}\underbrace{2\cdots2}_{b}\underbrace{3\cdots3}_{a}}$ and $\gamma_{i\underbrace{1\cdots1}_{a}\underbrace{2\cdots2}_{c}\underbrace{3\cdots3}_{b}}$ are also the same kind.

So we can get that the regimentation number of the spherical vortices is the solution number Q_n of the indefinite equation $a + b + c = n$, $(a \geqslant b \geqslant c)$. After analyzing, we can get the recurrence formula as follows: $Q_{6p+r} = Q_{6(p-1)+r} + 6p + r$, (9)

here $0 \leqslant r < 6$. So we have

$$Q_{6p} = 3p(p+1) + 1, \quad Q_{6p+1} = (3p+1)(p+1), \quad Q_{6p+2} = (3p+2)(p+1),$$
$$Q_{6p+3} = 3(p+1)^2, \quad Q_{6p+4} = (3p+4)(p+1), \quad Q_{6p+5} = (3p+5)(p+1).$$

Thus many kinds of spherical vortices can be incorporated. Let's see the following examples:

1) There is on kind when $n = 1$, α_{i1}, α_{i2}, α_{i3} are the same kind;

2) There are two kinds when $n = 2$, β_{i11}, β_{i22}, β_{i33} are the same kind, others belong to another kind.

4 The Analysis of the Chaotic Phenomena Produced by a Single Second-Order Nonaxisymmetric Spherical Vortex

We notic that only a single second-order nonaxisymmetric spherical vortex can produce the chaotic phenomena, and do not need to superpose different spherical vortices with different axis directions. We will analyze the chaotic phenomena produced by second-order nonaxisymmetric spherical vortex β_{i31}. The components of β_{i31} can be written as follows in the spherical coordinate:

$$\begin{cases} \dfrac{\mathrm{d}R}{\mathrm{d}t} = 9A^{(2)} R^{-3/2} \mathrm{J}_{5/2}(R) \cos\theta \sin\theta \cos\phi, \\ \dfrac{\mathrm{d}\theta}{\mathrm{d}t} = -C^{(2)}(1 - 2\cos^2\theta)\cos\phi - D^{(2)} R\cos\theta \sin\phi, \\ \dfrac{\mathrm{d}\phi}{\mathrm{d}t} = \dfrac{1}{\sin\theta}\{D^{(2)} R(1 - 2\cos^2\theta)\cos\phi - C^{(2)}\cos\theta\sin\phi\}. \end{cases} \tag{10}$$

We can prove that (10) is non-integrable except at the fixed points, or to say that it will produce the chaotic phenomena.

The chaotic phenomena above owe to the two restrictions of axis $\boldsymbol{l}$ and axis $\boldsymbol{m}$ in the velocity field $u_i = \beta_{i31} l_3 m_1$; when the particle rotates along one axis, it is pulled by another axis, so there is nothing to do but rotate chaotically, so chaotic phenomena occurred. In fact the example discussed above can be turned into the results discussed in paper [4] by coordinate transformation, and it can also be gotten by superposing two second-order axisymmetric spherical vortices perpendicularly with their strength ratio 2:1, respectively. It is also the superposition of two second-order spherical vortices $\beta_{iop} l_o m_p$ and $\beta_{iop} m_o l_p$ in paper [5]. For the chaotic pattern of the flow, you can see the Poincare section in paper [4]. There are also chaotic phenomena in the higher order Beltrami flow nonaxisymmetric spherical vortices, and we have much work to do. You can see the analysis of the superposition of higher order axisymmetric spherical vortices in paper [7] if you are interested in this subject.

References:

[1] SHI Chang-chun, HUANG Yong-nian. Some properties of three-dimensional Beltrami flows[J]. *Acta Mechanica Sinica*, 1991, **7**(4):289 - 294.

[2] SHI Chang-chun, HUANG Yong-nian, CHEN Yao-song. On the Beltrami flows[J]. *Acta Mechanica Sinica*, 1992, **8**(4):289 - 294.

[3] SHI Chang-chun, HUANG Yong-nian, ZHU Zhao-xuan, et al. Chaotic phenomena produced by the spherical vortices in the Beltrami flows[J]. *Chinese Physics Letters*, 1992, **9**(10):515 - 518.

[4] HUANG Yong-nian, SHI Chang-chun, ZHU Zhao-xuan, et al. Lagrange turbulence and Beltrami flow[A]. In: Chinese Society of Theoretical and Applied Mechancis, Ed. *Advances in Modern Fluid Mechanics* Ⅱ[C]. Beijing: Science Press, 1993, 1 - 15. (in Chinese)

[5] HUANG Yong-nian, HU Xin. Superposition about the 3D cortex solutions of the fluid dynamic equation[J]. *Applied Mathematics and Mechancis (English Edition)*, 2000, **21**(12):1359 - 1370.

[6] FENG Qing-zeng, Pattern formation in foree-free magnetic fields and Beltrami flows[J]. *Zeitschrift fur Naturforschung*, 1996, **51a**:1161 - 1169.

[7] SHI Chang-chun, HUANG Yong-nian, YE Yi, et al. The general properties of the spherical vortices (SV) of nth order and the chaotic phenomena and of the ordered structures of the SV of 3rd order [J]. *Applied Mathematics and Mechancis (English Edition)*, 1994, **15**(4):353 - 358.